acta phys. slov. 27. (1977), No. 1

ANALYTIC PARAMETRIZATION OF HIGH ENERGY CROSSING EVEN $\pi^{\pm}p$ FORWARD SCATTERING AMPLITUDE

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Recent results on rising total cross section for the $\pi^{\pm}p$ scattering are analyzed by the statistical extrapolation method. The assumed asymptotic behaviour is of the logarithmic form. The phase shift data and recent high energy measurements of σ_{tot} and of the ratio of real to imaginary part of the amplitude are used as the input. We conclude that the present data plus analyticity are not sufficient to determine the exact form of the logarithmic growth of the forward scattering F^{+} amplitude.

АНАЛИТИЧЕСКАЯ ПАРАМЕТРИЗАЦИЯ ВЫСОКОЭНЕРГЕТИЧЕСКОЙ КРОССИНГЧЁТХОЙ $\pi^{\pm}p$ АМПЛИТУДЫ РАССЕЯНИЯ ВПЕРЁД

В работе анализируются с помощью метода статистической экстраполяции последние результаты о возрастающем полном попечном сечении $\pi^{\pm}p$ рассеяния. Предполагаемое асимптотическое поведение имеет логарифмический вид. Фазовые сдвиги и последние измерения σ_{tot} при высоких энергиях и отношение действительной к мнимой частей амплитуды рассеяния используются как входные данные. Приходим к заключению, что доступные данные вместе с аналитичностью не являются достаточными для того, чтобы определить точную форму логарифмического роста амплитуды рассеяния вперёд F^{+}.

I. INTRODUCTION

The effect of the rising total cross section was first observed at CERN ISR [1] for the pp reaction.

Similar observations have recently been made for $\pi^{\pm}p$ and $K^{\pm}p$ reactions at the Fermi National Accelerator Laboratory [2]. They show that hadron-nucleon total cross sections in most cases decrease at low energies, pass through a minimum and then all of them start rising at energies p_L, $50 \leq p_L \leq 200$ GeV/c.

The behaviour of the total cross section is related to the sign of the real part of the forward scattering amplitude. Namely, if the total cross section grows with energy like $(\log E)^2$, the real part of the forward amplitude is such that $\varrho = \mathrm{Re}F/\mathrm{Im}F$ behaves like $\pi/\log E$. This is the well known result of Khuri and Kinoshita and follows from general assumptions of the local field theory.

It was recently confirmed by FNAL measurements [3] that $\varrho(E)$ of $\pi^{\pm}p$ and $K^{\pm}p$ reactions passes through zero at laboratory momenta around 100 GeV/c and asymptotically approaches zero from above. Any reasonable theoretical model which explains the rise of the hadron-nucleon total cross section should also give the correst position of zero of the ratio $\varrho = \mathrm{Re}F/\mathrm{Im}F$. This idea was followed by Bourrely and Fischer who parametrized the pp, $p\bar{p}$ [4] and $\pi^{\pm}p$, $K^{\pm}p$ [5] forward scattering amplitudes by real analytic functions which are crossing symmetric and the corresponding total cross sections behave like $(\log E/E_0)^{\gamma}$. They obtained a good fit for a large interval of the values γ. A remarkable result of their $\pi^{\pm}p$ parametrization was that the zero point of ϱ was almost independent of the growth rate γ. The position of zero stays within narrow limits 52—80 GeV/c of the laboratory momentum. The present FNAL data give this value around 100 GeV/c.

There are several other works which have the real analyticity and crossing symmetry as a main assumption. Jakob and Kroll [6] derived certain constraints on the asymptotic behaviour of the total cross section from dispersion relations. For crossing even $\pi^{\pm}p$ amplitude they assumed two types of high energy parametrization of σ_{tot}. The one with an asymptotically constant behaviour and the other with an $\ln^2(E/E_0)$ term. Although such parametrizations have quite different asymptotic limits they both fit present data in a certain energy range. It means that an asymptotic constant behaviour is not excluded.

