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A GENERAL METHOD FOR TESTING THE ANALYTICITY OF SCATTERING AMPLITUDES

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A general method for testing the analyticity of forward or fixed t amplitudes is presented. The method is based on the statistical approach to the data representation and analyticity testing and the results are formulated in terms of standard statistical concepts. The method is able to cope with the general case of not equal and correlated errors of real and imaginary parts. The practival usefulness of the method was tested by applying it to the πN forward scattering amplitude F^-.

ОБЩИЙ МЕТОД ДЛЯ ПРОВЕРКИ АНАЛИТИЧНОСТИ АМПЛИТУД РАССЕЯНИЯ

В работе приводится общий метод для проверки аналитичности амплитуды вперёд или амплитуды при фиксированном t. Метод основан на статистическом подходе к представлению данных и проверке аналитичности, причём результаты сформулированы на языке обычных статистических понятий. Метод позволяет рассмотреть общий случай неравных и коррелированных ошибок действительной и мнимой частей амплитуды. Практическая полезность данного метода проверена его применением к F^- амплитуде πN рассеяния вперёд.

I. INTRODUCTION

The statistical approach to the representation of data by analytic functions initiated by Cutkosky [1] has developed into an efficient and reliable method for analytic extrapolations [2], for the evaluation of coupling constants [3], [4], for the determination of resonance parameters [5, 6, 7], for testing the consistency of data on scattering amplitudes with analyticity assumptions [4, 5, 8, 9] and for the determination of parameters characterizing the amplitude in regions where data are not available [10, 11].

In the present paper we shall describe a general method for the determination of the most probable representation of data by an analytic function and for the testing of the consistency of data with assumed analyticity properties. The method fully

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exploits the information on the real and imaginary parts of the fixed t amplitudes and treats properly the general case of experimental errors (the real and imaginary parts may be correlated and may have different errors).

The method is, in a perspective, a continuation of the line starting with the Cutkosky's paper [1] and continuing further through Ref. [2], where a somewhat simplified but more manageable formulation of the problem was given.

In the statistical approach one looks for the function which is the "most probable" representation of the amplitude if the data and analyticity properties of the amplitude are known. Due to the statistical treatment of the data the estimates of the errors of the calculated quantities are truly statistical. In Ref. [2] the "most probable function" was constructed for the case of the errors of the real and imaginary parts at a given energy being equal and uncorrelated. This, of course, does not permit us to use the full information contained in the data.

The construction of the most probable representation of the amplitude was continued later on in papers by Ross [10, 12] and by Sheppard and Shih [13, 14]. In these papers it was shown how the most probable analytic function should be constructed when the errors of real and imaginary parts are unequal and their possible correlations are taken into account [14].

Solutions for cases of the values of the amplitude being exactly known in some points inside the analyticity region were studied in a related context by Nenciu [15] and Prešnajder [16].

A few very relevant and deep observations on the mathematics of the whole approach are due to Pietarinen [17], who also applied a statistical method to the determination of scattering amplitudes at a fixed t in the πN scattering [18].

In practice, the most probable function is not, hower, the full solution of the problem. What one needs in addition to it is the information about the consistency of the data with the assumed analyticity properties of the amplitude. Such a method has to be based also on the statistical treatment since otherwise the practical usefulness of the test would be rather doubtful and the error estimates unreliable.

The problem of constructing a suitable statistical test of the consistency of the data with analyticity properties was formulated in the case of equal errors in Refs. [4] and [8], and the argument appeared later on in Ref. [5]. In order to test the analyticity one needs a set of moments, which are given as inner products of the amplitude with a suitable set of functions. The weight in the scalar product is directly given by the smoothed errors of the data. For practical purposes it is desirable that the moments be uncorrelated, Gaussian distributed random variables with unit dispersions. This requirement, together with the weight given by errors, specifies almost uniqueely the moments to be evaluated. Any other set of moments will be much more difficult to handle properly from the statistical point of view.

In the present paper we shall describe the construction of a suitable set of moments for the case of unequal and correlated errors. This then permits us to test the analyticity of the πN forward scattering amplitude and determine reliably and with a truly statistical error the πN coupling constant.

