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PATH INTEGRALS AND THE KLEIN-GORDON EQUATION (II)

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Path integrals formalism presented previously is generalized for the cases in which we have to consider the amplitudes of the histories which change their orientation in time.

ИНТЕГРАЛЫ ПО ТРАЕКТОРИЯМ И УРАВНЕНИЕ КЛЕЙНА—ГОРДОНА

Формализм интегралов по траекториям, разработанный в предыдущих работах, обобщается на случаи амплитуд, изменяющих во времени своё направление.

I. INTRODUCTION

In our previous paper [4] we formulated the new path integrals representation of the Klein—Gordon (KG) equation. The above mentioned formalism does not offer the expression for the propagator function but it allows us to put into relation (by means of the path integrals) the solution of the KG equation and the boundary conditions (the wave function is given at two different temporal instants t_1 and t_2) and the solution we look for $t \in \langle t_1, t_2 \rangle$). The corresponding continual integrals can be transparently interpreted and they are in a close relation with those used by Feynman [1—2] in his formulation of nonrelativistic quantum mechanics.

The fields we considered in [4] were static magnetic fields and the continual integration had to be performed over histories which did not change their orientation in time. However, if we consider a spinless particle interacting with the field which is not a static magnetic one, we are forced to generalize our formalism. The generalization leads to the necessity to take into account the amplitudes of such histories which change their orientation in time (from a certain point the history can proceed in a time decreasing or increasing direction) and this generalization is the aim of this paper. In what follows we shall confine ourselves to the homogeneous electric fields and the paper is organized as follows. In Sect. II we present a simple example as the motivation for our speculations. In Sect. III we shall give the expression for the amplitude of the history. The solution of the KG

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equation in terms of the path integrals will be given in Sect. IV. The potentials we consider have the discontinuity point $t_n = t_0 + n\varepsilon$, $n = 0, 1, 2, \ldots$ and the formal limit $\varepsilon \to 0$ will be performed in Sect. V. Sect. VI contains some concluding remarks.

II. SPINLESS PARTICLE IN HOMOGENEOUS ELECTRIC FIELD

The dynamics of the spinless particle interacting with a homogeneous electric field can be given by the equation (we put $c = 1$)

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 \psi(\mathbf{x}, t) = [M_0^2 + (-i\hbar\,\mathrm{grad} - e\mathbf{A}(t))^2\, \psi(\mathbf{x}, t). \quad (1)$$

In what follows we shall look for $\psi(\mathbf{x}, t)$, $t \in \langle 0, \tau\rangle$ when $\psi(\mathbf{x}, 0)$ and $\psi(\mathbf{x}, \tau)$ are given. For this purpose let us consider the potential

$$\mathbf{A}(t) = \mathbf{A}_0 + (\mathbf{A}_1 - \mathbf{A}_0)\,\theta(t - t_1), \quad (2)$$

where $\mathbf{A}_{0,1}$ are constant vectors, $0 < t_1 < \tau$, and $\theta(t)$ is the step function. In this case ψ can be written as

$$\psi = \psi^{(+)} + \psi^{(-)},$$

where

$$\psi^{(\pm)}(\mathbf{x}, t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} C^{(\pm)}(\mathbf{p}) \exp\left[\frac{i}{\hbar}\left(\mathbf{px} \mp t\sqrt{M_0^2 + (\mathbf{p} - e\mathbf{A}_0)^2}\right)\right]$$

for $t \in \langle 0, t_1\rangle$ and likewise for $t \in \langle t_1, \tau\rangle$.

