acta phys. slov. 25 (1975), No. 1

NOTES ON THE TIME DEVELOPMENT OF CLASSICAL QUANTITIES

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A physical system is described by a C^*-algebra of observables $\mathfrak{A}$ and, in a certain GNS-representation $\pi_\omega(\mathfrak{A})$, by a one-parameter group of automorphisms α_t of the weak closure $\pi_\omega(\mathfrak{A})''$ of $\pi_\omega(\mathfrak{A})$ (α_t is the group of time development). A C^*-subalgebra $\mathfrak{M}$ ($\alpha_t\mathfrak{M} = \mathfrak{M}$) of the center $\mathfrak{Z}$ of π_ω $(\mathfrak{A})''$ is interpreted as the algebra of „macroscopic observables" (with respect to the given family of states) of the quantum system $\mathfrak{A}$. If ω, when restricted to the $\mathfrak{M}$, is an α_t-invariant state, the time development of „locally perturbed" states $\omega_x(x \in \mathfrak{A})$ on $\mathfrak{M}$ can be described by a one-parameter group of unitary operators U^t in a ceratin subspace of $\mathscr{H}_\omega$. If $U^t_{\mathfrak{M}} = \exp(-itH_{\mathfrak{M}})$ (strong continuity), the spectral properties of the generator $H_{\mathfrak{M}}$ determine the behaviour of $\omega_x \circ \alpha_t \in \mathscr{S}(\mathfrak{M})$ (= states on $\mathfrak{M}$) for $t \to \infty$. It is shown, e.g., that the nontriviality of $\alpha_t \in \mathrm{aut}\, \mathfrak{M}$ requires that the spectrum of $H_{\mathfrak{M}}$ is two-sidedly unbounded. Further conditions of nontriviality of α_t (on $\mathfrak{M}$) are given. Exact connections of „the rate of decay" ($t \to \infty$) of $\omega_x \circ \alpha_t$ with spectral properties of $H_{\mathfrak{M}}$ are derived. Results are illustrated by simple examples of classical mechanics. The concluding remark is devoted to the the application of the formalism to the quantum theory of measurement.

I. INTRODUCTION

In the quantum mechanical description of systems with infinitely many degrees of freedom it is useful to identify observables of a system with elements of an abstract C^*-algebra $\mathfrak{A}$ (i.e. a Banach symmetric algebra with a norm fulfilling $\|x^*x\| = \|x\|^2$ for all $x \in \mathfrak{A}$). We shall assume that $\mathfrak{A}$ has an identity. Since every C^*-algebra has a faithful *-representation in the algebra $\mathfrak{B}(\mathscr{H})$ of all bounded operators in some Hilbert space $\mathscr{H}$, the C^*-algebraic approach is a generalization of the conventional quantum mechanics. States of the system are described by the set of all positive normalized linear functionals on $\mathfrak{A}$, $\mathscr{S}(\mathfrak{A})$, i.e., the following conditions are satisfied:

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$$\omega \in \mathscr{S}(\mathfrak{A}) \Rightarrow \quad (i)\ \omega(x^*x) \geqslant 0 \quad (x \in \mathfrak{A}); \tag{1}$$

$$(ii)\ \omega(x + \lambda y) = \omega(x) + \lambda\omega(y) \quad (x, y \in \mathfrak{A}, \lambda \in \mathbf{C});$$

$$(iii)\ \omega(e) = 1 \quad (e (\in \mathfrak{A}) \text{ is the identity of } \mathfrak{A}.$$

(We denote by **C** the set of complex numbers and by **R** the reals). If $x^* = x$, $\omega \in \mathscr{S}(\mathfrak{A})$, then $\omega(x) \in \mathbf{R}$ and the number $\omega(x)$ is interpreted as the mean value of the observable $x \in \mathfrak{A}$ in the state ω. Each state $\omega \in \mathscr{S}(\mathfrak{A})$ generates canonically a cyclic *-representation $\pi_\omega(\mathfrak{A})$ of the algebra $\mathfrak{A}$ in a Hilbert space $\mathscr{H}_\omega$, called the *GNS*-representation (due to Gelfand, Najmark and Segal) and denoted by $(\mathscr{H}_\omega, \pi_\omega, \xi_\omega)$. The cyclic vector $\xi_\omega \in \mathscr{H}_\omega$ (i.e., $\pi_\omega(\mathfrak{A})\xi_\omega$ is norm-dense in $\mathscr{H}_\omega$) is such that

$$\omega(x) = (\xi_\omega, \pi_\omega(x)\xi_\omega) \qquad \text{for all } x \in \mathfrak{A}. \tag{2}$$

The property (2) determines the cyclic representation uniquely (up to the unitary equivalence).

