acta phys. slov. 28. (1978), No. 1

PATH INTEGRALS FOR INERTIALESS CLASSICAL PARTICLES UNDERGOING RAPID STOCHASTIC TREMBLING (I. EXAMPLES)

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Feynman's path integrals are studied in reference to the Fokker-Planck (Smoluchowski) equation. Examples are presented including the motion of an inertialess classical charged particle between electrodes in plate and cylindrical capacitors with charges fluctuating rapidly as Gaussian white-noise stochastic processes. Another example concerns magnetodiffusion of a charged particle in an unpolarized electromagnetic beam characterized by a white-noise spectrum.

ИНТЕГРАЛЫ ПО ТРАЕКТОРИЯМ ДЛЯ НЕИНЕРЦИОННЫХ ЧАСТИЦ, ПОДВЕРГАЮЩИХСЯ БЫСТРОМУ СЛУЧАЙНОМУ ВИБРАЦИОННОМУ ВОЗДЕЙСТВИЮ (I. ПРИМЕРЫ)

В работе рассматриваются фейнмановские интегралы по траекториям в связи с уравнением Фоккера-Планка (Смолуховского). В качестве примеров приводится движение неинерционной заряженной классической частицы между электродами конденсатора с плоскими и цилиндрическими досками, на которых заряды испытывают быстрие флуктуации типа гауссовского случайного шума с равномерным спектром. Следующий пример рассматривает магнитодиффузию заряженной частицы в неполяризованном электромагнитном пучке со спектром белого шума.

I. INTRODUCTION

The Feynman path integrals [1] were devised, and continue being used, as a to enabling to anatomize quantum-mechanical concepts. They are free of noncommutative operators and this is their most attractive property. Although in the majority of quantum-mechanical problems the usual operator treatment proved to be more effective than the path integral one, the latter did lead to new achievements at which it would be difficult to arrive in another way. This was first manifested by Feynman in his theory of liquid helium and polarons [1]. More recent efforts to use the path integrals concerned, after a paper by Edwards and

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Gulyaev [2], quantum-mechanical and quantum-statistical calculations of simplified models of disordered solids. One of them which was devised by the author with appreciation of its exact solvability — a model of semiconductors with randomly fluctuating bands [3] — yielded, when the path integral technique was applied, mathematically simple results (cf. e.g. [4], [5], [6], [7] for mathematical amendment and [8], [9] for applications).

There are, therefore, good reasons to look for other applications of the path integrals. In particular, the path integrals are naturally expected to be applicable in a theory paying heed to stochastic motions. The aim of the present contribution which consists of two papers (Part I and Part II) is to support the statement by some examples as well as by a general mathematical analysis.

The purpose of the examples presented in Part I is to give some sound motivation for the abstraction and mathematical details given in Part II including its Appendices. The basic point to which we will punctiliously adhere is the assertion that a certain class of Gaussian stochastic processes of white-noise type is equivalent to a class of the path integrals. With reference to such equivalence, it is easy to juxtapose each path integral to a partial differential equation identifiable with the Fokker-Planck (Smoluchowski) equation implied by the corresponding stochastic process.

The examples were intended to be addressed to theorists but they do offer also possibilities for some simple and interesting experimentation. In Sections III. 2 and III. 3 of the present paper we shall show that the mobility of a charged Brownian particle situated between electrodes of a plate or cylindrical capacitor, respectively, can be controllably enhanced if the charge on the electrodes is contrived to undergo violent Gaussian white-noise fluctuations, inducing thus an electrical, stochastically trembling force. Similarly, some enhancement of the Brownian motion can be induced by illuminating charged particles by intense white-noise electromagnetic radiation (not necessarily white light). Such a problem is shown to be easily solvable even when a longitudinal d. c. magnetic field is applied (the magnetodiffusion discussed in Section III. 4). For all the examples, explicit formulae are derived for the „diffusion coefficients" as functions of the magnitude of the stochastic force.