If

$$\psi^{(+)}(\mathbf{x}, 0) = \psi_0^{(+)}(\mathbf{x}) \qquad \psi^{(-)}(\mathbf{x}, \tau) \equiv 0, \quad (3)$$

then the solution of Eq. (1) (with the potential (2)) is

$$\psi_{0 \le t \le t_1}(\mathbf{x}, t) = \int d\mathbf{x}_0 \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \exp\left[\frac{i}{\hbar}(\mathbf{p}(\mathbf{x} - \mathbf{x}_0) - tH_0(\mathbf{p}))\right] \psi_0^{(+)}(\mathbf{x}_0) + \quad (4)$$

$$+ \int d\mathbf{x}_1 \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} R(\mathbf{p}) \exp\left[\frac{i}{\hbar}(\mathbf{p}(\mathbf{x}_1 - \mathbf{x}) - (t_1 - t)\,H_0(-\mathbf{p}))\right] \psi^{(+)}(\mathbf{x}_1, t_1)$$

$$\psi_{t_1 \le t \le \tau}(\mathbf{x}, t) = \int d\mathbf{x}_1 \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} D(\mathbf{p}) \exp\left[\frac{i}{\hbar}(\mathbf{p}(\mathbf{x} - \mathbf{x}_1) - (t - t_1)\,H_1(\mathbf{p}))\right] \psi^{(+)}(\mathbf{x}_1, t_1) \quad (4')$$

where

$$H_i(\mathbf{p}) = \sqrt{M_0^2 + (\mathbf{p} - e\mathbf{A}_i)^2}$$

$$\psi^{(+)}(\mathbf{x}_1, t_1) = \int d\mathbf{x}_0 \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \exp\left[\frac{i}{\hbar}(\mathbf{p}(\mathbf{x}_1 - \mathbf{x}_0) - t_1 H_0(\mathbf{p}))\right] \psi_0^{(+)}(\mathbf{x}_0)$$

and

$$1 + R(-\mathbf{p}) = D(\mathbf{p}) \quad (5)$$

$$1 - R(-\mathbf{p}) = D(\mathbf{p})\, H_1(\mathbf{p})/H_0(\mathbf{p}). \quad (6)$$

Eqs. (5—6) secure the continuity of ψ and $(\partial/\partial t)\,\psi$ at the point $t = t_1$.

Using the results obtained in [4] one can write

$$\psi_{0 \le t \le t_1}(\mathbf{x}, t) = \int d\mathbf{x}_0 \left[\int \mathscr{D}\sqrt{m} \int \mathscr{D}\mathbf{q} \int \mathscr{D}\mathbf{p}\, \delta\left(\mathbf{x} - \mathbf{x}_0 - \int_0^t d\xi\, \dot{\mathbf{q}}(\xi)\right) \times \exp\left\{\frac{i}{\hbar}\int_0^t d\xi\, [\mathbf{p}\dot{\mathbf{q}} - \mathscr{H}^{(+)}(\mathbf{p}, m, \xi)]\right\}\right] \psi_0^{(+)}(\mathbf{x}_0) + \quad (7)$$

$$+ \int d\mathbf{x}_0 \left[d\mathbf{x}_1 \int \mathscr{D}\sqrt{m} \int \mathscr{D}\mathbf{q} \int \mathscr{D}\mathbf{p}\, \delta\left(\mathbf{x}_1 - \mathbf{x} - \int_t^{t_1} d\xi\, \dot{\mathbf{q}}\right) \times \exp\left\{\frac{i}{\hbar}\int_t^{t_1} d\xi [\mathbf{p}\dot{\mathbf{q}} - \mathscr{H}^{(-)}(\mathbf{p}, m, \xi)]\right\} R\big(\mathbf{p}(t_1)\big) \int \mathscr{D}\sqrt{m_0} \int \mathscr{D}\mathbf{q}_0 \int \mathscr{D}\mathbf{p}_0 \times \delta\left(\mathbf{x}_1 - \mathbf{x}_0 - \int_0^{t_1} d\xi\, \dot{\mathbf{q}}_0\right) \exp\left\{\frac{i}{\hbar}\int_0^{t_1} d\xi\, [\mathbf{p}_0\dot{\mathbf{q}}_0 - \mathscr{H}^{(+)}(\mathbf{p}_0, m_0, \xi)]\right\}\right] \psi_0^{(+)}(\mathbf{x}_0).$$

