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NOTE ON THE STRUCTURE OF QEANTAL PROPOSITION SYSTEM

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It is shown that the motivation of the axioms of atomicity and of the covering law in the set L of all propositions, analogical with Jauch's and Piron's motivation in [3], is possible also in the probabilistic formulation. The new definition of state, mentioned in [3], reformulates to the statement that the state (pure) is unambiguously defined by the set $S_f = \{a \in L; f(a) = 1\}$. The introduction of the notion of the support of the state and the ascribing of atomic supports to the pure states enables us to explain the role of the dispersionfree states in the definition of the compatibility of propositions and in the determination of the partial ordering in L.

I. INTRODUCTION

The set L of all propositions of a physical system has according to Jauch [1] and Piron [2] the following properties: *i)* L is a partially ordered set; *ii)* for every proposition $a \in L$ there exists an orthocomplement $a' \in L$; *iii)* L is a complete lattice; *iv)* L is a weakly modular lattice; *v)* L is an atomic lattice; *vi)* for every proposition $b \in L$ and any atom $e \in L$, $b \leq x \leq e \vee b$ implies either $x = b$ or $x = e \vee b$ (the covering law).

While the properties *i)*, *ii)*, *iv)* can be easily derived from the natural physical properties of the system, it is difficult to find a natural physical motivation of the properties *iii)*, *v)* and *vi)*. Jauch and Piron [3] tried to justify these properties on the basis of a new definition of the state, not involving any probability statements. The state is defined as the subset of all true propositions in L. This definition is an analog of the classical notion of state and corresponds to the way in which the state is usually prepared.

The aim of this paper is to show that arguments for the motivation of *v)* and *vi)*, similar to that used in [3], can be found also in the probabilistic formulation. Some simple consequences of the Jauch's and Piron's new definition of the state are also mentioned.

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II. AXIOMS v) AND vi)

We will assume that the axioms *i)* through *iv)* are satisfied. In the probabilistic formulation the state is defined as the probability measure on L, that is, as the function $f: L \to R$, were R is the real line, with the following properties:

1. $0 \leq f(a) \leq 1$ for every $a \in L$;
2. $f(0) = 0, f(1) = 1$;
3. if $a_1, a_2, \ldots$ is the countable set of pairwise mutually disjoint (that is $a_1 \leq a_j'$ for $i \neq j$) elements of L, then

$$f(V_j a_j) = \sum_j f(a_j).$$

Each theory based on the experiment must involve additional assumptions

4. $a = b \langle = \rangle f(a) = f(b)$ for all f;
5. $a \leq b \langle = \rangle f(a) \leq f(b)$ for all f.

We shall further assume, as in [3], that:

6. $f(a_i) = 1$, for all $i \in T \Rightarrow f(\bigwedge_{i \in T} a_i) = 1$;
7. To every $a \in L$, there exists a state f such that $f(a) = 1$.

From these properties it follows

$$f(a') = 1 - f(a) \text{ for every } f; \tag{1}$$

and by this relation the orthocomplement a' to a is uniquely defined.

Jauch and Piron [3] defined the state as a set:

$$S = \{a \in L; a \text{ is true}\}. \tag{2}$$

In the probabilistic formulation proposition a is true in the state f if $f(a) = 1$. Let F be the set of all states. To every $f \in F$ let us define the set

$$S_f = \{a \in L; f(a) = 1\}. \tag{3}$$

The set S_f has evidently the following properties:

1. $a \in S_f$, $a \leq b$ implies $b \in S_f$,
2. $a \in S_f$, $b \in S_f$ implies $a \wedge b \in S_f$.

The element $\bigwedge_{a \in S_f} a$ is called the support of the state f, supp f. The set S_f can be expressed in the form

$$S_f = \{a \in L; \text{supp} f \leq a\}. \tag{4}$$

We will investigate under what conditions the set S_f uniquely defines the state f. Let f be a mixed state, that is

$$f = \lambda f_1 + (1 - \lambda) f_2; f_1, f_2 \in F, 0 < \lambda < 1. \tag{5}$$

If $a \in S_f \Rightarrow f(a) = 1 \Rightarrow f(a') = 0 \Rightarrow \lambda f_1(a') + (1 - \lambda) f_2(a') = 0 \Rightarrow f_1(a') =$