Part II (the sequel of the present paper) will deal with technical and conceptual nuances in the use of the path integrals in probabilistic applications. Detailed attention will be paid to mathematical questions arising from the non-differentiability of the paths of the path-integral theory. In particular, we shall show that the integrals written in the exponents of the probabilistic path integrals are not, in general, the same notion as the ordinary (Riemann-Lebesgue) integrals. We shall call them the "Markov process integrals" and suggest in what way they are related to the usual integrals for which the classical calculus applies.

Although the assumption of the white-noise property will be basic for our treatment, we shall also indicate difficulties arisen in attempts to omit this assumption. Finally, we shall also suggest a generalization of our theory to systems of an arbitrary number of degrees of freedom.

II. PROBABILISTIC EMPLOYMENT OF THE PATH INTEGRALS

A pattern of a stochastic theory related to the path integrals is the theory of the Brownian motion as presented by Wiener. The mathematical literature developing Wiener's ideas is extensive (of. e.g. the monography by Yeh [10]). However, much of the mathematical literature is hardly intelligible to physicists who need simple methods of calculations. Practical instructions how to develop stochastic theories in a language better comprehensible to physicists were given by Feynman [1]. For this reason, we will use Feynman's notation and when speaking about an "integral over Wiener measure" we have in mind the same item as the Feynman (functional) "path integral".

The Wiener stochastic process along a line (x-axis) is described by the diffusion (Einstein) equation

$$\frac{\partial P(x,t|x_0)}{\partial t}=D\,\frac{\partial^2 P(x,t|x_0)}{\partial x^2},\quad P(x,+0|x_0)=\delta(x-x_0). \tag{1}$$

The function $P(x,t|x_0)$ is the conditional probability density for a Brownian particle to find itself in the point x at the time $t>0$ if it was definitely in the point x_0 at $t_0=0$. It is given by the path integral

$$P(x,t|x_0)=\int_{x_0,0}^{x,t}\mathscr{D}x(\tau)\exp\left\{-\frac{1}{4D}\int_0^t d\tau\,\dot{x}^2(\tau)\right\}, \tag{2}$$

where $x(\tau)$ is a general continuous path connecting the points x_0, x so that $x(0)=x_0$, $x(t)=x$. The meaning of the symbol $\int\mathscr{D}x(\tau)$ for the functional "path integration" was thoroughly explained in [1]. If the motion of the particle is not restricted by boundaries, the solution to equation (1) is the well-known function

$$P(x,t|x_0)=\frac{1}{(4\pi Dt)^{1/2}}\exp\left\{-\frac{(x-x_0)^2}{4Dt}\right\}. \tag{3}$$

There is, however, also another type of a stochastic problem, where not an equation like (1) is prescribed but *vice versa* rather a *path integral can be derived in advance* for a stochastic process. If we have the path integral at our disposal, we may utilize its equivalence with the Fokker-Planck equation due to the process and solve preferably the latter instead of carrying out the functional path integration.

As an example of a probabilistic problem, Feynman and Hibbs discuss briefly (without entering mathematical details in Section 12.6, [1]) the path integral due to

a classical damped harmonic oscillator driven by a rapidly fluctuating force $f(t)$. The dynamics of the oscillator is described by the stochastic differential (Langevin) equation

$$m\ddot{x}+\gamma\dot{x}+kx=f(t), \tag{4}$$

where $f(t)$ is a Gaussian x-independent random function whose mean is zero and the autocorrelation function $\langle f(t')\,f(t'')\rangle$ is of the white-noise type:

$$\langle f(t')\,f(t'')\rangle=\eta^2\delta(t'-t''),\quad \langle f(t)\rangle=0. \tag{5}$$

The brackets $\langle\ \ \rangle$ denote averaging with respect to the fluctuations of $f(t)$. The corresponding path integral

$$P(x,t|x_0)=\int_{x_0,0}^{x,t}\mathscr{D}x(\tau)\exp\left\{-\frac{1}{2\eta^2}\int_0^t d\tau[m\ddot{x}(\tau)+\gamma\dot{x}(\tau)+kx(\tau)]^2\right\} \tag{6}$$

is clearly of a more complicated structure than the Wiener integral (2). Nevertheless, to exemplify the correspondence between a dynamic equation like (4) and a path integral for $P(x,t|x_0)$, it is sufficient to consider (for the sake of simplicity) an inertialess particle when $m=0$. Then the path integral (6), and more general path integrals related to it, remain to be functional integrals with respect to the Wiener measure. In such a case it is frequently only a matter of comparison to obtain the partial differential equation of the Fokker-Planck type for the probability density $P(x,t|x_0)$.

