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Letters to the Editor

HOMOGENEOUS OMNÈS-MUSKHELISHVILI EQUATION WITH INELASTIC UNITARITY

ОДНОРОДНОЕ УРАВНЕНИЕ ОМНЕСА-МУСХЕЛИШВИЛИ С НЕУПРУГОЙ УНИТАРНОСТЬЮ

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The procedure is found which leads to infinitely many inhomogeneous Omnès-Muskhelishvili equations for the pion form-factor, incorporating the inelastic unitarity condition. All these eqiations can be transformed into a homogeneous one.

Using the analytic properties of the pion form-factor $F(s)$, where s is the square of the momentum transfer, and assuming at most one subtraction for $F(s)$, we have the dispersion relation

$$F(s)=1+\frac{s}{\pi}\int_{4\mu^2}^{\infty}\frac{\operatorname{Im}F(s')\,ds'}{s'(s'-s-i\varepsilon)}. \tag{1}$$

In Eq. (1) we can substitute for Im $F(s)$ from the unitarity relation

$$\operatorname{Im}F(s)=h^*(s)F(s)+\sigma(s), \qquad s\geq 4\pi^2 \tag{2}$$

where

$$h(s)=\frac{\eta(s)\,e^{2i\delta(s)}-1}{2i} \tag{3}$$

is the $I=J=1$ partial wave $\pi\pi$ scattering amplitude with the real parameters $\eta(s)$ and $\delta(s)$ representing the inelasticity and the phase shift, respectively. $\sigma(s)$ represents the inelastic contribution from the higher mass intermediate states, $\sigma(s)\neq 0$ for $s\geq 16\mu^2$. In this case, the combination of analyticity and unitarity leads tu the inhomogeneous Omnès-Muskhelishvili (OM) equation for the form factor [1].

An alternative way in formulating the OM equation is to substitute for ImF into Eq. (1) one of the following expressions derived from Eq. (2) [2].

$$\operatorname{Im}F=\tan\alpha_i\operatorname{Re}F+\varkappa_i \qquad i=1,2 \tag{4}$$

where

$$\tan\alpha_1=\frac{\operatorname{Im}h}{\operatorname{Re}h},\qquad \varkappa_1=-\frac{\operatorname{Im}\sigma}{\operatorname{Re}h} \tag{5}$$

$$\tan\alpha_2=\frac{\operatorname{Re}h}{1-\operatorname{Im}h},\qquad \varkappa_2=\frac{\operatorname{Re}\sigma}{1-\operatorname{Im}\sigma}$$

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and the solutions to the OM equations are of the form

$$F(s)=F_0^{\alpha_i}(s)\left\{1+\frac{s}{\pi}\int_{16\mu^2}^{\infty}\frac{\varkappa_i(s')\cos\alpha_i(s')}{|F_0^{\alpha_i}(s')|\,s'(s'-s-i\varepsilon)}ds'\right\},\ i=1,2 \tag{6}$$

where

$$F_0^{\alpha_i}(s)=\exp\left\{\frac{s}{\pi}\int_{4\mu^2}^{\infty}\frac{\alpha_i(s')}{s'(s'-s-i\varepsilon)}ds'\right\} \tag{7}$$

is the Omnès function corresponding to the phase $\alpha_i(s)$. Here it is the term $\varkappa_i(s)$ from Eq. (4) which is responsible for the inhomogeneity of the OM equation.

The purpose of this note is to show that besides these two equations there can be formulated infinitely many equations of the inhomogeneous OM type incorporating the inelastic one channel initarity condition for the form factor, and to show that these equations can all be reduced to the homogeneous OM equation, which is easier to solve and which uses as an input only the phase of the pion-pion scattering amplitude and the phase of the inelastic spectral function $\sigma(s)$.

