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PATH INTEGRALS AND THE KLEIN—GORDON EQUATION (I.)

VILIAM PAŽMA*, Bratislava

We present the new path integrals representation of solutions of the Klein-Gordon (K.G.) equation. The path integrals in our formalism are the natural generalization (to the relativistic case) of those used by Feynman [1—2] in his formulation of nonrelativistic quantum mechanics.

ИНТЕГРАЛЫ ПО ТРАЕКТОРИЯМ И УПРАВЛЕНИЕ КЛЕЙНА-ГОРДОНА I.

В работе рассматривается новое представление для решений уравнения Клейна-Гордона помощью интегралов по траекториям. В данном формализме интегралы по траекториям представляют собой естественное обобщение (на релятивистский случай) интегралов по траекториям которые использовал Фейман [1, 2] в своей формулировке нерелятивистской квантовой механики.

I. INTRODUCTION

The formulation of nonrelativistic quantum mechanics in terms of path integrals [1—2] gives in a compact form the expressions for propagators or solutions of the nonrelativistic Schrödinger equation. This formulation does not contain the noncommutative mathematical objects (if we consider spinless particles only), does not require to know the Hamiltonian of the system and its conceptual framework is very near to that of classical mechanics. All this allows an elegant and relatively transparent interpretation of mathematical expressions.

The causal propagator of the K.G. equation can be expressed by means of path integrals, too [3]. The corresponding expression is elegant from the mathematical point of view but its interpretation is not as transparent as in the nonrelativistic case. This is apparently caused by the fact that the K.G. equation contains the partial time derivative of the second order. The particle described by the K.G. equation has further in addition to the nonrelativistic degrees of freedom also the degree of freedom connected with the existence of particles and antiparticles. From

* Ústav fyziky PFUK, Mlynská dolina, 816 31 BRATISLAVA, Czechoslovakia.

that there follows that if we want to find such a representation of the K.G. equation in terms of the path integrals which is in a simple relation with nonrelativistic path integrals, we have to divide (if possible) the solutions of the K.G. equation into two classes. One class of solutions corresponds to particles and other solutions are connected with antiparticles. The investigation of the time development of the solutions corresponding to particles or antiparticles would lead, at least in some special case, to a new path integrals formulation of the K.G. equation. The realization of this briefly sketched program in the case of the motion of the particle in a static magnetic field is the aim of this paper.

II. THE SPINLESS PARTICLE IN A STATIC MAGNETIC FIELD

The amplitude describing spinless particles in the static magnetic field satisfies the equation (we put $c=1$)

$$[i\hbar\partial_t]^2\,\Psi(\mathbf{x},t)=[M_0^2+(i\hbar\,\partial_x+e\mathbf{A}(\mathbf{x}))^2]\,\Psi(\mathbf{x},t). \tag{1}$$

The arbitrary solution of Eq. (1) can be written in the following form

$$\Psi(\mathbf{x},t)=\sum_{E>0}A(E)\,e^{-\frac{iEt}{\hbar}}\,\varphi_E(\mathbf{x})+\sum_{E<0}A(E)\,e^{-\frac{iEt}{\hbar}}\,\varphi_E(\mathbf{x}), \tag{2}$$

where $\varphi_E(\mathbf{x})$ are the solutions of the equation

$$E^2\varphi_E(\mathbf{x})=[M_0^2+(i\hbar\,\partial_x+e\mathbf{A}(\mathbf{x}))^2]\,\varphi_E(\mathbf{x}). \tag{3}$$

In what follows we shall use this designation

$$\Psi^{(\pm)}(\mathbf{x},t)=\sum_{E\gtrless 0}A(E)\,e^{-\frac{iEt}{\hbar}}\,\varphi_E(\mathbf{x}) \tag{4}$$

and interpret $\Psi^{(+)}$ as the wave function describing particles and $\Psi^{(-)}$ as the amplitude connected with antiparticles [4]. It is evident that if

$$\Psi^{(-)}(\mathbf{x},t_0)=\Psi^{(-)}(\mathbf{x},t_0+\tau)\equiv 0,$$

then for arbitrary $t\in\langle t_0,t_0+\tau\rangle$ we have $\Psi^{(-)}(\mathbf{x},t)\equiv 0$. From that it follows that in this case $\Psi^{(+)}$ and $\Psi^{(-)}$ develop (time development) independently each of another. Let us now investigate the time development of $\Psi^{(+)}$. There holds