We shall not discuss other works which use different dispersion relation techniques in order to extract some information about high energy behaviour from the present data. An important result of the recent calculations [7] is that the data are consistent with the dispersion relations.

In general the analyticity of the scattering amplitude is tested by using "ordinary" dispersion relations in which the real part is expressed as the principal value integral of the imaginary part. It means that one predicts the real part of the amplitude from the knowledge of its imaginary part over the same or larger energy region. One may ask whether it is not possible to predict the real and imaginary parts of the amplitude in one energy region if we know both the real and the imaginary parts of the amplitude in another region. If there were no errors it would be, of course in principle possible. In this paper we shall consider this question in a practicularly interesting case.

In fact we shall investigate whether it is possible to predict the special form of the high energy behaviour of the scattering amplitudes from the data available at present. Suppose that the amplitude (both the real and the imaginary parts) is known below some energy value, say E_0 and is denoted $F_0(E)$. We shall then specify a form for the amplitude at $E > E_0$. This form will contain free parameters $\{a_0, a_1, a_2, \beta\}$ so that the amplitude

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$$F(E)=\begin{cases} F_0(E) & \text{for } E<E_0 \\ F(E, a_0, a_1, a_2, \beta) & \text{for } E>E_0. \end{cases}$$

We shall try to choose the parameters $\{a_0, a_1, a_2, \beta\}$ in such a way that $F(E)$ be an analytic function in the whole energy plane. The method which we shall use is based on the Cutkosky statistical approach to the representation of data by analytic functions. It is described in more detail in [8].

The present paper is organized as follows. In part II we shall describe very briefly the method. The summary of the present data on the $\pi^{\pm}p$ forward amplitude will be given in section III. In the last section we shall present the results. The derivation of the asymptotic form can be found in the appendix A. In appendix B we shall clarify the definition of our χ^2.

II. DESCRIPTION OF THE METHOD

The crossing even $\pi^{\pm}p$ amplitude $F^{+}(E)=\frac{1}{2}(F_{\pi^-p}(E)+F_{\pi^+p}(E))$ (E is the laboratory pion energy) has two symmetric cuts $(-\infty; -m_\pi)$, (m_π, ∞) and a pair of poles at $E_p=\pm m_\pi/2M_p$.

We know from experiment the real and imaginary parts of that amplitude along the right-hand cut (m_π, ∞) up to about 150 GeV. Above 150 GeV we shall use a parametrization of the form

$$F^{+}(E, a_0, a_1, a_2, \beta)=\mathrm{i}s[a_0+a_1(\ln s-\mathrm{i}\,\pi/2)^{\beta}]+ \\ +a_2(1-\mathrm{i})\sqrt{s}, \qquad (1)$$

where s is the c.m.s. energy squared with a_0, a_1, a_2, β real. Such a parametrization will be tested by a method described in [8]. To apply it for our case, we first map the right half of the E-plane onto the unit disc using the following conformal mapping

$$x=(\alpha-\sqrt{1-E^2})/(\alpha+\sqrt{1-E^2})$$

where α is a real constant*. We work in units where $m_\pi=\hbar=c=1$. Now the upper and the lower parts of the right-hand cut in the E-plane will be mapped on the upper and lower parts of the unit circle in the x-plane. The pole E_p is mapped onto $x_p=x(E_p)$. Instead of the amplitude $F(x)$ we shall consider the function $y(x)=F(x)\,(x-x_p)\,(x+1)^2$, which is analytic inside the unit circle and has no pole. The factor $(x+1)^2$ prevents the function $y(x)$ from diverging at $x=-1$, which is the image of $E=\infty$.

* α is arbitrary and the result should be independent of α. We have chosen α in such a way that the high energy part $E=(150\text{ GeV}, \infty)$ was mapped onto the arc $\varphi=(165°, 180°)$.