The problems solved below involve both the presence of boundaries from which the particle cannot return back ("absorbing boundaries") and the presence of some deterministic force $F(x)$ sharing its exertion on the particle simultaneously with the stochastic force $f(t)$. The coefficient of friction γ will be taken in all cases constant.

III. EXAMPLES

III.1 One-dimensional Brownian motion without boundaries

The basic stochastic equation is reduced to the form

$$\gamma\dot{x}=f(t). \tag{7}$$

This equation is a good approximation to equation (4) for $k\neq 0$, but $k\to 0$, whenever the condition

$$m\ll\gamma^2/k \tag{8}$$

is satisfied [11]. The probability that the random force $f(t)$ has its actualization within an infinitesimal stripe $(f(\tau), f(\tau)+\delta f(\tau))$, $0<\tau<t$, $\delta f(\tau)>0$, is given by the expression

$$P_f[f(\tau)]=\frac{\delta f(\tau)}{C_f}\exp\left\{-\frac{1}{2\eta^2}\int_0^t d\tau f^2(\tau)\right\}, \tag{9}$$

where the normalization constant C_f is defined by the functional integral

$$C_f=\int_0^t \delta f(\tau)\exp\left\{-\frac{1}{2\eta^2}\int_0^t d\tau\, f^2(\tau)\right\}.$$

After replacing $f(\tau)$ by $\gamma\dot{x}(\tau)$ we obtain the probability that the path $x(\tau)$ of the particle has its actualization in the stripe $(x(\tau), x(\tau)+\delta x(\tau))$, $0<\tau<t$, $\delta x(\tau)>0$:

$$P_x[x(\tau)]=\frac{1}{C_f}\left|\frac{\delta f(\tau)}{\delta x(\tau)}\right|\exp\left\{-\frac{1}{2\eta^2}\int_0^t d\tau\, \dot{x}^2(\tau)\right\}. \tag{10}$$

The symbol $|\delta f(\tau)/\delta x(\tau)|=J$ denotes the absolute value of the Jacobian for the transformation $\{f(\tau)\}\to\{x(\tau)\}$ determined by relation (7). The Jacobian is no more path-dependent since the relation (7) is linear. Therefore, we may define the functional differential $\mathscr{D}x(\tau)=(J/C_f)\,\delta x(\tau)$ and take the path integral

$$P(x, t|x_0)=\int_{x_0\cdot 0}^{x,t}\mathscr{D}x(\tau)\exp\left\{-\frac{\gamma^2}{2\eta^2}\int_0^t d\tau\, \dot{x}^2(\tau)\right\} \tag{11}$$

as properly normalized so that

$$\int_{-\infty}^{\infty} dx\, P(x, t|x_0)=1 \tag{12}$$

for all values of x_0 and $t>0$. The path integral (11) fulfils the equation

$$\frac{\partial P(x, t|x_0)}{\partial t}=\frac{\eta^2}{2\gamma^2}\frac{\partial^2 P(x, t|x_0)}{\partial x^2}. \tag{13}$$

Its solution with respect to the initial condition $P(x, +0|x_0)=\delta(x-x_0)$ is

$$P(x, t|x_0)=\frac{1}{(2\pi t)^{1/2}}\frac{\gamma}{\eta}\exp\left\{-\frac{\gamma^2}{2\eta^2 t}(x-x_0)^2\right\}. \tag{14}$$

Hence, by comparing (14) with (3), (13) with (1) or (11) with (2), it is seen that

$$D=\frac{\eta^2}{2\gamma^2}. \tag{15}$$

In a physical application we might investigate a light — "massless" in the sense of the strong inequality (8) — particle subject to some rapid trembling force of a strength measured by η, whilst the environment resists the motion of the particle proportionally to γ.

It should be pointed out that D does not correspond to what used to be called "turbulent diffusion". Namely, the latter presumes random convections. This is not the case. Formula (15) corresponds to the diffusion called forth by some trembling force (say, electrical field) letting the liquid (assumed to be electrically neutral and unpolarizable) in which the particle is submersed, be practically quiet, except for a close region around the particle.