An arbitrary normed vector $\xi \in \mathscr{H}_\omega$ defines a vector state $\omega_\xi(x) \equiv (\xi, \pi_\omega \equiv x)\ \xi)$, $\omega_\xi \in \mathscr{S}(\mathfrak{A})$, in the representation $(\mathscr{H}_\omega, \pi_\omega, \xi_\omega)$. With some freedom of expression we can consider ω_ξ as a state on $\mathfrak{B}(\mathscr{H}_\omega)$, $\omega_\xi(B) \equiv (\xi, B\xi)$ for all $B \in \mathfrak{B}(\mathscr{H})_\omega)$. As a consequence of the finite precision in experiments we are not able to distinguish experimentally quantities in $\pi_\omega(\mathfrak{A})$ (by measurements in given states ω_ξ, $\xi \in \mathscr{H}_\xi$) from other ,,weakly-infinitely nearby" quantities described by operators in the operator-weak closure of $\pi_\omega(\mathfrak{A})$ in $\mathfrak{B}(\mathscr{H}_\omega)$. This leads us to define the "observables in the representation $(\mathscr{H}_\omega, \pi_\omega, \xi_\omega)$" by all selfadjoint operators of the von Neumann algebra (or the W^*-algebra) $\pi_\omega(\mathfrak{A})$. (We denote by $\mathfrak{N}'$ the commutant and by $\mathfrak{N}''$ the bicommutant of $\mathfrak{N} \subset \mathfrak{B}(\mathscr{H})$, in $\mathfrak{B}(\mathscr{H})$.) Similar considerations for states generate the concept of the "physical equivalence" of representations (compare [1]). The center $\mathfrak{Z}_\omega \equiv \pi_\omega(\mathfrak{A})'' \cap \pi_\omega(\mathfrak{A})'$ of $\pi_\omega(\mathfrak{A})''$ is a commutative W^*-algebra containing all such observables (in the representation π_ω) which commute with all other observables. The observables in $\mathfrak{Z}_\omega$ are "classical observables" of the system $\mathfrak{A}$ in the representation π_ω (or, equivalently, in the set of all states in $\overline{\text{co}\ \{\omega_\xi \mid \xi \in \mathscr{H}_\omega\}}$, where the closure of the convex hull is taken in the norm-topology of the dual space $\mathfrak{A}^*$ of $\mathfrak{A}$, with the norm $\|f\| = \sup_{\|x\|=1,\, x \in \mathfrak{A}} |f(x)|$ for $f \in \mathfrak{A}^*$). If $\pi_\omega(\mathfrak{A})$ is an irreducible representation, then $\pi_\omega(\mathfrak{A})'' = \mathfrak{B}(\mathscr{H}_\omega)$ and the centre $\mathfrak{Z}_\omega = \{\lambda 1_{\mathscr{H}_\omega} \mid \lambda \in \mathbf{C}\}$ is trivial. One can see from this, that in physical systems with finite number of degrees of freedom there are no nontrivial classical observables, which is the case of the conventional non-relativistic quantum mechanics.

The algebra of observables is usually constructed (in the case of an infinite number of degrees of freedom) as the C^*-inductive limit of the net of algebras describing finite systems (see e.g. [5, 11]). If for each $\alpha \in I$ (I is a directed index set) there is an algebra $\mathfrak{A}_\alpha$ describing a finite system and $\mathfrak{A}_\alpha \subset \mathfrak{A}_\beta$ for $\alpha < \beta$, $1_\alpha = 1_\beta$ (1_α is an identity of $\mathfrak{A}_\alpha$) for all $\alpha, \beta \in I$, then the C^*-inductive limit $\mathfrak{A}$ of the net $\{\mathfrak{A}_\alpha, \alpha \in I\}$ may be considered as the norm-closure of the union of all $\mathfrak{A}_\alpha$:

$$\mathfrak{A} = \overline{\bigcup_{\alpha \in I} \mathfrak{A}_\alpha}, \quad \mathfrak{A}_\alpha = \text{"local algebras"} \ (\alpha \in I) \tag{3}$$