The testing of analyticity is a rather general procedure which can also be used for the determination of unknown parameters in the parametrization of the scattering amplitude. This point is discussed in more detail in a review paper [19].

The outline of the paper is as follows. In the next Section we shall recapitulate the simpler problem of equal errors and formulate the desirable features of a solution to the problem of unequal correlated errors. In Section III and IV we shall discuss an explicitly solvable situation of the general problem, which may cover most if not all practical applications. Section V contains the application of the method to the πN forward scattering amplitude. Comments and conclusions are presented in the last Section.

II. FORMULATION OF THE PROBLEM

A fixed t two-body amplitude is analytic in the s plane with two cuts $(-\infty, s_2)$ and $(s_1; \infty)$. The calculations are considerably simplified if the cut s plane is conformally mapped onto the unit disc in, say, the complex x plane. The mapping is well known and for the special case of the forward pion-nucleon amplitude it is explicitly given in Ref. [3].

Because the amplitude $F(x)$ is real analytic, we shall suppose that the data $Y(x) = Y_1(x) + iY_2(x)$ and the corresponding error matrix $v_{ij}(x)$ $(i, j = 1, 2)$ obey the following symmetry relations with respect to the real axis:

$$Y_1(x^*) = Y_1(x) \qquad Y_2(x^*) = -Y_2(x)$$
$$v_{ii}(x^*) = v_{ii}(x) \qquad v_{12}(x^*) = -v_{12}(x)$$
$$v_{12}(x) = v_{21}(x).$$

Following Cutkosky's statistical approach [1] the data and the corresponding errors are used to define the probability in the space of functions real analytic inside the unit disc $\mathscr{D}$. The non-normalized probability $P(F/Y)$ for a function $F(x)$ is written in the form [14]:

$$P(F/Y) \sim \exp\left\{-\frac{1}{2}\chi_c^2(F/Y)\right\}, \tag{1}$$

where

$$\chi_c^2(F/Y) = \frac{1}{2\pi}\oint \sum_{i,j}[F_i(x) - Y_i(x)]\, W_{ij}(x)\, [F_j(x) - Y_j(x)]\, |dx|. \tag{2}$$

The symmetric matrix $W_{ij}(x)$ is the inverse of the error matrix $v_{ij}(x)$ and $F_1(x) = \mathrm{Re}\, F(x)$, $F_2(x) = \mathrm{Im}\, F(x)$. It should be noted that $v_{ij}(x)$ is the smoothed error matrix. Its relation to the actual error matrix measured in discrete points is discussed in detail in Refs. [2, 4] and [14]. In a similar way $Y(x)$ is a smooth interpolation of the data. If the data are not available along the whole boundary, one has to use some hypothesis on the behaviour of the amplitude in the unknown region and assign errors to that hypothesis (for more details see Ref. [2]).

The probability in Eqs. (1) and (2) may be interpreted in two ways. Either as a probability in the functional space; in that case $Y(x)$ is considered as fixed. Or, alternatively, we can think of $F(x)$ as of a true, though unknown, amplitude and regard (1) and (2) as a probability assigned to various possible experimental outcomes $Y(x)$. In the present paper we shall stick to the latter interpretations.

If the errors of the real imaginary parts are equal and uncorrelated, Eq. (2) simplifies to

$$\chi_c^2(F/Y)=\frac{1}{2\pi}\oint W(x)\,|F(x)-Y(x)|^2\,|dx| \qquad (3)$$

$$=(F-Y,F-Y),$$

where we have defined the inner product

$$(F,G)=\frac{1}{2\pi}\oint W(x)F^*(x)G(x)\,|dx|. \qquad (4)$$

For this particular case the test of analyticity and the determination of the coupling constant are rather simple. We introduce first the function $w(x)$, which is analytic and free of zeros in $\mathscr{D}$ and obeys the condition