III. 2 Presence of boundaries

The simplest problem of interest with boundaries is the Brownian motion between planar plates of a capacitor. Let the coordinates of the plates be $x_1=0$, $x_2=a$ and the medium between them a liquid dielectric of permitivity ε. We assume that a surface charge of density $\sigma(t)=\sigma_0+\Delta\sigma(t)$ is induced on the plate 1, the charge of the opposite plate 2 being — $\sigma(t)$. The electrical field between the plates is $E(t)=\sigma(t)/\varepsilon$ and a Brownian particle of charge q is driven by the force $f(t)+F(x)$, where

$$f(t)=q\Delta\sigma(t)/\varepsilon,\quad F(x)=q\sigma_0/\varepsilon\equiv F. \tag{16}$$

If the charge of the capacitor suffers Gaussian white-noise changes defined by the autocorrelation function

$$\langle\Delta\sigma(t')\Delta\sigma(t'')\rangle=\eta_\sigma^2\delta(t'-t''),\quad \langle\Delta\sigma(t)\rangle=0,$$

we have the dynamic (Langevin) equation

$$\gamma\dot{x}-F=f(t) \tag{17}$$

with the Gaussian random force $f(t)$ satisfying relations (5), where $\eta=\eta_\sigma q/\varepsilon$. The path integral

$$P(x, t|x_0)=\int_{x_0\cdot 0}^{x,t}\mathscr{D}x(\tau)\exp\left\{-\frac{1}{2\eta^2}\int_0^t d\tau[\gamma\dot{x}(\tau)-F]^2\right\} \tag{18}$$

can be rewritten, by using the transformation

$$x'(\tau')=x(\tau)-F\tau/\gamma,\quad \tau'=\tau, \tag{19}$$

into the Wiener form

$$P(x', t'|x_0')=\int_{x_0'\cdot 0}^{x',t'}\mathscr{D}x'(\tau)\exp\left\{-\frac{\gamma^2}{2\eta^2}\int_0^{t'} d\tau\, \dot{x}'^2(\tau)\right\}, \tag{20}$$

where $x_0'=x_0$, $-Ft'/\gamma<x'<a-Ft'/\gamma$, $-Ft'/\gamma<x_0'<a-Ft'/\gamma$.

The paths $x'(\tau)$ in integral (20) are confined in the interval $\Delta\equiv(-Ft'/\gamma, a-Ft'/\gamma)$. The confinement can be proven as follows. Let us define the function

$I_\Delta(x)$ equal to unity if x belongs to Δ and otherwise zero, and write down a more general path integral

$$P_V(x',t'|x_0') = \int_{x_0',0}^{x',t'} \mathscr{D}x'(\tau) \exp\left\{-\frac{\gamma^2}{2\eta^2}\int_0^{t'} d\tau \dot{x}'^2(\tau) - \int_0^{t'} d\tau\, V(x'(\tau))\right\} \tag{20a}$$

with unconfined paths: $-\infty < x'(\tau) < \infty$. (The path integral (20a) with the exponent enriched by a "potential-energy term" usually termed the "Feynman-Kac integral" by some authors.) The differential equation corresponding to (20a) is

$$\frac{\partial P_V(x',t'|x_0')}{\partial t'} = \frac{\eta^2}{2\gamma^2}\frac{\partial^2 P_V(x',t'|x_0')}{\partial x'^2} - V(x')\,P_V(x',t'|x_0'). \tag{21a}$$