Thus e.g. $\mathfrak{A}_\alpha$ describes a gas of the particles with hard cores in the finite volume V_α and $V_\alpha \subset V_\beta$ iff $\alpha < \beta$. The algebra $\mathfrak{A}$ constructed via (3) from the "local algebras" $\mathfrak{A}_\alpha$ is called the algebra of quasi-local observables. The time development of elements in $\mathfrak{A}$ is then obtained by a certain limiting procedure from the known time-developments in $\mathfrak{A}_\alpha$ ($\alpha \in I$) (see, e.g., [2]). The obtained development of $\mathfrak{A}$ need not be an automorphism group of $\mathfrak{A}$ even if the time development in each $\mathfrak{A}_\alpha$ is described by a oneparameter group of *-automorphisms. We can describe in many cases the time development of $\mathfrak{A}$ in some representation $\pi_\omega(\mathfrak{A})$ by a group of automorphisms of the W^*-algebra $\pi_\omega(\mathfrak{A})''$, [2]. In the latter case the time development of the vector state ω_ξ (if extended to $\pi_\omega(\mathfrak{A})''$) is

$$\omega_{\xi t}(x) \equiv (\xi, \alpha_t^{(\omega)} \circ \pi_\omega(x)\ \xi), \qquad \xi \in \mathscr{H}_\omega,\ x \in \mathfrak{A}, \tag{4}$$

where $\alpha_t^{(\omega)} \in \text{aut}\ \pi_\omega(\mathfrak{A})''$ for $t \in \mathbf{R}$, $\alpha_t^{(\omega)} \circ \alpha_{t'}^{(\omega)} = \alpha_{t+t'}^{(\omega)}$.

The vector states ω_ξ with $\xi \equiv \pi_\omega(x)\xi_\omega (x \in \mathfrak{A})$ are called "local perturbations of ω" and we denote them by ω_x:

$$\omega_x(y) \equiv (\pi_\omega(x)\xi_\omega, \pi_\omega(yx)\xi_\omega), \ (\|\pi_\omega(x)\xi_\omega\| = 1,\ x, y \in \mathfrak{A}). \tag{5}$$

The name "local perturbation" for ω_x comes from the case of the quasilocal algebra $\mathfrak{A}$ (3).

The state ω is t-invariant if $\omega_t \equiv \omega \circ \alpha_t = \omega$ for all $t \in \mathbf{R}$. The equilibrium states in the statistical physics constructed on the basis of the ergodic hypothesis by the time averaging are t-invariant. However, macroscopic quantities are mainly interesting from the thermodynamical point of view (or, more generally, from the point of view of some useful description of global characteristics of big systems). Since the relative fluctuations of (almost) all physically significant macroscopic quantities tend to zero in the thermodynamical limit of big systems, it is natural to suppose that bounded macroscopic quantities are constant in time in the equilibrium state of a big system. Such a characterization of the equilibrium is less dependent on the ergodic hypothesis than the usual one is. If the subalgebra $\mathfrak{M} (\subset \mathfrak{A})$ of the "macroscopic quantities" is known (it might be dependent on the representation π_ω and, in that case, $\mathfrak{M} \subset \pi_\omega(\mathfrak{A})''$) and if a time-developed macroscopic quantity is again a macroscopic one: $\alpha_t \mathfrak{M} = \mathfrak{M}$, we can define the macroscopically ($\mathfrak{M}$) — t-invariant state $\omega \in \mathscr{S}(\mathfrak{A})$:

$$\omega \in \mathscr{S}(\mathfrak{A}) \text{ is } \mathfrak{M}\text{-}t\text{-invariant iff } \omega(\alpha_t x) = \omega(x) \text{ for all } x \in \mathfrak{M}, \quad t \in \mathfrak{R}. \tag{6}$$

Definition (6) of the macroscopic t-invariance of the state seems to be useful from the point of view of the physical kinetics: the limit of $\omega_t(x)$ for $t \to \infty$ need not exist for all $x \in \mathfrak{A}$ (resp. for all $x \in \pi_\omega(\mathfrak{A})''$) but it might exist for $x \in \mathfrak{M}$: $\lim_{t\to\infty} \omega_t = \bar{\omega} \in \mathscr{S}(\mathfrak{M})$. The existence of the state $\bar{\omega}$ on $\mathfrak{M}$ is sufficient to determine macroscopic properties of the system in the limit $t \to \infty$.