$$|w(x)|=[W(x)]^{-1/2} \qquad \text{for } |x|=1.$$

The system of functions

$$P_k(x)=x^k w(x), \qquad k=0,-1,-2,\ldots$$

is then orthogonal and normalized with respect to the inner product (4). Supposing that $F(x)$ is analytic in $\mathscr{D}$ we can expand $F(x)$ and $Y(x)$ into the series

$$F(x)=\sum_{k=0}^{\infty} c_k P_k(x) \qquad (5)$$

$$Y(x)=\sum_{k=-\infty}^{\infty} q_k P_k(x). \qquad (6)$$

The probability (1) then becomes a product of the terms

$$\exp\left\{-\frac{1}{2}(q_k-c_k)^2\right\} \qquad (7)$$

and it is easy to see that $Q_k\equiv q_{-k}$ for $k=1,2,\ldots$ are Gaussian distributed with

vanishing mean values and unit standard deviations. To test the analyticity one thus only needs to calculate the coefficients

$$Q_k=(P_{-k},Y) \qquad k=1,2,\ldots$$

and check whether the values found are consistent with the $N(0,1)$ distribution.

If the amplitude has a single pole

$$F(x)=\frac{R}{x}+\sum_{k=0}^{\infty} a_k x^k, \qquad (8)$$

the residue R is determined as follows. The expansion (5) can be rewritten as

$$F(x)=R'P_{-1}(x)+\sum_{k=0}^{\infty} c_k P_k(x), \qquad (9)$$

where $R'=R/w(0)$. Inserting (6) and (9) into (3) we obtain the probability (1) as a product

$$P(F/Y)\sim\exp\left\{-\frac{1}{2}(Q_1-R')^2\right\}\prod_{k=2}^{\infty}\exp\left\{-\frac{1}{2}Q_k^2\right\}\prod_{k=0}^{\infty}\exp\left\{-\frac{1}{2}(q_k-c_k)^2\right\}.$$

The coefficient Q_1 thus determines the residue and the remaining Q_k, $k=2,3,\ldots$ can be used to test the analyticity of the amplitude.

The problem of unequal and correlated errors can be solved along the same lines provided that a suitable generalization of the set $\{P_k(x)\}$ is found. The problem is formulated as follows.

Let $\mathscr{L}^2$ be the set of all functions defined on the unit circle $\mathscr{C}$, fulfilling the conditions $F(x^*)=F^*(x)$ and having a finite norm induced by the inner product

$$(F,G)=\frac{1}{2\pi}\oint\sum_{i,j} F_i(x)\bar{G}_j(x)W_{ij}(x)\,|dx|. \qquad (10)$$

If we define

$$W_+(x)=\frac{1}{2}[W_{11}(x)+W_{22}(x)]$$

$$W_-(x)=\frac{1}{2}[W_{11}(x)-W_{22}(x)-2iW_{12}(x)],$$

the inner product (10) can be rewritten into the form [14]

$$(F,G)=\frac{1}{2\pi}\oint F^*(x)G(x)W_+(x)\,|dx|+\frac{1}{2\pi}\oint F(x)G(x)W_-(x)\,|dx|. \qquad (11)$$

Let $\mathscr{A}^2$ be the subspace of $\mathscr{L}^2$ consisting of boundary values of functions analytic in the unit disc $\mathscr{D}$ and let $\mathscr{N}^2$ be the orthogonal complement of $\mathscr{A}^2$ in the space $\mathscr{L}^2$.

Orthogonality is understood in the sense of the inner product (11). What we need is a basis $\{P_k(x)\}$ orthogonal and normalized with respect to (11) and such that

i) $\{P_k(x)\}$ $k=0, 1, 2, \ldots$ is complete in $\mathscr{A}^2$,
ii) $\{P_k(x)\}$ $k=-1, -2, \ldots$ is complete in $\mathscr{N}^2$.

Inserting the expansions (5) and (6) (where $P_k(x)$ denotes now the new basis) into the Eqs. (1) and (2) we obtain immediately the probability $P(F/Y)$ as a product of terms like (7). The coefficients

$$Q_k \equiv q_{-k} = (P_{-k}, Y)$$

can be used to test the analyticity of the amplitude in exactly the same way as before. To determine the residue of the amplitude we only need to find the expansion (9) in terms of the new basis and proceed as before.