Saving space we do not write down the equally complicated expression for (4′). In the equality (7)

$$\mathscr{H}^{(\pm)}(\mathbf{p}, m, \xi) = \frac{(\mathbf{p} \mp e\mathbf{A})^2}{2m} + \frac{1}{2}\left(m + \frac{M_0^2}{m}\right), \quad \dot{\mathbf{q}} = d\mathbf{q}/dt$$

and the definition of the continual integrals is the following

$$\int \mathscr{D}\sqrt{m} \int \mathscr{D}\mathbf{q} \int \mathscr{D}\mathbf{p}\, \delta\left(\mathbf{x} - \mathbf{x}_0 - \int_{t_0}^t d\xi\, \dot{\mathbf{q}}\right) \exp\left\{\frac{i}{\hbar}\int_{t_0}^t d\xi [\mathbf{p}\dot{\mathbf{q}} - \mathscr{H}^{(+)}(\mathbf{p}, m, \xi)]\right\} \equiv \lim_{N \to \infty}\left[\prod_{K=1}^{N} \int_0^\infty \frac{d\sqrt{m_K}}{\left(\frac{\pi\hbar}{i\varepsilon}\right)^{1/2}} \int \frac{d\mathbf{p}_K}{(2\pi\hbar)^3}\right]\left[\prod_{K=1}^{N-1} \int d\mathbf{q}_K\right] \times \exp\left\{\frac{i}{\hbar}[\mathbf{p}_K(\mathbf{q}_K - \mathbf{q}_{K-1}) - \varepsilon\mathscr{H}^{(+)}(\mathbf{p}_K, m_K, t_K)]\right\},$$

where $\varepsilon = (t - t_0)/N$, and the variables marked by the index K are taken at the time $t_K = t_0 + K\varepsilon$ and $\mathbf{q}(t_N) = \mathbf{x}$, $\mathbf{q}(t_0) = \mathbf{x}_0$.

We interprete the expression in the square bracket of the first term on the right-hand side of Eq. (7) as the sum of amplitudes of all possible histories connecting the points $(\mathbf{x}_0, 0)$, $(\mathbf{x}, t)$ and running in a time increasing direction. The amplitude of the history $(m, \mathbf{p}, \mathbf{q})$ has the form

$$\exp\left\{\frac{i}{\hbar}\int_0^t \mathrm{d}\xi\, [\mathbf{p}\dot{\mathbf{q}} - \mathcal{H}^{(+)}(\mathbf{p}, m, \xi)]\right\}$$

and $\mathcal{H}^{(+)}$ has the following meaning. In Petráš's [3] formulation of classical mechanics the action integral is given as (see [4] too)

$$S = \int_{t_0}^{t} \mathrm{d}\xi L = \int_{t_0}^{t} \mathrm{d}\xi \left[\frac{m\dot{\mathbf{q}}^2}{2} + e\dot{\mathbf{q}}\mathbf{A} - e\varphi - \frac{1}{2}\left(m + \frac{M_0^2}{m}\right)\right]$$

(we consider the interaction of a particle with the electromagnetic field) and

$$\mathcal{H}^{(+)} = \mathbf{p}\dot{\mathbf{q}} - L(\dot{\mathbf{q}}, \mathbf{q}, m, \xi), \qquad \mathbf{p} = \partial L/\partial\dot{\mathbf{q}}.$$

The second term of the right-hand side of Eq. (7) is interpreted as follows. The expression in square brackets is the sum of the amplitudes of all histories which connect the points in question but these histories change at the spacetime points $(\mathbf{x}_1, t_1)$ their orientation in time, i.e., from $(\mathbf{x}_1, t_1)$ they proceed backwards in time.

