acta phys. slov. 26 (1976), No.3

DIFFUSION APPROXIMATION OF THE KINETIC EQUATION FOR THE COSMIC RAY PARTICLES DISTRIBUTION FUNCTION

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The charged particle propagation in an irregular magnetic field is investigated. There was used the kinetic equation for the mean particle distribution function in an ensemble of random magnetic fields. The interaction of particles with a random magnetic field is specified by the collision integral and the correlation tensor of a random magnetic field. There is calculated the diffusion approximation of the kinetic equation in case of the particle propagation in a strong magnetic field of the interplanetary space, i.e. the particles are specified by helical motion in space.

I. INTRODUCTION

In the past few years the kinetic equation for the distribution function has been used more and more. It is the consequence of the fact that anisotropic currents of cosmic ray particles with the mean free path ~ 1 A.U. are being observed. It cannot be doubted that the ordinary diffusion theory cannot be used in such case and the kinetic investigation of processes in interplanetary space is necessary. The diffusion approximation of the kinetic equation for the mean distribution function presents the correct diffusion equation as well as the exact formulae for the diffusion process coefficients depending on the parameters of the medium.

Dolginov, Toptygin [1] carried aut the diffusion approximation of the kinetic equation (given in [1], too). The authors give the expression for the flux of high-energy particles, for which the Larmour radius $R \gg L_c$, where L_c is the auto-correlation length of a turbulent magnetic field, i.e. the momentum of the particle is constant in a regular magnetic field of the size L_c.

The same authors carried out in 1968 [2] the diffusion approximation in the case of a strong regular magnetic field, however, only the components of the diffusion tensor of the particle in space changed.

In this work there is presented the exact process of the diffusion approximation of the kinetic equation for the mean distribution function of the particle in an interplanetary magnetic field.

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II. THE KINETIC EQUATION

In interplanetary space there exist random, turbulent magnetic fields with the background regular field:

$$\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{H}_0(\boldsymbol{r}) + \boldsymbol{H}_1(\boldsymbol{r}, t), \tag{1}$$

where $\boldsymbol{H}(\boldsymbol{r}, t)$ is the full magnetic field, $\boldsymbol{H}_0(\boldsymbol{r})$ its regular components.

Turbulent fields are frozen into the plasma of the solar wind, mooving with the velocity $\boldsymbol{u}$. Therefore, in interplanetary space the induction electric field $-\frac{1}{c}[\boldsymbol{u}, \boldsymbol{H}]$ acts on the charged particles, too. The resulting electromagnetic force is:

$$\boldsymbol{\mu} = \frac{e}{c}[\boldsymbol{v}\ \boldsymbol{H}] - \frac{e}{c}[\boldsymbol{u}\ \boldsymbol{H}], \tag{2}$$

where $\boldsymbol{v} = c^2\boldsymbol{p}/\varepsilon(p)$ is the velocity of the particle, $\varepsilon(p) = c\sqrt{p^2 + m^2c^2}$ its energy.

For a random component of force, which acts on the particle we obtain from (1) and (2):

$$\boldsymbol{\mu}_1 = \frac{e}{c}[\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{H}_1]. \tag{3}$$

The interaction between particles of cosmic rays can be neglected, therefore we use the kinetic equation for the mean distribution function (in ensemble random fields):

$$\left(\frac{\partial}{\partial t} + \boldsymbol{L}_0\right) F(\boldsymbol{r}, \boldsymbol{p}, t) = \text{Col}\ F, \tag{4}$$

where we have

$$\hat{L}_0 = \boldsymbol{v}\nabla + \frac{e}{c}[\boldsymbol{w}\boldsymbol{H}_0]\ \boldsymbol{L} \tag{5}$$

$$L_\alpha = \frac{\partial}{\partial p_\alpha}$$

$$\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}.$$

The collision integral Col F represents the interaction particle-field and has by [3] the form:

$$\text{Col}\ F = \int_0^\infty \mathrm{d}\,\tau L_\alpha\, \mathrm{e}^{-\hat{L}_0\tau}\, D_{\alpha\lambda}\left(\frac{\boldsymbol{r} + \boldsymbol{r}_1}{2}, \boldsymbol{r}_1 - \boldsymbol{r} - \boldsymbol{u}\tau; \boldsymbol{p}, \boldsymbol{p}_1\right) L_\lambda F(\boldsymbol{r}, \boldsymbol{p}, t). \tag{6}$$

It is necessary to write $\boldsymbol{r}_1 = \boldsymbol{r}$ and $\boldsymbol{p}_1 = \boldsymbol{p}$ after the operator $\mathrm{e}^{-\hat{L}_0\tau}$ has acted on a certain function.