$$\Psi^{(+)}(\mathbf{x},t_0+\tau)=\sum_{E>0}A(E)\,e^{-\frac{iE\tau}{\hbar}}\,e^{-\frac{iEt_0}{\hbar}}\,\varphi_E(\mathbf{x})=$$
$$=\sum_{E>0}A(E)\int\mathcal{D}\sqrt{m(\xi)}\,e^{-\frac{i}{\hbar}\int_0^\tau\frac{d\xi}{2}\left(m(\xi)+\frac{E^2}{m(\xi)}\right)}\,e^{-\frac{iEt_0}{\hbar}}\,\varphi_E(\mathbf{x}), \tag{5}$$

where

$$\int\mathcal{D}\sqrt{m(\xi)}\,e^{\frac{i}{\hbar}\int_0^\tau\frac{d\xi}{2}\left(m(\xi)+\frac{E^2}{m(\xi)}\right)}\equiv$$
$$\equiv\lim_{\varepsilon\to 0}\prod_{n=1}^{N}\int_0^\infty\frac{d\sqrt{m(\xi)}}{\left(\frac{\pi\hbar}{i\varepsilon}\right)^{1/2}}\,e^{-\frac{i\varepsilon}{2\hbar}\left(m(\xi_n)+\frac{E^2}{m(\xi_n)}\right)},$$

$\varepsilon=\tau/N$ (N is integer) and $\xi_n=n\varepsilon$ (cl. [5]).

Using Eq. (3) we can rewrite (5) in the form

$$\Psi^{(+)}(\mathbf{x},t_0+\tau)=\int\mathcal{D}\sqrt{m(\xi)}\,e^{-\frac{i}{\hbar}\int_0^\tau\frac{d\xi}{2}\left(m(\xi)+\frac{M_0^2}{m(\xi)}\right)}\times$$
$$\times e^{-\frac{i}{\hbar}\int_0^\tau d\xi\frac{(\hat{P}_\xi-e\mathbf{A}_\xi(\mathbf{x}))^2}{2m(\xi)}}\,\Psi^{(+)}(\mathbf{x},t_0) \tag{6}$$

where $\hat{P}=-i\hbar\,\partial_x$ and ξ is the ordering index [1]. Using the equality

$$e^{-\frac{i}{\hbar}\int_0^\tau d\xi\frac{(\hat{P}_\xi-e\mathbf{A}_\xi(\mathbf{x}))^2}{2m(\xi)}}=\int\mathcal{D}\mathbf{v}(\xi)\,e^{-\frac{i}{\hbar}\int_0^\tau d\xi\mathbf{v}(\xi)(\hat{P}_\xi-e\mathbf{A}_\xi(\mathbf{x}))}\times$$
$$\times e^{\frac{i}{\hbar}\int_0^\tau d\xi\frac{m(\xi)\mathbf{v}^2(\xi)}{2}}, \tag{7}$$

where

$$\int\mathcal{D}\mathbf{v}(\xi)=\lim_{\varepsilon\to 0}\prod_{n=1}^{N}\int\frac{d^3\mathbf{v}(\xi_n)}{\left(\frac{2\pi i\hbar}{\varepsilon m(\xi_n)}\right)^{3/2}}$$

and rules of disentangling the operators [1], we obtain

$$\Psi^{(+)}(\mathbf{x},t_0+\tau)=\int d^3\mathbf{x}_0\int\mathcal{D}\sqrt{m(\xi)}\int\mathcal{D}\mathbf{v}(\xi)\,\delta(\mathbf{x}-\mathbf{x}_0-\int_0^\tau d\xi\mathbf{v}(\xi))\times$$
$$\times e^{\frac{i}{\hbar}\int_0^\tau d\xi\left[\frac{m\mathbf{v}^2}{2}+e\mathbf{v}\mathbf{A}(\mathbf{x}_0+\int_0^\xi d\eta\mathbf{v}(\eta))-\frac{1}{2}\left(m-\frac{M_0^2}{m}\right)\right]}\,\Psi^{(+)}(\mathbf{x}_0,t_0). \tag{8}$$