A suitable basis for $\mathscr{A}^2$ (in the case $v_{12}\equiv 0$) has already been constructed by Ross [10, 12] and an approach based on reproducing kernel functions was given by Sheppard and Shih [13] and by Nenciu [15]. A solution for the case of unequal errors on the boundary plus additional data inside the analyticity region was found by Prešnajder [16]. Recently, Shepard and Shih [14] generalized their approach [13] having included correlations between the real and the imaginary parts of the amplitude.

The construction of a suitable basis $\{P_k(x), k=-1, -2, \ldots\}$ which spans the space $\mathscr{N}^2$ is the problem discussed in the following two sections. The solutions which we shall present apply only to a restricted class of error matrices. The restriction is of minor importance for practical applications since the class of problems which can be treated by the method is sufficiently broad.

III. A SPECIAL CASE OF $W(x)$

In this section we shall construct the basis $\{P_k(x), k=0, \pm 1, \pm 2, \ldots\}$, which is orthonormal with respect to the inner product (11) for the case of

$$W_-(x) = \pi_1(\cos\varphi) + \mathrm{i}\sin\varphi\,\pi_2(\cos\varphi), \tag{12}$$

where π_i (.) are polynomials of the order n_i and

$$\cos\varphi = \frac{1}{2}(x + x^{-1}) \tag{13}$$

$$\sin\varphi = \frac{1}{2\mathrm{i}}(x - x^{-1})$$

for $|x|=1$. Making use of Eqs. (12) and (13) we have

$$W_-(x) = x^{-n}H(x), \tag{14}$$

where $n = \max(n_1, n_2+1)$ and $H(x)$ is by construction a polynomial of the order $2n$.

In the space $\mathscr{A}^2$ we shall start with the basis

$$A_k(x)\, x^k w(x) \qquad k=0, 1, 2, \ldots, \tag{15}$$

where $w(x)$ is analytic and free of zeros in $\mathscr{D}$ and such that

$$|w(x)| = [W_+(x)]^{-1/2} \qquad \text{for } |x|=1. \tag{16}$$

Inserting (14), (15) and (16) into the inner product (11) we find

$$(A_k, A_l) = \delta_{kl} + \alpha_{n-k-l}, \tag{17}$$

where α_m are the coefficients of the Taylor expansion of $H(x)w^2(x)$:

$$H(x)w^2(x) = \sum_{m=0}^{\infty} \alpha_m x^m .$$

By definition $\alpha_m = 0$ for $m<0$. Equation (17) then implies that the functions

$$A_k(x), \qquad k \geqslant n+1$$

form an orthonormal set. Moreover, any function of this system is orthogonal to the functions

$$A_k(x), \qquad k=0, 1, 2, \ldots, n. \tag{18}$$

The set (18) can be orthogonalized by standard procedures and the basis P_k constructed in this way is

$$P_k(x) = \sum_{l=0}^{n} C_l^k A_l(x) \qquad k=0, 1, 2, \ldots, n \tag{19}$$

$$P_k(x) = A_k(x) \qquad k \geqslant n+1 .$$

In order to construct a suitable basis in $\mathscr{A}^2$ we thus only need to calculate the $n+1$ coefficients $\alpha_0, \alpha_1, \ldots, \alpha_n$ and make a standard orthogonalization procedure*.

For our purposes, namely for testing the analyticity, it is more important to have a basis $P_k(x)$, $k=-1, -2, \ldots$ which spans the space $\mathscr{N}^2$. We start again with the set

$$A_k(x) = x^k w(x) \qquad k=-1, -2, \ldots$$

and subtract from each A_n its projection onto $\mathscr{A}^2$. Denoting such projection by $C_k(x)$ we have

* This basis in $\mathscr{A}^2$ is actually a special case of that constructed by Shepard and Shih [14] and (when putting $W_{12}\equiv 0$) of those constructed by Ross [10, 12], Shih [20], Shepard and Shih [13] and Prešnajder [16].