III. TENSOR OF RANDOM FIELDS

In a general case a tensor of random fields $\mu_1(x)$ is determined by the relation

$$D_{\alpha\lambda}(x_1\,;x_2) = \langle \mu_{1\alpha}(x_1)\,\mu_{1\lambda}(x_2)\rangle . \tag{7}$$

The square brackets are used to indicate the averaging over an ensemble of realizations of random magnetic fields. Substituting (3) in to (7) gives the expression for the correlation tensor of turbulent magnetic fields:

$$B_{\alpha\lambda}(x_1\,;x_2) = \langle H_{1\alpha}(x_1)\,H_{1\lambda}(x_2)\rangle . \tag{8}$$

We assume that turbulent fields are statistically isotropic and depend only on $x = x_1 - x_2$. We leave the argument $r = \frac{1}{2}(x_1 + x_2)$ as well, because the dependence $B_{\alpha\lambda}(r, x)$ from argument r describes low changes of $B_{\alpha\lambda}$ when we move from one turbulent cloud to another. In the case of a statistically isotropic field, we can write:

$$B_{\alpha\lambda}(r, x) = \frac{\langle H_1^2(r)\rangle}{3}\left\{\Psi_1\left(\frac{x}{L_c}\right)\delta_{\alpha\lambda} + \Phi_1\left(\frac{x}{L_c}\right)\frac{x_\alpha x_\lambda}{x^2}\right\} \tag{9}$$

$$\Psi_1\left(\frac{x}{L_c}\right) = \Psi\left(\frac{x}{L_c}\right) - \frac{2L_c^2}{x^2}\int_0^{x/L_c} dy\,\Psi(y)y .$$

The function Ψ is generally determined empirically. According to the majority of experimental results it is necessary for the function Ψ to have such a form that the spectral density of the energy of the magnetic field may have a power dependence on the wave vector. Frequently, the following form of the Ψ function is used:

$$\Psi\left(\frac{x}{L_c}\right) = \left(\frac{x}{L_c}\right)^{(\nu-1)/2} K_{(\nu-1)/2}\left(\frac{x}{L_c}\right) \tag{10}$$

whose Fourier-component is [4]:

$$\Psi(k) = \frac{A_\nu}{(k_0^2 + k)^{\nu/2+1}} \tag{11}$$

$$A_\nu = \frac{\nu\Gamma\left(\frac{\nu}{2}\right)}{4\pi^{3/2}L_c^{\lambda-1}\,\Gamma\left(\frac{\nu-1}{2}\right)} \tag{12}$$

and for ν the inequality $1 < \nu \leq 3.8$ holds.

IV. EQUATIONS OF THE DIFFUSION APPROXIMATION

The collision integral (6) is reduced to the form:

$$\text{Col}\,F = \int_0^\infty d\tau\,D_\alpha\,e^{-\hat{L}_0\tau}\,B_{\alpha\lambda}\left(\frac{r + r_1}{2}, r_1 - r - u\tau\right) D_\lambda F(r, p, t), \tag{13}$$

where

$$D_\alpha = \frac{e}{c}\,\varepsilon_{\alpha\mu\nu} w_\mu \frac{\partial}{\partial p_\nu}. \tag{14}$$

For the following calculations it is necessary to explain how the operator $e^{-\hat{L}_0\tau}$ influences the function. The second term in $\hat{L}_0$ is in fact $\Delta p\,\frac{\partial}{\partial p}$, and is the cause of the change of the momentum in a strong regular field. We shall assume the field H_0 to be homogeneous in a cloud of the dimension L_c.
The momentum P of the particle in the field H_0 is:

$$P(\tau) = h(ph) - [[ph]h]\cos\Omega\tau + [ph]\sin\Omega\tau , \tag{15}$$

where

$$\Omega = \frac{evH_0}{cp}, \qquad h = \frac{H_0}{H_0}.$$

The expression for the radius vector $R(\tau)$ is analogous. We write

$$V \equiv \exp\{-L_0\tau\}\; w = \frac{c^2}{\varepsilon(p)}\,P - u . \tag{16}$$

The process of the diffusion approximation means that the distribution function $F(r, p, t)$ is expanded in a series of spherical functions, and only the terms $l = 0{,}1$ are taken into account (see also [1]):

