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THE EVALUATION OF THE π N SCATTERING LENGTHS FROM DATA AND ANALYTICITY

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The S and P wave scattering lengths were evaluated by the method based on a statistical approach to the representation of data by analytic functions. Investigating the π N forward scattering amplitudes we obtained the following results:

$a_1 - a_3 = 0.298 \pm 0.017$ $\quad$ $a_{13} - a_{33} - a_{11} + a_{31} = -0.184 \pm 0.010$

$a_1 + 2a_3 = 0.026 \pm 0.043$ $\quad$ $a_{13} + 2a_{33} - a_{11} - 2a_{31} = 0.562 \pm 0.012.$

The scattering lengths themselves and their combinations were determined with less accuracy.

I. INTRODUCTION

It is well known that accurate values of scattering lengths cannot be obtained directly by extrapolating phase shifts. This is not only due to the difficulty of low energy experiments, but also to problems of separating out the dynamical effects of electromagnetic interactions — open so far — especially at low energies. The πN scattering lengths are therefore usually calculated from sum rules based on the dispersion relations (for a review and a representative list of references see [1]). Moreover, the statistical error estimates in calculations based on dispersion relations are known to be a delicate affair.

In the present paper we shall evaluate the πN scattering lengths by using a method which combines a truly statistical approach with the analyticity properties of scattering amplitudes. The present calculation is closely related to the work [2] done recently on the πN forward amplitudes.

The method which we shall use has been inspired by Cutkosky's paper [3] and is based on the development of his approach described in [2], [4] and [5]. More specifically, we shall determine the scattering lengths by the method of "analyticity testing" which, with minor amendments, can also be used for the determination of unknown parameters in the scattering amplitudes. The idea of the method will be

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described in the following section. The calculations and the results obtained are presented in Section III and commented upon in Section IV.

Before proceeding further it should be stressed that from the view-point of analytic extrapolation the scattering length is a very bad parameter, since strictly speaking it is given by the value of an amplitude at a single point on the boundary. Taken literally such a parameter can never be estimated on the basis of analyticity. When speaking about the scattering lengths, what we have in mind is rather something like "effective scattering lengths" which characterize (together with perhaps further parameters) the behaviour of the amplitudes between the threshold and the region where data are available. This means that the scattering lengths are understood here as parameters which determine a smooth low energy behaviour of the amplitudes. The term smooth is always hidden in the specific parametrization of the amplitudes at low energies.

II. THE METHOD

The basic idea of our method can briefly be described as follows. We are looking for a set of the scattering lengths for which πN amplitudes fulfil the required analyticity properties in the best possible way. The condition of "maximal analyticity" is formulated mathematically by means of the statistical method for analyticity testing.

At present, there are two methods for the testing of analyticity elaborated enough to be used in practical calculations. The former [2, 5] requires that the errors of (uncorrelated) real and imaginary parts of the amplitude be equal, while the latter [6] is suitable for handling the general case of unequal errors of (correlated) real and imaginary parts.

In our case there is not much to be gained by applying the more complicated though more general approach (see discussion in [6]), hence we shall use the simpler method of equal errors [2, 5]. In order to have the conservative error estimate we shall take as the error of both real and imaginary parts at a given value of energy the quantity $\varepsilon = \max(\varepsilon_R, \varepsilon_I)$ where ε_R, ε_I are errors of real and imaginary parts, respectively.

Let $F(\nu)$ be one of the πN crossing even forward amplitudes or a crossing odd amplitude divided by ν, where $\nu = (s-u)/4M_p$. Then $F(\nu)$ is analytic in the ν plane with two cuts $-\infty < \nu < -1$ and $1 < \nu < \infty$ (and perhaps a neutron pole at $\nu = \pm \nu_0$, $\nu_0 = (M_n^2 - M_p^2 - 1)/2M_p$, pion mass units are used). By performing the conformal mapping

$$x = x(\nu) = \frac{\sqrt{1-\nu_0^2} - \sqrt{1-\nu^2}}{\sqrt{1-\nu_0^2} + \sqrt{1-\nu^2}} \qquad (1)$$

we map the right half of the ν plane onto the unit disc. The cut $\nu > 1$ is mapped onto the unit circle, the neutron pole to the origin and $\nu = \infty$ to the point $x = -1$. The experimental data based on the phase shift analyses cover the arcs $60° < \varphi < 172°$ and $-172° < \varphi < -60°$, where φ is the angle of a point on the unit circle $x = \exp\{i\varphi\}$.

Let further $y(x)$ be the smooth representation of the experimental data and $\varepsilon(x)$ be the smoothed experimental error. The definition of the smoothed quantities is discussed in detail in [4] and for our particular case in [2].