The generalization of these results for the case when $\mathbf{A}(t)$ (a physically reasonable function) is approximated by $\mathbf{A}_\varepsilon(t)$ given as

$$\mathbf{A}_\varepsilon(t) = \mathbf{A}(n\varepsilon) \quad \text{for} \quad t \in \langle n\varepsilon, (n+1)\,\varepsilon\rangle$$

$$n = 0, \pm 1, \pm 2, \ldots$$

will be performed in the following sections.

III. AMPLITUDE OF THE HISTORY

Let us consider the continual curve (history) in an eight dimensional space $(t, m, \mathbf{p}, \mathbf{q})$ and let the curve start at the point $(t_0, m_0, \mathbf{p}_0, \mathbf{q}_0)$ and end at $(t_1, m_1, \mathbf{p}_1, \mathbf{q}_1)$. It means that if $t = t(\lambda)$, $m = m(\lambda)$, $\mathbf{p} = \mathbf{p}(\lambda)$, $\mathbf{q} = \mathbf{q}(\lambda)$, $\lambda \in \langle 0, 1\rangle$ are parametric equations of the curve, then $t_0 = t(0)$, $m_0 = m(0)$, ... $t_1 = t(1)$, $m_1 = m(1)$, ... If for example $t(\lambda)$ is a decreasing function for $\lambda \in \langle \lambda_1, \lambda_2\rangle$, then we shall say that the history we consider runs backwards in time between the points $(t(\lambda_1), m(\lambda_1), \ldots)$, $(t(\lambda_2), m(\lambda_2), \ldots)$. If t decreases (or increases) along certain parts of the history, then for this part the inverse function $\lambda = \lambda(t)$ exists and we shall formally represent the whole history by the "functions" $(m(t), \mathbf{p}(t), \mathbf{q}(t))$.

On the basis of the previous results we assign to the history $s \equiv (m, \mathbf{p}, \mathbf{q})$ connecting the points $(\mathbf{x}_0, t_0)$, $(\mathbf{x}, t)$ the amplitude

$$A(s|\mathbf{A}_\varepsilon|\,(\mathbf{x}, t), (\mathbf{x}_0, t_0)) = a(s|\mathbf{A}_\varepsilon) \exp\left(\frac{i}{\hbar} S(s)\right). \tag{8}$$

We shall calculate the functional $S(s)$ in the following way. Let us divide the history s into the parts s_i, $i = 1, 2, \ldots, k$, where s_i represents the history running only forwards or only backwards in time. We assign to s_i the action

$$S(s_i) = \int_{t_i}^{t_i' > t_i} \mathrm{d}t\, [\mathbf{p}\dot{\mathbf{q}} - \mathcal{H}^{(\pm)}(\mathbf{p}, m, t)], \tag{9}$$

where the sign + (−) is taken as that time when s_i runs forwards (backwards) in time. Then

$$S(s) = \sum_{i=1}^{k} S(s_i).$$

The number $a(s|\mathbf{A}_\varepsilon)$ is given as follows.

i) We assign to the point $t_n = t_0 + n\varepsilon$ at which the history s changes its orientation in time the number

$$R^{(\pm)}(\mathbf{p}, t_n) = \pm\frac{H(\pm\mathbf{p}, t_n) - H(\pm\mathbf{p}, t_{n-1})}{H(\pm\mathbf{p}, t_n) + H(\pm\mathbf{p}, t_{n-1})},$$

where

$$H(\mathbf{p}, t_n) = \sqrt{M_0^2 + (\mathbf{p} - e\mathbf{A}(t_n))^2}.$$

ii) We assign to the point $t_k = t_0 + k\varepsilon$ at which the history s does not change its orientation in time the number

$$D^{(\pm)}(\mathbf{p}, t_n) = 1 + R^{(\mp)}(-\mathbf{p}, t_n). \tag{11}$$

$R^{(-)}$, $D^{(-)}$ ($R^{(+)}$, $D^{(+)}$) are taken at that time when the history proceeds from t_n backwards (forwards) in time. Now $a(s|\mathbf{A}_\varepsilon)$ is equal to the product of all such numbers which must be taken into account for s. If s does not change its orientation in time and does not contain any point of discontinuity of $\mathbf{A}_\varepsilon$, then $a(s|\mathbf{A}_\varepsilon) = 1$. If s changes its orientation in time at the point which is not a discontinuity point of $\mathbf{A}_\varepsilon$, then $a(s|\mathbf{A}_\varepsilon) = 0$.