It is quite natural to ask whether Eqs. (4) for $i=1,2$ are the only possible ones expressing ImF in terms of ReF. In order to answer this question we shall first calculate $F(s)$ from the unitarity. Eq. (2) represents the set of two linear algebraic equations in which six quantities ReF, ImF, Reh, Imh, Reσ and Imσ appear. We thus have two constraints acting on these six quantities and therefore, any two of them can be expressed in terms of the remaining four. For the form factor we then have

$$\operatorname{Re}F=\frac{\operatorname{Re}h\operatorname{Re}\sigma+(1-\operatorname{Im}h)\operatorname{Im}\sigma}{\operatorname{Im}h-|h|^2}, \tag{8}$$

$$\operatorname{Im}F=\frac{\operatorname{Re}h\operatorname{Im}\sigma+\operatorname{Im}h\operatorname{Re}\sigma}{\operatorname{Im}h-|h|^2} \tag{9}$$

providing that $h(s)$ does not obey the elastic unitarity relation

$$\operatorname{Im}h\neq|h|^2, \tag{10}$$

which is true for $\sigma(s)\neq 0$, $s\geq 16\mu^2$.

Thus, in the time-like inelastic region we can directly calculate the pion formfactor $F(s)$ from $h(s)$ and $\sigma(s)$ by means of these algebraic equations[1]), while in the elastic region ($\sigma=0$, $\eta=1$) the unitarity enables us to calculate only the phase of the formfactor but not its absolute value.

For the phase of $F(s)$ we have

$$\tan\alpha_0=\frac{\operatorname{Re}\sigma\operatorname{Im}h+\operatorname{Im}\sigma\operatorname{Re}h}{\operatorname{Re}\sigma\operatorname{Re}h+\operatorname{Im}\sigma(1-\operatorname{Im}h)},\qquad s\geq 16\mu^2. \tag{11}$$

This equation defines the phase of $F(s)$ in the inelastic region. However, by taking the limit $\sigma\to 0$, $\eta\to 1$ we find that

$$\lim_{\substack{\sigma\to 0\\ \eta\to 1}}\tan\alpha_0=\frac{\operatorname{Im}h}{\operatorname{Re}h}=\tan\delta,\qquad 4\mu^2\leq s<16\mu^2, \tag{12}$$

which is consistent with elastic unitarity and we can write the phase of $F(s)$ as $\alpha_0(s)$ in both the elastic and the inelastic regions.

[1]) During the preparation of the manuscript we have learned that Pham and Truong found a similar algebraic expression for the form factor, however, in terms of $\sigma\,e^{i\delta}$ and η [3].

Thus

$$\mathrm{Im}F = h_0^* F = \tan\alpha_0 \,\mathrm{Re}F, \qquad s \geq 4\mu^2, \tag{13}$$

where

$$h_0 = h + \frac{\sigma^*}{F^*} = e^{i\alpha_0} \sin\alpha_0 \tag{14}$$

and Eq. (13) is also of the form of Eq. (4) (with $i=0$), only now $\varkappa_0 = 0$. This means that instead of the inelastic unitarity (2) we have now an equivalent formulation which looks like the elastic unitarity (Eq. (13)) but with a new "effective" amplitude $h_0(s)$, which in the elastic region is identical with the elastic amplitude $h(s)$. Therefore the standard procedure leads to the corresponding OM equation of the homogeneous type, (since $\varkappa_0 = 0$) with the solution $F_0^{\alpha_0}(s)$ given by Eq. (7) with $\alpha_0(s)$ from Eq. (11) and this solution is equivalent to the solutions given by Eq. (6).

Having in mind Eqs. (8) and (9) we can find arbitrarily many expressions for ImF of the form of Eq. (4) and therefore also arbitrarily many equivalent inhomoneous OM equations. We have freedom to choose for tg α_i, $i = 3, 4, \ldots$, etc. any function containing some of the four quantities Reh, Imh, Reσ and Imσ, $\varkappa_i$ is then given by Eq. (4) in this way:

$$\varkappa_i = \frac{\mathrm{Re}\sigma(\mathrm{Im}h - \tan\alpha_i \,\mathrm{Re}h) + \mathrm{Im}\sigma[\mathrm{Re}h - \tan\alpha_i(1-\mathrm{Im}h)]}{\mathrm{Im}h - |h|^2} \tag{15}$$

or vice versa, we can choose for $\varkappa_i$ and calculate $\tan\alpha_i$. For $\tan\alpha_i$, $i=1,2$ we get $\varkappa_i$, $i=1,2$ of Eq. (5). It is obvious that for $\tan\alpha_0$ the corresponding $\varkappa_0 = 0$.