$= f_2(a') = 0 \Rightarrow f_1(a) = 1, f_2(a) = 1 \Rightarrow a \in S_{f_1}, a \in S_{f_2}$, that is $S_f \subseteq S_{f_1} \cap S_{f_2}$. Conversely, if $a \in S_{f_1}$ and $a \in S_{f_2}$, then $\lambda f_1(a) + (1-\lambda) f_2(a) = f(a) = 1 \Rightarrow a \in S_f$.
Thus we have

$$S_f = S_{f_1} \cap S_{f_2}. \tag{6}$$

From (6) can be derived the following assertion about the supports a_f, a_{f_1}, a_{f_2} of the states f, f_1 and f_2, respectively,

$$a_f = a_{f_1} \vee a_{f_2}. \tag{7}$$

Indeed, by (4) $a_f \leq a \Longleftrightarrow a \in S_f$, by (6) this is equivalent to $a \in S_{f_1}$ and $a \in S_{f_2} \Longleftrightarrow a_{f_1} \leq a$, $a_{f_2} \leq a \Longleftrightarrow a_{f_1} \vee a_{f_2} \leq a$, that is $a_f = a_{f_1} \vee a_{f_2}$.

Proposition a_f in (7) is the support of the mixture of states f_1 and f_2 for every λ. The mixed state is then not uniquely determined by the set S_f.

Now let us consider the pure state f_0 with the support a_{f_0}. If there exists $b \in L$, $b \leq a_{f_0}$, then the state f_1, in which $f_1(b) = 1$, has the support $a_{f_1} \leq b$. But the state

$$f = \lambda f_0 + (1-\lambda) f_1, \quad 0 < \lambda < 1,$$

has the support

$a_f = a_{f_0} \vee a_{f_1} = a_{f_0}$, because $a_{f_1} \leq b \leq a_{f_0}$.

The states f and f_0 have then the same support, and consequently the state f_0 is not uniquely determined by the set S_{f_0}. For the set S_{f_0} uniquely to define the state f_0, a_{f0} must be an atom in L, that is, there must exist a one to one correspondence between the pure states in F and the atoms in L. A similar situation is in the standard logics, that is in the proposition systems consisting of all closed subspaces of the separable Hilbert space [7].

Let us consider now the axiom *vi*). It is shown in [3] that the covering law $x \leq y \leq x \vee q$, $x, y, q \in L$, q is an atom $\Rightarrow$ imply

$$\text{either } y = x \text{ or } y = x \vee q, \tag{8}$$

is equivalent to the property

element $(q \vee a') \wedge a$, $q, a \in L$, q is an atom, is an atom in L. (9)

For the motivation analogical to that in [3], let us define operations on the set F, that is to every $a \in L$ let us assign the transformation T_a from F into F [4, 9], where

$$T_a f = f(a) f_{[a]}, \; f_{[a]} \in F_a^1 \text{ and } F_a^1 = \{f \in F; f(a) = 1\}. \tag{10}$$

The operation T_a can be interpreted as the change of the state which occurs by the measurement of the proposition a. In the case of the result of the experiment being 1, the original state f changes into the state $f_{[a]}$. This is in

agreement with the von Neumann postulate of repetition [8, p. 177, postulate (M)]. Concerning operations T_a we shall assume:

1. T_a, $a \in L$ is a pure operation, that is if f is a pure state, then the final state $f_{[a]}$ will be also pure.
2. $a \leq b \Rightarrow T_a T_b = T_b T_a = T_a$.

Assumption 1. can be similarly as in [3], motivated so that it is the expression of the requirement that no loss of information should occur during the measu rement; the pure state provides us namely with the maximal possible information about the system.

Assumption 2. is motivated by the case of the classical system: axiom *iv*) namely implies that a and b are compatible if $a \leq b$.

We shall show, similarly as in [6], that if

$$T_b f = f(b) f_{[b]}; \; f_{[b]} \in F_b^1, \; f(b) \neq 0 \text{ and} \tag{11}$$

$\operatorname{supp} f = a$, then

$$\beta = \operatorname{supp} f_{[b]} \leq (a \vee b') \wedge b. \tag{12}$$

Indeed, $(a \vee b')' = a' \wedge b \leq b$, therefore by the property 2 of the operation T_a we have
$T_{a' \wedge b} T_b f = f(b) f_\beta(a' \wedge b) f_{[a' \wedge b]} = T_{a' \wedge b} f = f(a' \wedge b) f_{[a' \wedge b]}$,
that is

$f_\beta(a' \wedge b) = \dfrac{f(a' \wedge b)}{f(b)}$. But $a' \wedge b \leq a'$, $f(a') = 0 \Rightarrow f(a' \wedge b) = 0$, therefore $f_\beta(a \vee b') = 1 \Rightarrow \beta \leq a \vee b'$. Because $\beta \leq b$, we have $\beta \leq (a \vee b') \wedge b$.