$$C_k(x)=\sum_{l=0}^{\infty}(P_l, A_k)\,P_l(x)=$$

$$=\sum_{l=0}^{n}(P_l, A_k)\,P_l(x)+\sum_{l=n+1}^{n+k}\alpha_{n-k-l}A_l(x)$$

for $k=-1, -2, \dots$, where we have used the relation (17), which is valid for any integer k, l (including negative values) and the second relation of Eq. (19). The system of functions

$$B_k(x)=A_k(x)-C_k(x) \qquad k=-1, -2, \dots$$

is by construction orthogonal to $\mathscr{A}^2$ and spans the space $\mathscr{N}^2$. In order to find explicitly the system $B_k(x)$ we need to calculate first the projections $C_k(x)$ and for that we only need to calculate the inner products (P_l, A_k) and determine the coefficients α_j for $n+1\leq j\leq n+k$. The calculation of these quantities numerically is an easy matter.

In the next step we orthogonalize the system $\{B_k(x)\}$ finding thus the desired basis $\{P_k(x)\}$ in $\mathscr{N}^2$. It is useful to make the orthogonalization by choosing suitably the coefficients d_k^l in the expansions

$$P_k(x)=\sum_{l=1}^{k} d_k^l B_l(x), \qquad k=-1, -2, \dots \tag{20}$$

In this case

$$P_k(x)=x^k T_k(x), \qquad k=-1, -2, \dots,$$

where $T_k(x)$ are functions regular in the unit disc. The orthogonalization indicated in (20) presents no practical difficulties. A similar procedure can be used also in the case when

$$W_-(x)=\frac{\pi_1(\cos\varphi)}{\pi_2(\cos\varphi)}+\mathrm{i}\sin\varphi\,\frac{\pi_3(\cos\varphi)}{\pi_4(\cos\varphi)}$$

where π_i are polynomials. The problem is only formally more complicated and we shall not discuss it in detail.

IV. SIMPLE EXAMPLES

Let the scattering amplitude we are interested in have a pole in the origin of the x plane:

$$F(x)=\frac{R}{x}+\textit{analytic function}. \tag{21}$$

We wish to determine the residue R which is proportional to the coupling constant. We shall first expand $F(x)$ into the set of functions $P_k(x)$ described above. In this way we have

$$F(x)=\frac{R}{k}P_{-1}(x)+\textit{analytic function},$$

where $k=T_{-1}(0)$. From the properties of the function $P_k(x)$ it follows that

$$(F, P_{-1})=\frac{R}{k} \tag{22}$$

$$(F, P_{-n})=0 \qquad n=2, 3, \dots$$

If $Y(x)$ is the smoothed interpolation of the experimental data about the amplitude $F(x)$, then $Q_n=(Y, P_{-n})$ are random Gaussian distributed variables with a unit standard deviation around the mean (F, P_{-n}).

The first relation in (22) then gives the values of R and the corresponding error:

$$R=kQ_1\pm k\equiv R_{\exp}\pm k.$$

Let us consider now the simple case of constant, equal and uncorrelated errors $v_{11}(x)=v_{22}(x)=\varepsilon^2$, $v_{12}(x)=0$. Then

$$P_{-n}(x)=x^{-n}\varepsilon, \qquad n=1, 2, 3, \dots$$

so that $T_{-1}(0)\equiv k=\varepsilon$. As expected the error of the result is equal to ε

$$R=R_{\exp}\pm\varepsilon. \tag{23}$$

Let now the errors be constant, uncorrelated but not equal: $v_{11}(x)=\varepsilon_1^2$, $v_{22}(x)=\varepsilon_2^2\neq\varepsilon_1^2$, $v_{12}(x)=0$. It is easy to show that in this case

$$P_{-n}(x)=ax^{-n}+bx^n, \qquad n=1, 2, 3, \dots,$$

where

$$a=\left(\frac{\varepsilon_1^2+\varepsilon_2^2}{2}\right)^{1/2}, \qquad b=\frac{\varepsilon_1^2-\varepsilon_2^2}{2a}.$$

Now $T_{-1}(0)\equiv k=a$ and instead of Eq. (23) we get

$$R=R_{\exp}\pm a.$$

In a method capable of dealing only with equal errors one has to enhance the smaller of the two errors to the value of the bigger one

$$\varepsilon=\max(\varepsilon_1, \varepsilon_2)\leq\sqrt{\varepsilon_1^2+\varepsilon_2^2}=a\sqrt{2}.$$

It is easy to see that by using the method of unequal errors we can at most gain a factor of $\sqrt{2}$ in the accuracy of the final result.