In the low energy region we shall parametrize the amplitude in each partial wave by an effective range expansion, which for our purpose can be written as*

$$q^{2l+1} \operatorname{cotg} \delta_l = \frac{1}{a_l} + \frac{r_l}{2} q^2 \tag{2}$$

where q is the centre-of-mass momentum. The method also requires that the smoothed error $\varepsilon(x)$ be defined in this region. This $\varepsilon(x)$ has to be defined beforehand and kept fixed during a single calculation. In fact, $\varepsilon(x)$ specifies the weight attached to each function analytic in the unit disc (see Ref. [4]) and this weight is necessary for the problem to be well defined. We shall return to this question in the next section.

The detailed behaviour of the amplitudes and their smoothed errors at "high energies" (above 2 GeV, i.e., for $\varphi > 172°$) was not important for the calculation of the scattering lengths. We have taken the same high energy parametrization of the amplitude as in Ref. [2].

If the smoothed error function $\varepsilon(x)$ is known we can construct the weight function $g(x)$, which is analytic and free of zeros in the unit disc and which satisfies the condition $|g(x)| = \varepsilon(x)$ for $|x| = 1$. As shown in Refs. [4] and [5] the quantities

$$Q_n = \frac{1}{2\pi} \oint_B \frac{y(x)}{g(x)} x^n \, |dx|, \qquad n = 1, 2, \ldots \tag{3}$$

are Gaussian random distributed quantities with vanishing mean value (for the amplitudes containing a pole term it is true for $n \geq 2$) and unit standard deviations. Here $y(x)$ is a smooth function representing the data and the hypotheses about the behaviour of the amplitude at low and high energies. The quantity

$$\chi^2(a_i, r_i, h) = \sum_{n=1(2)}^{N} Q_n^2(a_i, r_i, h) \tag{4}$$

* The results obtained below show that data and/or the method are anyway not accurate enough to determine the parameters r_l. Due to this the differences among various parametrizations of the phase shifts are of no importance.

(where h represents the set of high energy parameters) has the standard χ^2 distribution with the number of degrees of freedom given by N and by the number of free parameters. The values of a_i, r_i and h are then determined by minimization of χ^2 and their errors are given by standard statistical procedures.

III. CALCULATIONS AND RESULTS

III. 1. Input data

In our calculations we have used two sets of phase shifts and elasticities. The first one has been the recent Saclay analysis [7], which takes into account also accurate measurements by Bussey et al. [8]. As the second set we have choosen the combination of phase shift analysis by Carter et al. [9] (88 MeV $\leq T_\pi \leq$ 310 MeV), which is mainly based on [8], and that of Almehed and Lovelace (outside this region). It is clear that at the present stage of experimental and theoretical knowledge on πN scattering the question of unambigous separation of electromagnetic and strong effects is important and requires careful treatment. The same holds true for the contribution from the region below the π^-p threshold arising from processes $\pi^- p \to \pi^0 n$ and $\pi^- p \to \gamma n$. The inclusion of this contribution in our method is even more complicated than in the dispersion relation technique (see, e.g., [11]), because the present method requires the knowledge of the real part of the amplitude in this region.

The main aim of the present paper is to show how statistical analytic methods can be used in determining unknown parameters in scattering amplitudes and to obtain realistic estimates of the statistical errors of the πN scattering lengths. We therefore disregarded the question of contribution from the unphysical continuum and supposed the phase shifts quoted in Refs. [7, 9, 10] to be the true nuclear phase shifts.

III. 2. Choice of the function to be minimized

The low energy behaviour of different πN forward scattering amplitudes is determined by different combinations of scattering lengths. For example, the invariant amplitudes A_l and B_l (in a given isospin channel) depend mainly on the difference of the P wave scattering lengths $a_{21,1} - a_{21,3}$. On the other hand, the amplitude $F = A + \nu B$ is sensitive to the value of the S wave scattering length. To take full advantage of our method, we minimized the sum of the quantities (4) constructed from four independent amplitudes. If we took as an independent set the amplitudes $A^+, B^+/\nu, A^-/\nu, B^-$ the S wave scattering lengths would be determined with a small accuracy. Therefore, we minimized the sum of χ^2 s (4) constructed from amplitudes $F^+, B^+/\nu, F^-/\nu, B^-$. Calculations have shown that results remain unchanged if the amplitudes B are replaced by the amplitudes A in this set.

III. 3. Smoothed error in the low energy region

As it has already been stressed in the previous section, we have to assign some errors to scattering lengths before starting the calculation in order to be able to evaluate the smoothed error in the low energy region. If the starting errors were chosen too small we might find an unacceptably large χ^2 (which gives the confidence level of the hypothesis of the analytic properties of amplitudes). In such a case the output errors (after being multiplied by $\sqrt{\chi^2_R}$) would be also considerably higher than the input ones. If the initial errors were too large, the resulting error estimates of the calculated scattering lengths would turn out to be considerably lower than these starting values. This is caused by the fact that the output errors are given by the smoothed error $\varepsilon(x)$ in the whole region.