According to [8] we can now say that the sum $\sum_1^N |y_K|^2$, where

$$y_K=\frac{1}{2\pi}\oint\frac{y(x)x^K}{g(x)}\,\mathrm{d}|x|, \qquad (2)$$

has a chi-squared distribution with N degrees of freedom. $g(x)$ is the Ciulli-Fischer [9] weight function constructed in such a way that it is analytic and free of zeros within the unit disc and, on the boundary, it obeys the condition $|g(x_i)|=\varepsilon_{ef}(x_i)$. ε_{ef} is the effective error defined as $\varepsilon_{ef}=\varepsilon/\sqrt{\sigma}$, where ε is an experimental error and σ is the density of experimental points.

The integration in (2) proceeds along the unit circle. In the upper half of the circle we know the function $y(x)$ from 0° up to 165°. It is an interpolation of the data. (It corresponds to the region from the threshold up to 150 GeV in the E plane). From 165° up to 180° we use our parametric form $y(x, a_0, a_1, a_2, \beta)$, which corresponds to the asymptotic form (1) of the amplitude. Since the function $y(x)$ is real analytic, integration along the lower part of the circle is trivial.

Minimizing $\chi^2=\sum_1^N y_K^2$, where y_K are calculated from (2), we get optimal values of the parameters a_0, a_1, a_2, β of our asymptotic form.

We stress once more that we do not fit the last data by our parametrization (1) but expect such behaviour above the last data point.

III. DATA ON $F^{+}(E)$

Before presenting the results we shall shortly summarize what we know from experiment about the even $\pi^{\pm}p$ forward amplitude in the energy plane. Since the data come from different sources we can divide the right-hand cut into three regions:

1) From the threshold up to the point where the data of the phase shift analysis start. The amplitude can be calculated in this region by use of the effective range approximation. Using our conformal mapping with $\alpha=142.07$, this region was mapped onto a very small arc of the unit circle, so that the contribution to y_K from this part was of the order 10^{-4} and could be neglected.

2) The region from $E=0.17$ up to $E=2.78$ GeV. Here we have used the phase shift analysis Saclay 72. The amplitude was calculated by a simple interpolation of data. Since the density of data points is quite large in the x-plane, the linear interpolation should not distort too much the "true" amplitude.

3) The interval from the last Saclay data point up to 150 GeV. The imaginary part of $F^{+}(E)$ has been determined from the measurement of the total cross section [10] via the optical theorem. The real part was calculated from $\varrho=\mathrm{Re}F/\mathrm{Im}F$,

which is measured with the help of the Coulomb interference [11]. Here the simple linear interpolation of data may not be quite correct, since the density of data points, particularly for $\mathrm{Re}F^+$, is much lower than in the previous region. On the other hand this part will be suppressed by the weight function $g(x)$. A more serious drawback of this region is that the data come from five different sources and they may not be quite consistent. This influences the effective error and therefore increases χ^2.

IV. THE RESULTS

As shown above, χ^2 was defined as a sum $\sum_1^N y_K^2$, where y_K, s are given by (2). N was equal to 12. By choosing a higher N we could get more reliable results, but on the other hand with a high N we have problems with the integration of rapidly oscillating functions. The results are shown in Table 1. If we leave out the logarithmic term, i.e. we test the form

$$F(E, a_0, a_2) = \mathrm{i}a_0 s + a_2(1-\mathrm{i})\sqrt{s}, \tag{3}$$

we get $\chi^2 = 30.33$ and for the coefficients the values $a_0 = -0.0011$, $a_2 = 0.00057$.

For the case of the total constant cross section, i.e.

$$F(E, a_0) = \mathrm{i}a_0 s, \tag{4}$$

the minimization gives $\chi^2 = 20.56$ and $a_0 \approx 10^{-5}$.

The difference between the χ^2 values of parametric forms (1) and (4) is surprisingly small, since in the case (4) we have only one free perameter a_0 in contrast to (1) where there are three free parameters.