If, however, some additional information about the amplitude $F(x)$ is known beforehand, we can gain considerably by using the method of unequal errors. This is in particular true for superconvergent amplitudes. To make the point clear let us consider firstly the function

$$G(x) = g + a_1 x + a_2 x^2 + \dots$$

and let g be the parameter we wish to determine. If the errors are uncorrelated, constant and equal, we have

$$P_0(x) = \varepsilon$$

and by the same procedure as above we get

$$g = g_{\exp} \pm \varepsilon .$$

Let now the errors be unequal, $v_{11}(x) \equiv \varepsilon_1^2$ and $v_{22}(x) \equiv \varepsilon_2^2$, $\varepsilon_2 \gg \varepsilon_1$. In that case

$$P_0(x) = \varepsilon_1 \qquad (24)$$

and

$$g = g_{\exp} \pm \varepsilon_1 \qquad \varepsilon_1 \ll \max(\varepsilon_1, \varepsilon_2).$$

In this way the unequal error procedure leads to a much more accurate result.

If the amplitude is superconvergent, the determination of the coupling constant can be performed with the accuracy of the (better known) imaginary part. To see it let us suppose that $F(x)$ is given by Eq. (21) and that for $x \to -1$ ($x = -1$ is the image of $\omega = \infty$ in the original energy plane) its absolute value behaves like

$$|F(x)| \sim |1 - x|^{2+\varepsilon}, \qquad \varepsilon > 0.$$

Further, let us introduce a function $H(x)$, which is real analytic within the unit disc, purely imaginary on the unit circle and has a simple zero in the origin. Such a function must have a singularity at the unit circle (since otherwise the non-vanishing imaginary part of $H(x)$ implies that Re $H(x)$ cannot be identically zero). An example of a suitable function $H(x)$ is

$$H(x) = \frac{x}{1-x^2}\left(\frac{1-x}{1+x}\right)^2 .$$

For $x = \exp(\mathrm{i}\,\varphi)$ we have

$$H(e^{\mathrm{i}\varphi}) = -\frac{\mathrm{i}}{4}\,\mathrm{tg}\,\frac{\varphi}{2}\left(1 + \mathrm{tg}^2\,\frac{\varphi}{2}\right).$$

The product $G(x) = H(x)F(x)$ is real analytic inside the unit disc and its behaviour near $x = -1$ is controlled by the relation

$$|G(x)| \sim |1 + x|^{\varepsilon - 1}, \qquad \varepsilon > 0 .$$

so that the moments (G, P_k) are well defined. The error of the Re $G(x)$ is proportional to the error ε_I of the imaginary part of $F(x)$. According to the result obtained above (24), the value of the function G in the origin $G(0) = H(0)R$ (and therefore also the residue R of the amplitude $F(x)$) can be determined to the accuracy which is proportional to ε_I. This, of course, is what we have to expect. If $F(s)$ is superconvergent, the residue in $s = s_0$ can be calculated simply by the Cauchy theorem and in this way the residue is given as an integral over the imaginary part of the amplitude. If the amplitude is not superconvergent, this approach cannot be used. A subtraction constant crawls into the dispersion relation, this subtraction constant can be evaluated only by using the information about the real part of the amplitude and the coupling constant can be determined only with the accuracy with which the real part is known.