IV. THE SOLUTION OF THE KG EQUATION IN TERMS OF PATH INTEGRALS

According to the previous results the solution of Eq. (1) (with the potential $\mathbf{A}_\varepsilon$ (for $t \in \langle 0, \tau\rangle$ can be formally written in the following form

$$\psi(\mathbf{x},t)=\int d\mathbf{x}_0 \sum_s A(s|\mathbf{A}_\varepsilon|(\mathbf{x},t),(\mathbf{x}_0,0))\,\psi^{(+)}(\mathbf{x}_0,0)+ \tag{12}$$

$$+\int d\mathbf{x}_1 \sum_s A(s|\mathbf{A}_\varepsilon|(\mathbf{x},t),(\mathbf{x}_1,\tau))\,\psi^{(-)}(\mathbf{x}_1,\tau).$$

The summation in Eq. (12) must be performed over all the histories which connect the points in question and all the points of which belong to the time interval $\langle 0,\tau\rangle$ (it means that if we have the parametric expression of histories, then $t(\lambda)\in\langle 0,\tau\rangle$ for all values of λ).

V. FORMAL LIMIT $\varepsilon\to 0$

In this section the formal limit $\varepsilon\to 0$ will be performed. This step seams to be the weakest point of this paper but the final results indicate that the limit $\varepsilon\to 0$ as given below, is not meaningless.

For a sufficiently small ε we can write ($\mathbf{A}$ is assumed to be differentiable)

$$R^{(\pm)}(\mathbf{p},t_n)\approx \pm\frac{\varepsilon}{2}\frac{\partial}{\partial t_n}\ln H(\pm\mathbf{p},t_n)= \tag{13}$$

$$=\frac{e\mathbf{E}(t_n)(\mathbf{p}\mp e\mathbf{A}(t_n))\,\varepsilon}{2H^2(\pm\mathbf{p},t_n)},$$

where $\mathbf{E}=-d\mathbf{A}/dt$.

Hence the sum of amplitudes of all histories which connect the points $(\mathbf{x}_0,0)$, $(\mathbf{x},t)$ and run forwards in time is equal to

$$K_0^{(+)}(\mathbf{x},t;\,\mathbf{x}_0,0)=\int \mathscr{D}\sqrt{m}\int \mathscr{D}\mathbf{q}\int \mathscr{D}\mathbf{p}\;\delta\Big(\mathbf{x}-\mathbf{x}_0-\int_0^t d\xi\,\dot{\mathbf{q}}(\xi)\Big)\times \tag{14}$$

$$\times\exp\Big\{\frac{i}{\hbar}\int_0^t d\xi\Big[\mathbf{p}\dot{\mathbf{q}}-\mathscr{H}^{(+)}(\mathbf{p},m,\xi)+$$

$$+ie\hbar\frac{\mathbf{E}(\xi)(\mathbf{p}(\xi)-e\mathbf{A}(\xi))}{2H^2(\mathbf{p}(\xi),\xi)}\Big]\Big\},$$

where we used (n is running over a corresponding set of indices)

$$\prod_n D^{(+)}(\mathbf{p},t_n)\approx\prod_n\Big(1-\frac{e\mathbf{E}(t_n)(\mathbf{p}(t_n)-e\mathbf{A}(t_n))\,\varepsilon}{2H^2(\mathbf{p}(t_n),t_n)}\Big)\approx$$

$$\approx\exp\Big\{-\int_0^t d\xi\,\frac{e\mathbf{E}(\mathbf{p}-e\mathbf{A})}{2H^2(\mathbf{p},\xi)}\Big\}.$$