In conclusion we can say that so far the data plus analyticity do not give any convincing results concerning the logarithmic growth of the form of Eq. (1) of the total cross section and even the constant asymptotic behaviour cannot be excluded.

Table 1

β	a_0	a_1	a_2	χ^2
1/4	1.7	−0.978	2.5	19.40
1/2	0.84	−0.28	2.33	19.39
3/4	0.55	−0.11	2.174	19.37
1	0.41	−0.05	2.04	19.35
5/4	0.33	−0.02	1.92	19.33
6/4	0.27	−0.011	1.81	19.30
7/4	0.23	−0.006	1.71	19.30
2	0.2	−0.003	1.63	19.25

ACKNOWLEDGEMENT

I would like to thank Dr. J. Pišút for drawing my attention to this problem and for the careful reading of the manuscript. I should also like to thank Dr. Kwiečinski for useful suggestions and Dr. J. Boháčik and I. Furdík for interesting discussions and computational help.

APPENDIX A

In this part we shall shortly explain why we have used the parametrization of the specific form (1).

We want to fulfil two requirements:

$$\text{the crossing symmetry } F^+(s) = F^+(-s) \tag{5}$$

$$\text{and the real analyticity } F^{+*}(s) = F^+(s^*) \tag{6}$$

Strictly speaking the crossing symmetry requires $F^+(E) = F^+(-E)$, where E is the laboratory pion energy but asymptotically $E \approx s/2M$ and therefore (5) should be valid too.

Besides conditions (5) and (6) we also require a certain asymptotic behaviour. Namely, we want the function $F(s)$ to contain three types of terms:

i) One with the asymptotic behaviour ias. (It corresponds to the constant total cross section).

ii) The term of the type $s(\log s)^\beta$, which represents the logarithmic growth of the total cross section.

iii) The term representing the Regge behaviour, $-a\sqrt{s}$.

Condition i) is fulfilled by the function

$$L(s) = 1 - \mathrm{i}a\sqrt{s^2 - s_0^2},$$

which has also two symmetric cuts $(-\infty, -s_0)$, (s_0, ∞). Similarly, the function $\ln L(s)$ fulfils crossing and real analyticity and asymptotically behaves like $\ln s$.

Constructing from function $L(s)$ the analytic function $F(s)$ which contains terms of all three types: i), ii), iii) and then performing the limit $s \to \infty$, we come to the form:

$$F^+(s) = \mathrm{i}s(a_0 + a_1(\ln s - \mathrm{i}\,\pi/2)^\beta) + (\mathrm{i}-1)\,a_2\sqrt{s},$$

where a_0, a_1, a_2, β are real.

APPENDIX B

In the following we shall show that the sum $\Sigma\, y_K^2$ is consistent with the currently used definition of χ^2.

Let us start with the simplest example. (What follows is explained in more detail in [12]).

Let B denote the boundary of a region $\mathcal{D}$ and let y_i and ε_i be the data and the corresponding errors in measuring the function $F(x_i)$ in points x_i. The values y_i are understood as the random gaussian distributed variables with the means $F(x_i)$ and the dispersion ε_i.

Suppose that y_i, $F(x_i)$ are real. Let $\varphi(x)$ be a real function defined on B. Then using standard statistical methods it can be shown that the random variable

$$S = \sum_{i=1}^{N} y_i \varphi(x_i) \varepsilon_i^{-2},$$

where N is the number of data points, is gaussian distributed with the mean value

$$E(S) = \sum_{i=1}^{N} F(x_i)\, \varphi(x_i)\, \varepsilon_i^{-2}$$

and with the variance

$$D(S) = \sum_{i=1}^{N} \varepsilon_i^{-2}\, \varphi^2(x_i).$$

If the data are dense enough we can rewrite the previous sum into the integral

$$S = \int_B y(x)\varphi(x)\varepsilon^{-2}(x)\, |dx|, \tag{7}$$

where $\varepsilon(x)$ is a smooth interpolation to $\varepsilon_i/\sqrt{\sigma(x_i)}$, where $\sigma(x_i)$ is the density of data points around x_i. The mean value of the gaussian distributed variable defined by (7) is