V. APPLICATION TO THE πN FORWARD SCATTERING AMPLITUDE F^-

We shall describe here an application of our method to a realistic situation. The most suitable candidate is apparently the πN scattering which is well understood theoretically and where the data are relatively copious. We shall discuss here only the crossing odd forward scattering amplitude F^-, which is less sensitive to assumptions of the high energy behaviour than the remaining forward amplitudes.

We are mostly interested in comparing the present general method with the equal errors version used in Ref. [9]. We shall therefore use here only the data obtained from the phase shift analyses in spite of the fact that more accurate data about the imaginary part can be extracted from total cross-section measurements.

The calculation was analogous to that of Ref. [9], wherever it was possible. For details, in particular for the parametrization of the amplitude in regions not covered by phase shift analyses, the reader is referred to our recent work [9].

The most delicate point in using the data from phase shift analyses is the question of errors and correlations. For example, the CERN 67 analysis [21] gives the errors of both phase shift and elasticities, but the complete covariance matrix is not available. Because of that and for the sake of simplicity we shall assume that the real and imaginary parts are uncorrelated. In order to compensate for the possible lowering of actual errors introduced by this assumption we shall take a rather conservative error estimate of both real and imaginary parts. The errors are calculated by using the following formulae:

$$\sigma_R = \sum_{i=1}^{2l_{\max}+1} \left\{ \left| \frac{\partial \mathrm{Re} F}{\partial \delta_i} \Delta\delta_i \right| + \left| \frac{\partial \mathrm{Re} F}{\partial \eta_i} \Delta\eta_i \right| \right\}$$

$$\sigma_I = \sum_{i=1}^{2l_{\max}+1} \left\{ \left| \frac{\partial \mathrm{Im} F}{\partial \delta_i} \Delta\delta_i \right| + \left| \frac{\partial \mathrm{Im} F}{\partial \eta_i} \Delta\eta_i \right| \right\},$$

where σ_R and σ_I stand for (non-smoothed) errors of the real and the imaginary parts of the amplitude F, and $\Delta\delta_i$ and $\Delta\eta_i$ denote the errors of phase shifts and elasticities, respectively.

The CERN 71 analyses [22] is clearly an improvement of the previous CERN 67 one. The former, however, does not quote the errors. In working with the CERN 71 analysis we have therefore taken the function W_+ and W_- (which are determined by the errors and the density of data) directly from similar calculations using the CERN 67 analysis.

The numerical procedures needed for the application of the general method (see Section III) are considerably more complicated than those needed for the simpler case [9]. In order to test the calculations based on the general method we have first inserted the data (with equal errors) used in the previous calculations [9] into the new programs. In this way we were testing both the general method and the numerical procedures. In the next stage we have used the general method with unequal errors in order to obtain the more sensitive test of analyticity and the more reliable estimate of the coupling constant. The results of the two calculations are summarized in the Table 1. where, in order to facilitate the comparison with the previous work, we also present the results obtained by the equal errors method [9].

Table 1

	CERN 1967		CERN 1971	
	f^2	χ^2_{2-20}	f^2	χ^2_{2-20}
Equal errors method [9]	0.0812 ±0.0018	6.0	0.0803 ±0.0018	6.0
Present method with equal, uncorrelated errors	0.0812 ±0.0016	7.5	0.0804 ±0.016	7.4
Present method with non-equal, uncorrelated errors	0.0813 ±0.0014	13.0	0.0804 ±0.0014	13.6

The comparison of the analyticity testing and estimates of f^2 based on the method of equal errors (first row), the general method with the equal errors input (second row) and the general method with different but uncorrelated errors (third row). The subscripts on χ^2 denote from which $Q_n\chi^2$ was calculated.

The results in the Table clearly show that for the case of equal errors both methods lead essentially to the same results. This also shows that the numerical procedures used in the more general case are reliable and do not introduce additional biases.

Finally, the results indicate that the method of unequal errors will lead to a more

accurate determination of the coupling constant when the error of the imaginary part will be such smaller than that of the real part. The discussion in Section IV, however, indicates that one can at most gain a factor of about $2^{-1/2}$ in the error estimate (and a much larger improvement for superconvergent amplitudes). In our case the improvement is only marginal (compare the second and the third rows of the Table) since both the real and the imaginary parts were taken from phase shift analyses, where the errors σ_R and σ_I are of comparable magnitudes.