Let f be a pure state. Then a is an atom. By the property 1 of the operations T_a, β is an atom, too. For the operations T_a we can prove that if $a, b \in L$ and a is compatible with b, $a \leftrightarrow b$, then

$$T_a T_b = T_b T_a = T_{a \wedge b} \text{ [4], [5].} \tag{13}$$

The elements $a, b \in L$ are compatible iff there exists $a_1, b_1, c \in L$ pairvise mutually disjoint and such that

$$a = a_1 \vee c, \quad b = b_1 \vee c. \tag{14}$$

Because $\beta \leq b$, by *iv*) $\beta \leftrightarrow b$. Then also $\beta' \leftrightarrow b$. By (13)
$T_{\beta'} T_b f = f(b) f_\beta(\beta') f_{[\beta']} = 0 = T_{\beta' \wedge b} f = f(\beta' \wedge b) f_{[\beta' \wedge b]}$,
that is $f(\beta' \wedge b) = 0 \Rightarrow a \leq (\beta' \wedge b)' = b' \vee \beta$. Therefore we have

$$a \leq b' \vee \beta,\; b' \leq b' \vee \beta \Rightarrow a \vee b' \leq \beta \vee b' \Rightarrow (a \vee b') \wedge b \leq (\beta \vee b') \wedge b = \beta. \tag{15}$$

The last equality follows from *iv*):
$\beta \leq b \Rightarrow b = \beta \vee (b \wedge \beta') \Rightarrow \beta = b - (b \wedge \beta') = b \wedge (b \wedge \beta')' = b \wedge (b' \vee \beta)$.

From (12) and (15) it follows that $\beta = (a \vee b') \wedge b$, and β is an atom in L if a is an atom, that is, the covering law holds.

III. THE NOTION OF THE COMPATIBILITY OF PROPOSITIONS

We shall show that for the compatible propositions a, b in L there exists a common set of dispersionfree pure states in F, which is complete on the Boolean sublattice of L, generated by a, b. We recall, that the set P of states is complete on some set M of propositions, if for each two propositions $x, y \in M$, $x \neq y$; there exists a state $f \in P$, for which $f(x) \neq f(y)$.

Let us write

$$F_a^{1p} = \{f \in F; f \text{ pure}; f(a) = 1\},$$

$$F_a^{0p} = \{f \in F; f \text{ pure}; f(a) = 0\},\ a \in L. \tag{16}$$

Then the set of all dispersionfree pure states of a is

$$F_a^p = F_a^{1p} \cup F_a^{0p}. \tag{17}$$

Let $a, b \in L$, $a \leftrightarrow b$. By (14) is $a = a_1 \vee c$; $b = b_1 \vee c$; a_1, b_1, c are pairwise disjoint and $a_1 = a \wedge b'$, $b_1 = a' \wedge b$, $c = a \wedge b$. It is known [1, 2, 7] that $a \leftrightarrow b \Rightarrow a \leftrightarrow b'$;

$$a = (a \wedge b) \vee (a \wedge b') = c \vee a_1, \tag{18}$$

$$b' = (a' \wedge b') \vee (a \wedge b') = d \vee a_1,$$

where $d = a' \wedge b'$, a_1, b_1, c, d are pairwise disjoint.

Let us consider the following set of pure states from F:

$$S = \{f; \operatorname{supp} f \leq a_1\} \cup \{f; \operatorname{supp} f \leq b_1\} \cup \{f; \operatorname{supp} f \leq c\} \cup$$

$$\cup \{f; \operatorname{supp} f \leq d\},\ f \in F, f \text{ pure}. \tag{19}$$

The set S is complete for the Boolean lattice generated by (a, b), which is composed from the propositions $(a, b, a', b', a \vee b, a \wedge b, a \wedge b', \ldots)$ because for each two propositions there exists $f \in S$ with the different probabilities.