The continual integral (14) can be easily calculated. Performing, at first, the integration over m by means of formula [6]

$$\int_0^\infty \frac{d\sqrt{m}}{\left(\frac{\pi\hbar}{i\varepsilon}\right)^{1/2}}\exp\Big\{-\frac{i\varepsilon}{2\hbar}\Big[\frac{\mathbf{a}^2}{m}+m\Big]\Big\}=\exp\Big\{-\frac{i\varepsilon}{\hbar}\sqrt{\mathbf{a}^2}\Big\}$$

and then the additional integrations, we obtain

$$K_0^{(+)}(\mathbf{x},t;\,\mathbf{x}_0,0)=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\sqrt{\frac{H(\mathbf{p},0)}{H(\mathbf{p},t)}}\times \tag{15}$$

$$\times\exp\Big\{\frac{i}{\hbar}\Big[\mathbf{p}(\mathbf{x}-\mathbf{x}_0)-\int_0^t d\xi H(\mathbf{p},\xi)\Big]\Big\}.$$

Likewise the sum of amplitudes of all histories running backwards in time and connecting the points $(\mathbf{x}_1,\tau>t)$, $(\mathbf{x},t)$ is equal to

$$K_0^{(-)}(\mathbf{x},t;\,\mathbf{x}_1,\tau)=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\sqrt{\frac{H(\mathbf{p},\tau)}{H(\mathbf{p},t)}}\times \tag{16}$$

$$\times\exp\Big\{\frac{i}{\hbar}\Big[\mathbf{p}(\mathbf{x}-\mathbf{x}_1)-\int_t^\tau d\xi H(\mathbf{p},\xi)\Big]\Big\}.$$

The sum of amplitudes of histories which connect the points $(\mathbf{x}_0,0)$, $(\mathbf{x},t)$ and change at the points $(\mathbf{x}_1,t_1\in\langle t,\tau\rangle)$ their orientation in time (from $(\mathbf{x}_1,t_1)$ they proceed backwards in time) is equal to

$$\int d\mathbf{x}_1\int_t^\tau dt_1\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\sqrt{\frac{H(\mathbf{p},t_1)}{H(\mathbf{p},t)}}\cdot(-1)\frac{\dot H(\mathbf{p},t_1)}{2H(\mathbf{p},t_1)}\times \tag{17}$$

$$\times\exp\Big\{\frac{i}{\hbar}\Big[\mathbf{p}(\mathbf{x}-\mathbf{x}_1)-\int_t^{t_1} d\xi\, H(\mathbf{p},\xi)\Big]\Big\}\cdot K_0^{(+)}(\mathbf{x}_1,t_1;\,\mathbf{x}_0,0)$$

($\dot H$ as usual means $(d/dt)\,H$).

These results allow us to write the corresponding expression for any sum of amplitudes and indicate that Eq. (1) will be equivalent to the pair of the following integral equations.

Let us write

$$\psi(\mathbf{x},t)=\chi^{(+)}(\mathbf{x},t)+\chi^{(-)}(\mathbf{x},t), \tag{18}$$

where

$$\chi^{(\pm)}(\mathbf{x},t)=\int d\mathbf{x}_0\sum_s^{(\pm)} A(s|\mathbf{A}|(\mathbf{x},t),(\mathbf{x}_0,0))\,\psi^{(+)}(\mathbf{x}_0,0)+ \tag{19}$$

$$+\int d\mathbf{x}_1\sum_s^{(\pm)} A(s|\mathbf{A}|(\mathbf{x},t),(\mathbf{x}_1,\tau))\,\psi^{(-)}(\mathbf{x}_1,\tau)$$

and $\Sigma^{(+)}(\Sigma^{(-)})$ means the summation over all histories which come to $(\mathbf{x}, t)$ in the time increasing (decreasing) direction. According to the results of this section