For our proof, we specify the "potential energy":

$$V(x) = [1 - I_\Delta(x)]\,V_0, \qquad V_0 = \text{const} > 0.$$

Clearly, we can realize the confinement of the paths $x'(\tau)$ into the region Δ by letting V_0 tend to infinity. However, from the presence of V in the *exponent* of (20a) we can deduce that, for x' lying out of Δ (and then also for x_0' lying out of Δ),

$$V(x')\,P_V(x',t'|x_0') \to 0$$

if $V_0 \to \infty$. Thus, we may omit the last term of equation (21a) and write the equation

$$\frac{\partial P(x',t'|x_0')}{\partial t'} = \frac{\eta^2}{2\gamma^2}\frac{\partial^2 P(x',t'|x_0')}{\partial x'^2}, \quad P(x', +0|x_0') = \delta(x'-x_0'), \tag{21}$$

or equation

$$\frac{\partial P(x,t|x_0)}{\partial t} = \frac{\eta^2}{2\gamma^2}\frac{\partial^2 P(x,t|x_0)}{\partial x^2} - \frac{F}{\gamma}\frac{\partial P(x,t|x_0)}{\partial x}, \tag{22}$$

$$P(x, +0|x_0) = \delta(x - x_0), \tag{22a}$$

for $0<x<a$, $0<x_0<a$. The limiting procedure $V_0 \to \infty$ is tantamount to the statement that the surfaces $x=0$, $x=a$ are absorbers of the Brownian particles. Say, whenever a particle touches the plate, it remains stuck on it. Then we have the boundary conditions

$$P(0,t|x_0) = P(a,t|x_0) = 0. \tag{23}$$

Equations (22), (22a) and (23) determine the conditional probability density $P(x,t|x_0)$ completely. We may use the function $P(x,t|x_0)$, for instance, in calculating the probability

$$\int_0^a dx\, P(x,t|x_0)$$

that a particle starting from the point x_0 at the time $t_0 = 0$ will escape absorption either on the boundary $x=0$ or $x=a$, i. e. it will survive in the interior space $0<x<a$, up to the time instant t.

Of course, if the boundaries $x=0$, $x=a$ were ideally reflecting, i. e. not absorbing, boundary conditions (23) would be replaced by zero values of the normal derivatives $\partial P(x,t|x_0)/\partial x$ on the boundaries and the integral $\int_0^a dx\, P(x,t|x_0)$ would become equal to unity for all values of x_0 and $t \geq 0$.

The unique solution to equation (22) with respect to conditions (22a) and (23) is

$$P(x,t|x_0) = \sum_{n=-\infty}^{\infty} [P_F^\infty(x+2na, t|x_0) - P_{-F}^\infty(-x+2na, t|x_0)], \tag{24}$$

where we have denoted

$$P_F^\infty(x,t|x_0) = \frac{1}{(2\pi t)^{1/2}}\frac{\gamma}{\eta}\exp\left\{-\frac{\gamma^2}{2\eta^2 t}\left(x - \frac{Ft}{\gamma} - x_0\right)^2\right. \tag{25}$$

(a function defined on the whole real axis, $-\infty < x < \infty$).

Notice that if equation (22) were solved directly by the Fourier method, the result would look differently (involving trigonometric functions) but could be transformed into the form (24) by using the Poisson summation formula. (This was shown by the author for $F=0$ upon another occasion elsewhere [12]).

Thus we may conclude that the random temporal changes of the capacitor charge can be used for separation of colloidal particles. By intensifying the strength η of the trembling, the diffusion can be considerably enhanced at will.

III. 3 A cylindrical capacitor

The idea of section III. 2 can be further developed for a cylindrical capacitor. We have in mind a thin wire along the z-axis with some charge density $\zeta(t)$ and the second electrode charged oppositely, determined by the radius $\varrho = a$. The electrical force $f(\varrho,t) + F(\varrho)$ acting upon a charge q of a Brownian particle is directed radially. Again, if $\zeta(t) = \zeta_0 + \Delta\zeta(t)$ where $\Delta\zeta(t)$ is a Gaussian random component, $\langle\Delta\zeta(t)\rangle = 0$, $\langle\Delta\zeta(t')\,\Delta\zeta(t'')\rangle = \eta_\zeta^2\delta(t'-t'')$, we obtain

$$f(\varrho,t) = \frac{q\Delta\zeta(t)}{2\pi\varepsilon\varrho}, \qquad F(\varrho) = \frac{q\zeta_0}{2\pi\varepsilon\varrho} \tag{26}$$

and the dynamic stochastic equation reads

$$\gamma\dot{\varrho} - \frac{q\zeta_0}{2\pi\varepsilon\varrho} = \frac{q\Delta\zeta(t)}{2\pi\varepsilon\varrho}. \tag{27}$$