III. 4. Results

As starting values of the scattering lengths and their errors we have taken the following numbers (inspired by the "recommended values" from Ref. [1])

$$a_1 - a_3 = 0.290 \pm 0.020 \qquad a_1 + 2a_3 = -0.045 \pm 0.045 \tag{5}$$
$$a_{13} - a_{33} - a_{11} + a_{31} = -0.204 \pm 0.013$$
$$a_{13} - a_{33} + a_{11} - a_{31} = -0.282 \pm 0.013$$
$$a_{13} + 2a_{33} - a_{11} - 2a_{31} = 0.567 \pm 0.030$$
$$a_{13} + 2a_{33} + a_{11} + 2a_{31} = 0.231 \pm 0.030.$$

In the case of the input data calculated from the phase shift analysis Saclay 1973 [7], the output errors were larger than the input ones by a factor of 5

$$a_1 - a_3 = 0.27 \pm 0.10 \qquad a_1 + 2a_3 = 0.22 \pm 0.21$$
$$a_{13} - a_{33} - a_{11} + a_{31} = -0.29 \pm 0.09$$
$$a_{13} - a_{33} + a_{11} - a_{31} = -0.17 \pm 0.24$$
$$a_{13} + 2a_{33} - a_{11} - 2a_{31} = 0.40 \pm 0.09$$
$$a_{13} + 2a_{33} + a_{11} + 2a_{31} = -0.13 \pm 0.37.$$

But the scattering lengths and their errors calculated from the second set of phase shifts (see III. 1.) did not differ conspicuously from the starting values. Encouraged by this fact, we inserted the numbers obtained as an input for further calculation in an attempt to obtain a "self-consistent" solution. After three such steps we succeeded in finding the following "self-consistent" values

$$a_1 - a_3 = 0.298 \pm 0.017 \qquad a_1 + 2a_3 = 0.026 \pm 0.043 \tag{6}$$

$$a_{13} - a_{33} - a_{11} + a_{31} = -0.184 \pm 0.010$$
$$a_{13} - a_{33} + a_{11} - a_{31} = -0.277 \pm 0.078$$
$$a_{13} + 2a_{33} - a_{11} - 2a_{31} = 0.562 \pm 0.012$$
$$a_{13} + 2a_{33} + a_{11} + 2a_{31} = 0.10 \pm 0.12.$$

We would like to stress that the πN coupling constant did not play any role in calculating the scattering lengths (because we worked only with Q_n, $n \geq 2$ for amplitudes having a pole) — but, of course, not vice versa. The values of the πN coupling constant corresponding to the set of scattering lengths (6) are $f^2 = 0.0798 \pm 0.0010$ (from the amplitude B^+/ν) and $f^2 = 0.0837 \pm 0.0022$ (from the amplitude F^-/ν).

Keeping the smoothed error $\varepsilon(x)$ on its "self-consistent" solution and minimizing with respect to the scattering lengths themselves and to the crossing even and odd combinations of P wave scattering lengths we obtained the following estimates

$$a_1 = 0.208 \pm 0.020 \qquad a_3 = -0.091 \pm 0.017$$
$$a_{11} = -0.109 \pm 0.035 \qquad a_{31} = -0.063 \pm 0.022$$
$$a_{13} = -0.045 \pm 0.035 \qquad a_{33} = 0.186 \pm 0.022$$

$$a_{11} - a_{31} = -0.046 \pm 0.040$$
$$a_{13} - a_{33} = -0.231 \pm 0.040$$
$$a_{11} + 2a_{31} = -0.235 \pm 0.060$$
$$a_{13} + 2a_{33} = 0.327 \pm 0.060.$$

The minimized function turned out to be almost insensitive to the values of the parameters r_l in the effective range expansion (2), thus we kept them fixed at zero value during all calculations.

IV. COMMENTS

The combinations of S and P wave scattering lengths (6) evaluated by our method are consistent with the "recommended values" (5) of Ref. [1] within the errors. The combinations $a_{13} - a_{33} + a_{11} - a_{31}$ and $a_{13} + a_{11} + 2a_{33} + 2a_{31}$ are determined with much less accuracy in our approach due to the small dependence of low energy forward amplitudes on them. As a consequence, the crossing odd and even combinations of the P wave scattering lengths are also determined less accurately. The results would have been probably better if we had worked with fixed t amplitudes for $t \neq 0$.

Our values are also compatible with the results of two recent works [12, 13].

It should be stressed that our errors reflect only statistical errors of the experimental data. Neither scattering lengths nor their errors have been corrected for some (probably important) effects (see the detailed discussion in [14]). Here the situation is far from being clear and all calculations of πN low energy parameters are unfavourably influenced by systematic uncertainties which seem to be larger than the pure statistical errors arising from the experimental data.

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