$$\chi^{(+)}(\mathbf{x}, t) = \chi_0^{(+)}(\mathbf{x}, t) + \int_0^t \mathrm{d}\xi \int \mathrm{d}\mathbf{y} \int \frac{\mathrm{d}\mathbf{p}}{(2\pi\hbar)^3} \sqrt{\frac{H(\mathbf{p}, \xi)}{H(\mathbf{p}, t)}} \times \tag{20}$$

$$\times \frac{\dot{H}(\mathbf{p}, \xi)}{2H(\mathbf{p}, \xi)} \exp \left\{ \frac{\mathrm{i}}{\hbar} \left[\mathbf{p}(\mathbf{x}-\mathbf{y}) - \int_\xi^t \mathrm{d}\eta \, H(\mathbf{p}, \eta) \right] \right\} \chi^{(-)}(\mathbf{y}, \xi)$$

$$\chi^{(-)}(\mathbf{x}, t) = \chi_0^{(-)}(\mathbf{x}, t) - \int_t^\tau \mathrm{d}\xi \int \mathrm{d}\mathbf{y} \int \frac{\mathrm{d}\mathbf{p}}{(2\pi\hbar)^3} \sqrt{\frac{H(\mathbf{p}, \xi)}{H(\mathbf{p}, t)}} \times \tag{20'}$$

$$\times \frac{\dot{H}(\mathbf{p}, \xi)}{2H(\mathbf{p}\xi)} \exp \left\{ \frac{\mathrm{i}}{\hbar} \left[\mathbf{p}(\mathbf{x}-\mathbf{y}) - \int_t^\xi \mathrm{d}\eta \, H(\mathbf{p}, \eta) \right] \right\} \chi^{(+)}(\mathbf{y}, \xi),$$

where

$$\chi_0^{(+)}(\mathbf{x}, t) = \int \mathrm{d}\mathbf{x}_0 \, K_0^{(+)}(\mathbf{x}, t;\ \mathbf{x}_0, 0) \, \psi^{(+)}(\mathbf{x}_0, 0)$$

$$\chi^{(-)}(\mathbf{x}, t) = \int \mathrm{d}\mathbf{x}_1 \, K_0^{(-)}(\mathbf{x}, t;\ \mathbf{x}_1, \tau) \, \psi^{(-)}(\mathbf{x}_1, \tau).$$

Eqs. (20, 20′) can be regarded as the new representation of Eq. (1).

To comment these results let us write ψ in the form

$$\psi(\mathbf{x}, t) = \int \frac{\mathrm{d}\mathbf{p}}{(2\pi\hbar)^3} \Phi(\mathbf{p}, t) \exp\left(\frac{\mathrm{i}}{\hbar} \mathbf{p}\mathbf{x}\right). \tag{21}$$

Then

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + \frac{1}{\hbar^2} H^2(\mathbf{p}, t) \, \Phi = 0. \tag{22}$$

From the formal point of view Eq. (22) is identical with the onedimensional time-independent Schrödinger equation. Using results obtained in [5] or [6] we almost immediately obtain Eqs. (20, 20′).

VI. CONCLUSIONS

The generalization of our formalism for the case of the interaction of a spinless particle with arbitrary electromagnetic fields is not trivial. Our speculations indicate that the generalization leads to very complicated mathematical expressions but the formalism preserves its basic features.

It seems that the formalism can be practically used in these cases only when the effects connected with the histories which change their orientation in time can be neglected. Namely in our formalism the amplitude of the history has the form

$$\exp\left[\frac{\mathrm{i}}{\hbar}(S + \hbar S_1)\right],$$

where $S = \int \mathrm{d}t \, [\mathbf{p}\dot{\mathbf{q}} - \mathscr{H}^{(\pm)}]$ and S_1 is connected with the possibility of the change of orientation in time of the history. It is natural to expect that in the case of the "almost" classical motion of a particle one can put $S_1 = 0$, or equivalently $R^{(\pm)} = 0$.

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