Fortunately, we can rewrite this equation into the form

$$\frac{1}{2}\gamma\frac{\mathrm{d}}{\mathrm{d}t}\varrho^2-\frac{q\zeta_0}{2\pi\varepsilon}=\frac{q}{2\pi\varepsilon}\Delta\zeta(t), \tag{28}$$

where the "random force term" (the r.h.s) is ϱ-independent. We can, therefore, write down the path integral

$$\tilde{P}(\varrho^2, t|\varrho_0^2)=\int_{\varrho_0{}^2,0}^{\varrho^2,t}\mathscr{D}\varrho^2(\tau)\exp\left\{-\frac{2\pi^2\varepsilon^2}{q^2\eta_\zeta^2}\int_0^t \mathrm{d}\tau\left[\frac{1}{2}\gamma\frac{\mathrm{d}}{\mathrm{d}\tau}\varrho^2(\tau)-\frac{q\zeta_0}{2\pi\varepsilon}\right]^2\right\}. \tag{29}$$

However, since $P(\varrho, t|\varrho_0)\,\mathrm{d}\varrho_0=\tilde{P}(\varrho^2, t|\varrho_0^2)\,\mathrm{d}(\varrho^2)$ and

$$P(\varrho, +0|\varrho_0)=\delta(\varrho-\varrho_0) \tag{30}$$

we may write the initial condition (30) reformulated

$$\tilde{P}(\varrho^2, +0|\varrho_0^2)=\delta(\varrho^2-\varrho_0^2). \tag{30a}$$

The path integral (29) implies the equation

$$\frac{\partial\tilde{P}(\varrho^2, t|\varrho_0^2)}{\partial t}=\frac{q^2\eta_\zeta^2}{2\pi^2\varepsilon^2\gamma^2}\frac{\partial^2\tilde{P}(\varrho^2, t|\varrho_0^2)}{\partial(\varrho^2)^2}-\frac{q\zeta_0}{\pi\varepsilon\gamma}\frac{\partial\tilde{P}(\varrho^2, t|\varrho_0^2)}{\partial(\varrho^2)}. \tag{31a}$$

If the electrodes ideally absorb the particles, we have the boundary conditions

$$\tilde{P}(+0, t|\varrho_0^2)=\tilde{P}(a^2, t|\varrho_0^2)=0 \tag{32a}$$

and the solution $\tilde{P}(\varrho^2, t|\varrho_0^2)$ is similar to solution (24). Hence, the function $P(\varrho, t|\varrho_0)$ satisfies the equation

$$\frac{\partial P(\varrho, t|\varrho_0)}{\partial t}=\frac{C_1}{\varrho}\frac{\partial^2 P(\varrho, t|\varrho_0)}{\partial\varrho^2}+\frac{1}{\varrho}\left(\frac{3C_1}{\varrho^2}+C_2\right)\times \left[-\frac{\partial P(\varrho, t|\varrho_0)}{\partial\varrho}+\frac{1}{\varrho}P(\varrho, t|\varrho_0)\right], \tag{31}$$

where $C_1=\frac{1}{2}\left(\frac{q\eta_\zeta}{2\pi\varepsilon\gamma}\right)^2$, $C_2=\frac{q\zeta_0}{2\pi\varepsilon\gamma}$, and the boundary conditions

$$P(0, t|\varrho_0)=P(a, t|\varrho_0)=0. \tag{32}$$

Equations (31) and (31a) correspond to the assumption that the Brownian particles move (in our idealized case) along radial rays $0<\varrho<a$ with fixed coordinates z and fixed azimuthal angles α.

It is striking indeed that equation (31) does not resemble (unlike equation (22) which is identical with) the usual diffusion equation, right when cylindrical coordinates are used.