Likewise for $\Psi^{(-)}$ we obtain

$$\Psi^{(-)}(\mathbf{x},t_0-\tau)=\int d^3\mathbf{x}_0\int\mathcal{D}\sqrt{m}\int\mathcal{D}\mathbf{v}\,\delta(\mathbf{x}-\mathbf{x}_0-\int_0^\tau d\xi\mathbf{v})\times$$
$$\times e^{\frac{i}{\hbar}\int_0^\tau d\xi\left[\frac{m\mathbf{v}^2}{2}-e\mathbf{v}\mathbf{A}(\mathbf{x}_0+\int_0^\xi d\eta\mathbf{v})-\frac{1}{2}\left(m-\frac{M_0^2}{m}\right)\right]}\,\Psi^{(-)}(\mathbf{x}_0,t_0). \tag{9}$$

The expressions (8—9) can be interpreted as follows: Let us put in (8)

$$t=t_0+\xi,\quad \mathbf{v}(\xi)=\frac{d\mathbf{q}(\xi)}{d\xi}\equiv\dot{\mathbf{q}}(\xi)$$

and consider the histories $\mathbf{q}_s(t)=\mathbf{q}_s(t_0+\xi)$ which connect the space-time points $(\mathbf{x}_0=\mathbf{q}_s(t_0),\ t_0)$ and $(\mathbf{x}=\mathbf{q}_s(t_0+\tau),\ t_0+\tau)$. We assign to the history $\mathbf{q}_s(t)$ the amplitude

$$\exp\left(\frac{i}{\hbar}\int_0^\tau \mathrm{d}\xi L(\dot{\mathbf{q}}_s,\mathbf{q}_s,m_s)\right), \tag{10}$$

where

$$L=\frac{m\dot{\mathbf{q}}^2}{2}-e\dot{\mathbf{q}}\mathbf{A}(\mathbf{q})-\frac{1}{2}\left(m+\frac{M_0^2}{m}\right) \tag{11}$$

and we shall call expression (10) the transition probability amplitude of the particle (or the amplitude of the history) from $(\mathbf{x}_0,t_0)$ to $(\mathbf{x},t_0+\tau)$ along the history $\mathbf{q}_s(t)$. The function $m(t)$ is regarded as an instantaneous mass of the particle. In accordance with (5) $m(t)$ can arbitrary alter in the interval $\langle 0,\infty\rangle$. It will be shown later that (11) can be regarded as the relativistic Lagrange function of the particle moving in the static magnetic field. On the basis of the previous the expression

$$\int\mathscr{D}\sqrt{m}\int\mathscr{D}\mathbf{q}\,\delta\left(\mathbf{x}-\mathbf{x}_0-\int_{t_0}^{t_0+\tau}\mathrm{d}t\dot{\mathbf{q}}\right)e^{\frac{i}{\hbar}\int_{t_0}^{t_0+\tau}\mathrm{d}t\left[\frac{m\dot{\mathbf{q}}^2}{2}+e\dot{\mathbf{q}}\mathbf{A}(\mathbf{q})-\frac{1}{2}\left(m+\frac{M_0^2}{m}\right)\right]} \tag{12}$$

is interpreted as the sum of amplitudes of all possible histories connecting the points in question and running in the time increasing direction.

The situation is negligibly different in the case of Eq. (9). In this case we put $t=t_0-\xi$ and the expression

$$\int\mathscr{D}\sqrt{m}\int\mathscr{D}\mathbf{v}\,\delta\left(\mathbf{x}-\mathbf{x}_0-\int_0^\tau\mathrm{d}\xi\mathbf{v}\right)\times e^{\frac{i}{\hbar}\int_0^\tau\mathrm{d}\xi\left[\frac{m v^2}{2}+e\mathbf{v}\mathbf{A}(\mathbf{x}_0+\int_0^\xi\mathrm{d}\eta\mathbf{v})-\frac{1}{2}\left(m+\frac{M_0^2}{m}\right)\right]} \tag{13}$$

is interpreted as the sum of amplitudes of all possible histories connecting the given two points and running in the time decreasing direction. If we put

$$\mathbf{v}(\xi)=\frac{\mathrm{d}\mathbf{q}(\xi)}{\mathrm{d}\xi}=-\frac{\mathrm{d}\mathbf{q}(t)}{\mathrm{d}t},$$

then the expression (13) can be interpreted as the transition probability amplitude of the particle with the charge $(-e)$ from $(\mathbf{x},t_0-\tau)$ to $(\mathbf{x}_0,t_0)$.

The relation between the amplitudes (12—13) and Feynman's expression for the nonrelativistic propagator is very simple. To obtain the nonrelativistic amplitude it is sufficient to omit the integration over $m(t)$ and to put $m(t)=M_0$.