The solution of the OM equation, where ImF is given by Eq. (4) with $i \neq 0, 1, 2$, is of the following form:

$$F(s) = \exp\left\{\frac{s}{\pi}\int_{4\mu^2}^{16\mu^2} \frac{\delta(s')\,ds'}{s'(s'-s-i\varepsilon)}\right\} \exp\left\{\frac{s}{\pi}\int_{16\mu^2}^{\infty} \frac{\alpha_i(s')\,ds'}{s'(s'-s-i\varepsilon)}\right\} \times \tag{16}$$

$$\times \left\{1 + \frac{s}{\pi}\int_{16\mu^2}^{\infty} \frac{\varkappa_i(s')\cos\alpha_i(s')\,ds'}{|F_0^\delta(s')\,F_0^{\alpha_i}(s')|\,s'(s'-s-i\varepsilon)}\right\}$$

where the first exponential defines $F_0^\delta(s)$ and the second defines $F_0^{\alpha_i}(s)$; $(i = 3, 4, 5, \ldots)$.

Hence, for ImF we have infinitely many different expressions of the form of Eq. (4) and there correspond to them different formulations of the OM equation, as well as different forms of solutions to these equations, which are, however, mathematically all equivalent. The solution for any particular case is easily obtained from this general form of solutions (Eq. (16)). For instance if we choose $\alpha_i = 0$, then $\varkappa_i = \mathrm{Im}F$ and we get the solution in terms of the Omnès function for the phase $\delta(s)$ along the elastic cut and in terms of the dispersion integral with ImF over the inelastic cut. Or, if we require for the phase $\alpha_i(s)$ to be equal to the phase shift $\delta(s)$ in the elastic region, i.e. we choose such phase $\alpha_i(s)$ that

$$\lim_{\substack{\sigma\to 0\\ \eta\to 1}} \tan\alpha_i = \tan\delta, \qquad 4\mu^2 \leq s < 16\mu^2,$$

then the solution given by Eq. (16) goes over to the solution given by Eq. (6).

Let us note that in order to find the phase $\alpha_0(s)$ we do not need to know the whole complex function $\sigma(s)$, the knowledge of the phase of $\sigma(s)$ is sufficient.

Eq. (11) can be rewritten in the following form

$$\tan\alpha_0 = \frac{\tan\alpha_1 + \tan\omega}{1 + \frac{1-\mathrm{Im}h}{\mathrm{Im}h}\tan\alpha_1\tan\omega} = \tan(\alpha_1+\omega)\,\frac{1 - \tan\alpha_1\tan\omega}{1 + \frac{1-\mathrm{Im}h}{\mathrm{Im}h}\tan\alpha_1\tan\omega} \tag{17}$$

where $\omega(s)$ is the phase of the inelastic term $\sigma(s)$, i.e. $\sigma(s) = |\sigma(s)| \exp i\omega(s)$. Since $(1-\mathrm{Im}h)/\mathrm{Im}h$ never equals -1, it follows from there that α_0 can never be equal to $\alpha_1 + \omega$, except in the trivial case when $\tan\omega = 0$.

In this way the phase of $F(s)$ is directly related to the phase of $h(s)$ and to the phase of $\sigma(s)$, and the knowledge of $\alpha_1(s)$ and $\omega(s)$ permits us to determine the whole formfactor $F(s)$.

The existence of various equivalent solutions gives us freedom to use the most convenient from the physical point of view. For instance, in [2] it is noted that because of the stange behaviour of α_2 in the resonance region the formula with $\tan\alpha_2$ is practically less convenient than the formula with $\tan\alpha_1$. Moreover, the trouble with the concrete calculation of $F(s)$ is that $\sigma(s)$ is not known in practice and different assumptions and approximations for $\sigma(s)$ are used. It may happen in future that we shall have at least some partial knowledge of $\sigma(s)$, say, we might get the data only on the phase of $\sigma(s)$ or only on the real or the imaginary part of $\sigma(s)$, and then the existence of various solutions will enable us to choose the most appropriate in order to calculate the pion form factor.

REFERENCES

[1] Omnès R.: Nuovo Cim. *8* (1958), 316;
Muskhelishvili N. I.: *Singuljarnyje uravnenia.* Gos. izd. phys.-mat. lit. Moskva 1962.

[2] Pham T. N., Truong T. N.: Phys. Rev. D *14* (1976), 185.

[3] Pham T. N., Truong T. N.: Phys. Rev. *D 16* (1977), 896.

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