In analogy to function (25) it is natural to define the function

$$\tilde{P}_{\zeta_0}^{\infty}(\omega, t|\omega_0)=\left(\frac{\pi}{2t}\right)^{1/2}\frac{\varepsilon\gamma}{q\eta_\zeta}\exp\left\{-\frac{\pi^2\varepsilon^2\gamma^2}{2q^2\eta_\zeta^2 t}\left(\omega-\frac{q\zeta_0 t}{2\pi\varepsilon\gamma}-\omega_0\right)^2\right\} \tag{33}$$

with the definition region on the whole real axis: $-\infty<\omega<\infty$, $-\infty<\omega_0<\infty$. The solution to equation (31) with respect to conditions (30), (32) — in correspondence to solution (24) — is

$$P(\varrho, t|\varrho_0)=2\varrho\sum_{n=-\infty}^{\infty}[\tilde{P}_{\zeta_0}^{\infty}(\varrho^2+2na^2, t|\varrho_0^2)-\tilde{P}_{-\zeta_0}^{\infty}(-\varrho^2+2na^2, t|\varrho_0^2)]. \tag{34}$$

It should be emphasized, anyway, that the transition from equation (27) to equation (28) was a lucky step that cannot be repeated generally[1)]. In general, we should consider a path integral containing a "kinetic energy term" dependent not only on the time derivatives $\dot{x}(\tau)$ of the paths but also explicitly on $x(\tau)$ (or $\varrho(\tau)$, $r(\tau)$, etc.). Path integrals of such a sort occur commonly if curvilinear coordinates are used. It is not at all always easy to find a transformation to Cartesian coordinates for which one could utilize results known from the theory of Wienerian functional integrals.

III. 4 Magnetodiffusion in an unpolarized "white" electromagnetic radiation

A charged particle of charge q finds itself in a uniform, temporally constant magnetic field $\mathbf{B}=(0, 0, B)$. Moreover, the particle is placed in a beam, parallel to $\mathbf{B}$, of unpolarized electromagnetic waves whose distribution in frequencies mimics a white-noise spectrum. Then the fluctuating force $\mathbf{f}(t)$ is perpendicular to $\mathbf{B}$ and the components $f_x(t)=qE_x(t)$, $f_y(t)=qE_y(t)$ represent independent random functions of zero mean ($\langle f_x(t)\rangle=\langle f_y(t)\rangle=0$, $\langle f_x(t')f_y(t'')\rangle=0$ with a common white-noise autocorrelation function

$$\langle f_x(t')f_x(t'')\rangle=\langle f_y(t')f_y(t'')\rangle=q^2\eta_E^2\delta(t'-t''). \tag{35}$$

In this case, the stochastic equations of motion read

$$\gamma\dot{x}-qB\dot{y}=f_x(t),\quad \gamma\dot{y}+qB\dot{x}=f_y(t). \tag{36}$$

If the random force $\mathbf{f}(t)$ is Gaussian, the corresponding path integral is

$$P(xy, t|x_0y_0)=\int_{x_0,0}^{x,t}\mathscr{D}x(\tau)\int_{y_0,0}^{y,t}\mathscr{D}y(\tau)\exp\left\{-\frac{\gamma^2+q^2B^2}{2q^2\eta_E^2}\times \int_0^t \mathrm{d}\tau[\dot{x}^2(\tau)+\dot{y}^2(\tau)]\right\}. \tag{37}$$

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1) However, it can easily be repeated for a spherical capacitor, say.

(Note the identity $(\gamma\dot{x} - qB\dot{y})^2 + (\gamma\dot{y} + qB\dot{x})^2 = (\gamma^2 + q^2B^2)(\dot{x}^2 + \dot{y}^2)$. So we can establish the Fokker-Planck (Smoluchowski) equation to our problem:

$$\frac{\partial P(xy, t|x_0y_0)}{\partial t} = \frac{q^2\eta_E^2}{2(\gamma^2 + q^2B^2)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) P(xy, t|x_0y_0), \tag{38}$$

$$P(xy, +0|x_0y_0) = \delta(x - x_0)\,\delta(y - y_0). \tag{38a}$$

As seen readily by comparing equation (38) with equation (1), we have, in fact, derived an effective coefficient of "magnetodiffusion"