$$F(r, p, t) = \frac{1}{4\pi}\left\{N(r, p, t) + \frac{3p}{vp}\,J(r, p, t)\right\}, \tag{17}$$

where the functions $N(r, p, t)$ and $J(r, p, t)$ signify the concentration and the current density of the particles and depend on $p = |p|$ only. Since in the magnetic field $p = P$ holds, the collision integral (13), using (15—17), is:

$$\text{Col}\,F = \frac{1}{4\pi}\left(\frac{e}{c}\right)^2 \varepsilon_{\alpha\beta\gamma} w_\beta \frac{\partial}{\partial p_\mu}\int_0^\infty d\tau\,B_{\alpha\lambda}(r, w_{ll}\,\tau)\times \tag{18}$$

$$\times\left\{\frac{3}{vp}[VJ]_\lambda - \frac{1}{p}[uP]_\lambda\left(\frac{\partial N}{\partial p} + \frac{3}{vp}\left(P\frac{\partial J}{\partial p}\right) - \frac{6}{vp^2}\left(1 - \frac{v^2}{2c^2}\right)(PJ)\right)\right\}.$$

Since the dependence of the tensor $B_{\alpha\lambda}(\mathbf{r}, \mathbf{x})$ on the first argument, which describes low changes of the intensity of turbulent fluctuations when passing from one turbulant range to another, is slight, we have used in the expression (18) an approximation:

$$e^{-\hat{L}_0\tau}\, B_{\alpha\lambda}\left(\frac{\mathbf{r}+\mathbf{r}_1}{2}, \mathbf{r}_1-\mathbf{r}-\mathbf{u}\tau\right) \rightarrow B_{\alpha\lambda}(\mathbf{r}, \mathbf{w}_{II}\, \tau), \tag{19}$$

where $\mathbf{w}_{II}=\mathbf{h}(v\mathbf{h})-\mathbf{u}$.

The non-diagonal components of the tensor, proportionate to Ψ_1, give a small deposit, and we obtain:

$$\text{Col}\, F=\frac{1}{12}\langle H_i^i(\mathbf{r})\rangle \left(\frac{e}{c}\right)^2 \varepsilon_{\alpha\beta\gamma} w_\beta \frac{\partial}{\partial p_\gamma}\int_0^\infty d\tau\, \Psi\left(\frac{w_{II}\tau}{L_c}\right)\times \tag{20}$$

$$\times\left\{\frac{3}{vp}[\mathbf{VJ}]_\alpha-\frac{1}{p}[\mathbf{uP}]_\alpha\left(\frac{\partial N}{\partial p}+\frac{3}{vp}\left(\mathbf{P}\frac{\partial \mathbf{J}}{\partial p}\right)-\frac{6}{vp^2}\left(1-\frac{v^2}{2c^2}\right)(\mathbf{PJ})\right)\right\}.$$

If we integrate the Eq. (4) with the collision integral (20) over an angle space with the polar axis along the vector $\mathbf{p}$, the equation may split into two equations for the functions $N(\mathbf{r}, p, t)$ and $\mathbf{J}(\mathbf{r}, p, t)$. Because u/v is a small parameter, we expand the collision integral in series [5]. We obtain the following equation for the density particles:

$$\frac{\partial N}{\partial p}+\text{div}\, \mathbf{J}-\frac{1}{vR}[\mathbf{uh}]\left(p\frac{\partial \mathbf{J}}{\partial p}+(1+\beta^2)\,\mathbf{J}\right)= \tag{21}$$

$$=a\frac{\partial^2 N}{\partial p^2}+(b+2a)\frac{1}{p}\frac{\partial N}{\partial p}+\frac{1}{p^2}\mathbf{J}(\mathbf{u}g+\mathbf{h}(\mathbf{uh})k+[\mathbf{uh}]l)+$$

$$+\frac{1}{p}\frac{\partial \mathbf{J}}{\partial p}(\mathbf{u}m+\mathbf{h}(\mathbf{uh})n+[\mathbf{uh}]o),$$

where the coefficients are as follows:

$$a=\tfrac{1}{3}(u^2\Psi_+-(\mathbf{uh})^2\,\Psi_-), \tag{22}$$

$$b=\tfrac{1}{3}(u^2+(\mathbf{uh})^2)\,\bar{\Psi}_s,$$

$$g=\Psi_++2\bar{\Psi}_{sc},$$

$$h=-\Psi_-+2\bar{\Psi}_{sc}-\Psi'_-,$$

$$l=(5-\beta^2)\,\Psi_s-2(2-\beta^2)\,\Psi_{sc}+\bar{\Psi}_{ss},$$

$$m=\Psi_+,$$

$$n=-\Psi_-,$$

$$o=\Psi_s+2\Psi_{sc},$$

$$\Psi_\pm=\hat{T}\Psi\left(\frac{v\tau}{L_c}\right)(1\pm\cos\Omega\tau), \tag{23}$$

$$\Psi_s=\hat{T}\Psi\left(\frac{v\tau}{L_c}\right)\sin\Omega\tau,$$

$$\Psi_{sc}=\hat{T}\Psi\left(\frac{v\tau}{L_c}\right)\sin\Omega\tau\cos\Omega\tau,$$

$$\Psi'_\pm=\hat{T}v\,\frac{\partial\Psi(v\tau/L_c)}{\partial v}(1\pm\cos\Omega\tau),$$

$$\bar{\Psi}_s=\hat{T}\frac{v^2}{c^2}\,\Omega\tau\Psi\left(\frac{v\tau}{L_c}\right)\sin\Omega\tau,$$

$$\bar{\Psi}_{ss}=\hat{T}\frac{v^2}{c^2}\,\Omega\tau\Psi\left(\frac{v\tau}{L_c}\right)\sin^2\Omega\tau,$$

$$\bar{\Psi}_{sc}=\hat{T}\frac{v^2}{c^2}\,\Omega\tau\Psi\left(\frac{v\tau}{L_c}\right)\sin\Omega\tau\cos\Omega\tau;$$

$$\hat{T}\ldots=\tfrac{1}{3}\langle H_i^i(\mathbf{r})\rangle\left(\frac{e}{c}\right)^2\int_0^\infty d\tau\ldots$$

In the expansion of the collision integral we neglect higher order derivates. The second equation is:

$$\frac{1}{v}\frac{\partial \mathbf{J}}{\partial t}+\frac{v}{3}\,\text{grad}\, N-\frac{1}{R}[\mathbf{uh}]\frac{p}{3}\frac{\partial N}{\partial p}+\frac{1}{R}[\mathbf{hJ}]=$$

$$=\frac{v}{3p}\frac{\partial N}{\partial p}\left\{\mathbf{h}(\mathbf{uh})\,\Psi_--\mathbf{u}\Psi_++[\mathbf{uh}]\,\Psi_s-\mathbf{h}\frac{(\mathbf{uh})}{v}\,\Psi'_-\right\}+ \tag{24}$$

$$+\frac{v}{p^2}\left\{\mathbf{h}(\mathbf{Jh})\,\Psi_--\mathbf{J}\Psi_++[\mathbf{hJ}]\,\Psi_s-\frac{\mathbf{u}}{v}(\mathbf{Jh})\,\Psi'_++\frac{1}{v}\mathbf{h}(\mathbf{uh})\,\Psi'_-\right\}.$$

In the case of the current density being independent of time, the latter equation gives an explicit expression for $\mathbf{J}$ with the help of the gradient density N and the energy spectrum $\partial N/\partial p$:

$$J_\alpha=-\kappa_{\alpha\beta}\frac{\partial N}{\partial x_\beta}-\bar{w}_\alpha\frac{p}{3}\frac{\partial N}{\partial p}. \tag{25}$$

The diffusion tensor $\kappa_{\alpha\beta}$ depends on both the parameters of medium and the momentum of the particle, which holds also for $\bar{\mathbf{w}}$ at "drift" velocity.

The expression (25) may be used for determining some quantities difficult to measure, for example ∇N. In case of low energies of the particles, with the Larmour radius $R \ll L_c$, direct measurements in interplanetary space are evidently necessary.

Apart from that, in the process of the calculation of expression (25) we can obtain explicit expressions for the components of the diffusion tensor $\kappa_{\alpha\beta}$ and the mean free paths.

ACKNOWLEDGEMENT

The authors want to thank Professor L. I. Dorman and M. E. Kats, CSc. for many helpful discussions and suggestions.

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Received April 28th, 1975