The increase of the quantity

$$\chi^2 = \Sigma\, Q_n^2,$$

(see Table) is due to the narrowing of the error corridor. It is worth stressing that all the χ^2 in the Table are perfectly acceptable from the statistical point of view. This also shows that the data about the amplitude F^- fulfill the expected analyticity requirements. Apart from the values of χ^2 the analyticity of the data is also indicated by the fact that three estimates of the coupling constant based on the completely different numerical procedures lead to the same result. But the comparison of the first and the second rows of the Table shows that the present method probably slightly underestimates the error of the coupling constant. We shall therefore take as a final error estimate the value 0.0016.

VI. COMMENTS AND CONCLUSIONS

The general method for testing analyticity, for the search of the most probable analytic representation of data and for the calculation of coupling constants presented above is able to make full use of the information contained in the experimental data*. The method is based on the statistical treatment of the experimental data and the results of the testing of analyticity can thus be expressed in statistical terms well known from the testing of hypotheses. The procedure described in this paper can also be used in constructing the basis in $\mathscr{A}^2$ (the space of analytic fuctions), i. e., in looking for the most probable analytic representation of the data. Comparing it with the method based on reproducing kernel functions [14] which requires solving integral equations one can immediately see the advantage of our method being in simpler mathematics. On the other hand, in the Shepard and Shih approach [14] no special form of the error matrix is required.

It is worth stressing that passing from the simplified [4] to the general (and more complicated) method of analyticity, the testing described in this paper is not always very advantageous. In determining the value of the coupling constant one can gain

* Strictly speaking, the method requires that the inverse smoothed error matrix can be approximated by polynomials. In the majority of practical applications this can always be done.

only a factor of about $2^{-1/2}$ in the error estimate even if the accuracy of the imaginary part is much higher than that of the real part. The only exception here is represented by the superconvergent amplitudes when one can determine the residue with the accuracy of the better known imaginary part even if the errors of the real part are much larger. On the other hand, we hope (see discussion in Section IV) that advantage will be taken of the present method in determining the coefficients of an expansion of the amplitude around the crossing symmetry point $\nu=0$, $t=0$. The importance of this problem (and the related calculation of the σ term) has been stressed by Höhler et al. [23] (see also Refs. [24] and [25]).

The data on the πN scattering amplitudes we have used in our calculations confirm that the πN interaction is consistent with the locality of the interaction (microcausality). As a byproduct we have obtained the value of the coupling constant together with its statistical error estimate. The value found

$$f^2=0.0805\pm0.0016 \qquad \text{(from CERN 1971 analysis)}$$

is consistent with the recommended value [26] $\left(f^2=0.081\begin{matrix}+0.003\\-0.004\end{matrix}\right)$, with Sznajder Hald's calculations [27] ($f^2=0.0816\pm0.029$) and also with the recent calculation [28] scattering from the phase shift analysis [29] based on the precise low energy πN data [30] ($f^2=0.79\pm0.001$). On the other hand, it does not agree very well with the value of Samaranayake and Woolocock [31] ($f^2=0.0763\pm0.0020$) and it seems to be in complete disagreement with the Saclay 1973 result [32] ($f^2=0.0742\pm0.0013$). This disagreement is, perhaps, partially caused by the fact that new data have appeared after the CERN 1971 analysis was completed. But, on the other hand, the new total cross-section measurements [33] have been included in calculations of Ref. [27] which give a f^2 very near to our value.

All this clearly shows that in the present stage the systematic errors (for example, those connected with the Coulomb barrier corrections and with the contribution of the unphysical continuum arising from the processes $\pi^-p\to\gamma n$ and $\pi^-p\to\pi^0 n$) are much more important than the purely statistical errors of the data. This question has been discussed very thoroughly in Ref. [31] and in a very recent paper by Woolcock [34]. According to the results of Ref. [9] the statistical errors of data seem to be small enough to allow investigation of the isospin invariance breaking effects.

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