$$D_B = \frac{q^2\eta_E^2}{2(\gamma^2 + q^2B^2)}. \tag{39}$$

IV. CONCLUDING REMARKS

The present paper has been meant as an introduction to the subsequent one (Part II). Having followed the basic idea that each solution of the Fokker-Planck (Smoluchowski) equation could be represented by a Feynman path integral in correspondence to a stochastic Langevin equation with a Gaussian random force of the white-noise, we dealt only with special cases by having chosen the deterministic component F of the force in the Langevin equation either absent (Sections III. 1 and III. 4) or constant (Section III. 2), or at least by having assumed that the problem could be transformed in such a manner that any x-dependence of the deterministic term in the Langevin equation would be cancelled out (Section III. 3). In all such cases, and even more generally, if $F(x)$ is a linear function in argument x, the use of the path integrals can be effectively avoided when we are interested in simple average values like $\langle (x(t) - \langle x(t) \rangle)^2 \rangle$. Such, or similar, values can be calculated, owing to the linearity of the Langevin equation, by independent analytical methods which *eo ipso* offer excellent possibilities for checking the correctness of the path-integral calculations. For linear Langevin equations, moreover, analytical calculations are feasible both when the stochastic force $f(t)$ is not of the white-noise type and even when it is not Gaussian (cf. Appendix I after the subsequent paper). No such simple possibility is available if the deterministic force $F(x)$ makes the Langevin equation substantially nonlinear with respect to x and the path integrals, as well as the Fokker-Planck (Smoluchowski) equation associated with them, turn out considerably more valuable.

Our final remark should touch upon a question which was not treated throughout the present paper, namely the thermalization of the Brownian particles. Being aware of thermal fluctuations, we ought to add, besides the diffusion coefficient D (of Section III. 1) due to the external trembling force, the thermal diffusion coefficient given by the Einstein formula

$$D_T = \frac{k_B T}{\gamma}, \tag{40}$$

where k_B is the Boltzmann constant, T absolute temperature and γ the friction coefficient. Obviously, the theorizing given in the present paper is justified by the condition

$$D_T \ll D, \tag{41}$$

which means that the strength η of the external trembling force has been chosen sufficiently intense in order to surpass the ubiquitous thermal Brownian motion. Taking the Brownian particles as a statistical ensemble, we may define an "effective temperature" $T_{eff}(\eta)$ corresponding to the "trembling-assisted diffusion":

$$T_{eff}(\eta) = \frac{\eta^2}{2k_B\gamma}. \tag{42}$$

(In this definition, we have respected the Einstein formula (40) and formula (15) for D). The condition (41) may be rewritten in the form

$$T \ll T_{eff}(\eta), \tag{43}$$

which indicates that the ensemble of the Brownian particles as a whole might be interpreted in our theory as "hot". The "effective temperature" $T_{eff}(\eta)$, of course, does not refer to the true temperature (measuring the internal energy) of the single Brownian particles. The amount by which the particles are actually warmed up would be calculable by a painstaking analysis taking into account the kinetics of the heat exchange between the particles and their environment.

REFERENCES

[1] Feynman, R. P., Hibbs, A. R.: *Quantum Mechanics and Path Integrals.* McGraw-Hill, New York 1965.
[2] Edwards, S. F., Gulyaev, Y. B.: Proc. Phys. Soc. *83* (1964), 495.
[3] Bezák, V.: Proc. Roy. Soc. A *315* (1970), 339.
[4] Bezák, V.: J. Phys. *A 4* (1971), 324.
[5] Papadopoulos, G. I.: J. Phys. *A 7* (1974), 183.
[6] Barta, Š.: *Habilitation Thesis.* Techn. Univ. Bratislava.
[7] Drchal, V., Mašek, J.: Czech. J. Phys. *B 26* (1976), 967.
[8] Bezák, V., Banský, J.: Phys. Stat. Sol. *(b) 76* (1976), 569.
[9] Majerníková, E.: Phys. Stat. Sol. *(b) 79* (1977), 339.
[10] Yeh, J.: *Stochastic Processes and the Wiener Integral.* Dekker, New York 1973.
[11] Andronov, A. A., Vitt, A. A., Khaikin, S. E.: *Theory of Oscillators.* Pergamon Press, New York 1966; translation from Russian, Moskva 1959.
[12] Bezák, V.: Proc. Cambr. Phil. Soc. *64* (1968), 787.

Received July 19th, 1977