III. CLASSICAL LIMIT

Let us now consider the formal limit $\hbar\to 0$. In this case only a small vicinity of that history for which the functional

$$\int=\int_0^\tau\mathrm{d}t\left[\frac{m\dot{\mathbf{q}}^2}{2}+e\dot{\mathbf{q}}\mathbf{A}(\mathbf{q})-\frac{1}{2}\left(m+\frac{M_0^2}{m}\right)\right] \tag{14}$$

is extremal contributes to the total transition probability amplitude. As every history is given by means of two functions $(\mathbf{q}(t), m(t))$ we have to vary not only $\mathbf{q}(t)$ but $m(t)$, too if we want to find the extremal history. We obtain from $(\delta L/\delta m)=0$

$$m(t)=\frac{M_0}{\sqrt{1-\dot{\mathbf{q}}}}$$

and from $(\delta L/\delta\mathbf{q})=0$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\mathbf{q}})=e\dot{\mathbf{q}}\,\mathrm{rot}\mathbf{A}(\mathbf{q}).$$

From that it follows that

$$L=\frac{m\dot{\mathbf{q}}^2}{2}+e\dot{\mathbf{q}}\mathbf{A}(\mathbf{q},t)-e\varphi(\mathbf{q},t)-\frac{1}{2}\left(m+\frac{M_0^2}{m}\right)$$

can be regarded as the Lagrance function of the particle interacting with the electromagnetic field. This form of L and the variation principle mentioned above were first formulated by Petráš (unpublished).

IV. THE SOLUTION OF THE K.G. EQUATION IN TERMS OF PATH INTEGRALS

Let the boundary conditions of the K.G. equation be given in such a way

$$\Psi(\mathbf{x},0)=\Psi_0(\mathbf{x})\qquad\Psi(\mathbf{x},\tau)=\Psi_\tau(\mathbf{x}). \tag{15}$$

The results of Sect. II. allow us to write for $\Psi(\mathbf{x},t)\,t\in\langle 0,\tau\rangle$ the following expression

$$\begin{aligned}\Psi(\mathbf{x},t)=&\int\mathrm{d}^3\mathbf{x}_0\sum_s A(s|\mathbf{A}|(\mathbf{x},t),(\mathbf{x}_0,0))\,\Psi_0^{(+)}(\mathbf{x}_0)+\\&+\int\mathrm{d}^3\mathbf{x}_1\sum_s A(s|\mathbf{A}|(\mathbf{x},t),(\mathbf{x}_1,\tau))\,\Psi_\tau^{(-)}(\mathbf{x}_1),\end{aligned} \tag{16}$$

where

$$\sum_s A(s|\mathbf{A}|(\mathbf{x},t),(\mathbf{x}_0,0))$$

represents the sum of amplitudes of all possible histories connecting the points in question. The second term on the right-hand side of Eq. (16) contains the sum of amplitudes of all histories connecting the points $(\mathbf{x}_1, \tau)$ and $(\mathbf{x}_0, 0)$ and running in the time decreasing direction.

The functions $\Psi_0^{(+)}$ and $\Psi_\tau^{(-)}$ are given by these equations

$$\Psi_0^{(+)}(\mathbf{x}) = \Psi(\mathbf{x}, 0) - \Psi^{(-)}(\mathbf{x}, 0) = \Psi_0(\mathbf{x}) - \\ - \int d^3\mathbf{x}_1 \sum_s A(s|\mathbf{A}|(\mathbf{x}, 0), (\mathbf{x}_1, \tau))\, \Psi_\tau^{(-)}(\mathbf{x}_1) \tag{17}$$

$$\Psi_\tau^{(-)}(\mathbf{x}) = \Psi(\mathbf{x}, \tau) - \Psi^{(+)}(\mathbf{x}, \tau) = \Psi_\tau(\mathbf{x}) - \\ - \int d^3\mathbf{x}_0 \sum_s A(s|\mathbf{A}|(\mathbf{x}, \tau), (\mathbf{x}_0, 0))\, \Psi_0^{(+)}(\mathbf{x}_0). \tag{18}$$

Eqs. (16—18) can be regarded as the new path integrals representation of Eq. (1).

V. CONCLUSION

The above presented path integrals formalism for the special case of the K.G. equation seems to be very transparent and exhibits, we believe, the closest continuity with the Feynman path integrals in the nonrelativistic case. Although the nonstandard formulation of the extremal action principle, following from the presented formalism in a natural way, is not relativistically covariant, yet the classical equations of motion are relativistically covariant. The same can be affirmed about the final results of the presented formalism.

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