$$E(S) = \int_B F(x)\varphi(x)\varepsilon^{-2}(x)\, |dx|$$

and the variance is

$$D(S) = \int_B \varphi^2(x)\varepsilon^{-2}(x)\, |dx|.$$

The previous example can be easily generalized for the case of a complex function $F(x_i)$ with complex values y_i. We suppose for simplicity that both the real and the imaginary parts of y_i have the same errors.

The previous arguments are valid for both the real and the imaginary components of $F(x)$ and of y so that we can say that the variables Q_R and Q_I defined as

$$Q_R + iQ_I = \int_B y(x)\varphi(x)\varepsilon^{-2}(x)$$

are gaussian distributed.

Instead of one function $\varphi(x)$ we can choose a suitable set of functions — $\varphi_K(x)$ and then construct Q_K 's as

$$Q_K = \int_B y(x)\varphi_K(x)\, dx. \tag{8}$$

If $F(x)$ is analytic and $\varphi_K(x)$ are suitably normalized, then the real and imaginary parts of Q_K 's are gaussian, $N(0, 1)$ distributed and hence the sum $\sum_1^N |Q_K|^2$ is the usual χ^2 with N degrees of freedom.

So far we have assumed that the data are given along the whole boundary B. Minimizing χ^2 then we could say whether the data y_i are consistent with the analyticity of the function $F(x)$.

However, in our case there is a part of the boundary with no data — the asymptotic region. There we have to use a certain hypothesis. In this situation we can interpret our χ^2 — test as follows: We suppose that in the asymptotic region the data follow the curve given by formula (1). Then we ask how are such "data" plus data from the region up to 150 GeV consistent with the analyticity of the amplitude in the whole energy plane.

REFERENCES

[1] Amaldi et al., PL *44 B* (1973), 112; Amendolis S. R., PL *44 B* (1973), 119.

[2] Baker W. F. et al., *Proc. XVIIth intern. Conf. on High Energy Physics*, London 1974.

[3] Ankenbrandt C. et al., Preprint FERMILAB-Conf-75/61-EXP, August 1975.

[4] Bourrely C., Fischer J., Nucl. Phys. *B 61* (1973), 513.

[5] Bourrely C., Fischer J., Sekera Z., Nucl. Phys. *B 67* (1973), 452.

[6] Jakob H. P., Kroll P., CERN preprint TH 1968.

[7] Höhler G., Jakob H. P., Z. Phys. *268* (1974), 75; Hendrich R. E., Lautrup B., Phys. Rev. *D 11* (1974), 529.

[8] Nogová A., Pišút J., Prešnajder P., Nucl. Phys. *B 61* (1973), 438.

[9] Ciulli S., Fischer J., Nucl. Phys. *B 24* (1970), 537.

[10] Carroll A. S., Palermo Conf. 1975; Foley K. J., Phys. Rev. *19* (1967), 330; Giacomelli S., Prog. Nucl. Phys. *12* (2).

[11] Foley K. J., Phys. Rev. *181* (1969), 1775; Vorobev I. A., Sov. Journ. Nucl. Phys. *19* (1974), 433; Nurushev K. P., Proc. London Conf. 1974; Ankenbrandt C. et al., FERMILAB-Conf-75/61-EXP; References [10] and [11] (except for Ankenbrandt) are taken from Höhler, Jakob, Kaiser TKP 13/75 preprint.

[12] Pázman A. et al., *On the Determination of Singularities of Scattering Amplitudes from the data.* Bratislava, Inst. of Phys. of the Slov. Acad. Sci. preprint 1971.

Received March 23rd, 1976