It is clear, that

$$S \subset F_a^p \text{ and } S \subset F_b^p. \tag{20}$$

Indeed, if $f \in S$, supp $f \leq c$, then supp $f \leq a$ and supp $f \leq b$, that is $f \in F_a^{1p}$, $f \in F_b^{1p}$; if supp $f \leq d \Rightarrow$ supp $f \leq a'$, supp $f \leq b' \Rightarrow f(a) = 0$, $f(b) = 0$, and $f \in F_a^{0p}$, $f \in F_b^{0p}$. If supp $f \leq a_1 \Rightarrow$ supp $f \leq a$, supp $f \leq b'$ and $f \in F_a^{1p}$ and $f \in F_b^{0p}$. Analogically, supp $f \leq b_1 \Rightarrow f \in F_a^{0p}$, $f \in F_b^{1p}$.

On the other hand, if the sublattice generated by (a, b) has a common set

of dispersionfree pure states, it is a Boolean lattice (that is, $a \leftrightarrow b$) [1]. We can give another proof in our special case. Let S be the complete set. Then for each $f \in S$

$$f(a) = 1 \Rightarrow f(b) = 1 \text{ or } f(b) = 0,$$

$$f(b) = 1 \Rightarrow f(a) = 1 \text{ or } f(a) = 0. \tag{21}$$

Let us denote supp $f = q$, q is an atom in L. Then (21) can be rewritten in the form

$$q \leq a \Rightarrow q \leq b \text{ or } q \leq b',$$

$$q \leq b \Rightarrow q \leq a \text{ or } q \leq a'. \tag{22}$$

Let us write

$$c = \bigvee_{\{q;\, q \leq a,\, q \leq b\}} q, \tag{23}$$

$$a_1 = \bigvee_{\{q;\, q \leq a,\, q \leq b'\}} q, \qquad b_1 = \bigvee_{\{q;\, q \leq a',\, q \leq b\}} q,$$

where $q =$ supp f and $f \in S$.

Then a_1, b_1 and c are paerwise disjoint. Evidently, $a_1 \vee c \leq a$; $b_1 \vee c \leq b$.

Then for $f \in S$; $f(a) = 0 \Rightarrow f(a_1 \vee c) = 0$, $f(b) = 0 \Rightarrow f(b_1 \vee c) = 0$. But if $f \in S$, $f(a) = 1$, then by (22), (23) supp $f \leq c$ or supp $f \leq a_1$; in both cases $f(a_1 \vee c) = f(a_1) + f(c) = 1$. We see, that for each $f \in S$ is $f(a_1 \vee c) = f(a)$, $f(b_1) \vee c) = f(b)$, and because the set S is complete, we have $a_1 \vee c = a$, $b_1 \vee c = b$; that is $a \leftrightarrow b$.

Further we can show, that the sets F_a^{1p}, $a \in L$, determine the partial ordering in L, that is

$$F_a^{1p} \leq F_b^{1p} \Leftrightarrow a \leq b,\ a, b \in L. \tag{24}$$

Indeed, $f \in F_a^{1p}$, iff $q =$ supp $f \leq a$. By the assumption, it is also $q \leq b$. But then $\bigvee_{\{q \leq a,\, a \text{ atom}\}} q \leq b$, that is $a \leq b$,

because we can show, that for each $a \in L$:

$$a = \bigvee_{\{p \in M_a\}} p, \text{ where}$$

$$M_a = \{p \text{ is an atom in } L;\ p \leq a\}.$$

Indeed,

$$\bigvee_{\{p \in M_a\}} p \leq a \Rightarrow a = (\bigvee_{\{p \in M_a\}} p) \vee (a \wedge (\bigvee_{\{p \in M_a\}} p)').$$

If $a \wedge (\bigvee_{\{p \in M_a\}} p)' \neq 0$, then by $v)$ there should exist an atom $q \in L$, $q \leq a \wedge (\bigvee_{\{p \in M_a\}} p)' \Rightarrow q \leq a, q \leq (\bigvee_{\{p \in M_a\}} p)'$, which is in contradiction with the definition of $\bigvee_{\{p \in M_a\}} p$. That is $\bigvee_{\{p \in M_a\}} p = a$. The converse implication in (24) is evident from the definition of the sets F_x^{1p}, $x \in L$.

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