In the present paper we are intersted in the existence of the limits ω and in the speed of convergence of $\bar{\omega}_t \to \bar{\omega}$ ($t \to \infty$) for "partial states" on $\mathfrak{M}$ (for the notion of partial states and for a relevant discusion see [19]). The state $\omega_{t=0}$ is supposed to be a local perturbation of some $\mathfrak{M}$-t-invariant state ω_{ξ_0}. We shall work in a fixed cyclic representation ($\mathscr{H}$, π, ξ_0) of $\mathfrak{A}$ all the time. Thus, we can (and we shall) write $\mathfrak{A}$ instead of $\pi(\mathfrak{A})$, so that $\mathfrak{A} \subset \mathfrak{B}(\mathscr{H})$ and the center $\mathfrak{Z}_\pi \equiv \mathfrak{Z} = \mathfrak{A}' \cap \mathfrak{A}''$ (commutants $\mathfrak{A}'$ and $\mathfrak{A}'' \equiv (\mathfrak{A}')'$ are taken in $\mathfrak{B}(\mathscr{H})$). We shall further assume that $\mathfrak{M} \subset \mathfrak{Z}$ ($\mathfrak{M}$ is a C^*-algebra), i.e., that "macroscopic quantities" are classical ones commuting with all observables. It is known, that in infinite systems described by quasilocal algebras macroscopic quantities belong to the "observables at infinity" [3], which in turn belong to the center $\mathfrak{Z}$ [4]. The assumption $\alpha_t \mathfrak{M} = \mathfrak{M}$ is both natural and useful for further considerations[1]. Now we arrive at the problem of the convergence of states $\omega \circ \alpha_t \equiv \omega_t$ (for $t \to \infty$) of the classical system described by the commutative C^*-algebra $\mathfrak{M}$ and the one-parameter group of *-automorphisms $\alpha_t \in \operatorname{aut} \mathfrak{M}$ (all the automorphisms dealt with further on are *-automorphisms).

In Sec. II. a connection is formulated between the time development in $\mathfrak{A}''$ and that in $\mathfrak{M}$. We shall derive there some conditions for the existence of $\lim_{t\to\infty} \omega_t \in \mathscr{S}(\mathfrak{M})$. In the proposition II. 5. it is shown that the group $\alpha_t \in \operatorname{aut} \mathfrak{M}$ cannot be "too continuous" (we mean here the continuity of $t \to \omega(\alpha_t x)$ for all $x \in \mathfrak{M}''$ and all $\omega \in \mathscr{S}(\mathfrak{M}'')$) if it should not be trivial (i.e. $\alpha_t \equiv 1$). Further conditions on α_t are formulated in terms of spectral properties of the generator of time development if α_t is unitarily implemented.

Sec. III. contains a list and a brief discussion of some further conditions of the nontriviality of α_t. The proposition III. 1. is interesting from the point of view of the usual interpretation of the generator of time development as the energy operator: for a nontrivial α_t the spectrum of the generator cannot be bounded either from below, or from above. The next almost trivial example taken from the classical mechanics shows that the generator is not Hamiltonian, but a Liouville operator. The then following proposition shows that the nontriviality of α_t presupposes $\|\alpha_t - 1\| = 2$. For the main parts of proofs in this Section we refer the reader to the literature.

The relation between the speed of the convergence $\omega_t \to \bar{\omega}$ (for $t \to \infty$) and the spectral properties of the generator of time development is investigated in Sec. IV. Speaking loosely, the better the analytic properties of the spectral measure of the generator are, the faster is the convergence. An example is given of a freely moving classical particle on a finite closed curve.

The concluding note (Sec. V.) deals with the possible application of the formalism to the quantum theory of measurement.

[1] In the following we denote restrictions of mappings α_t, ω, ... by the same symbols as the respective mappings defined on a bigger algebra.

II. THE MACROSCOPIC SUBSYSTEM OF A QUANTUM SYSTEM

The interpretation of the subsequent considerations ought to be understood according to the previous Section. Let $\mathfrak{A}$ be a C^*-algebra of operators of a Hilbert space $\mathscr{H}$, $\mathfrak{A} \subset \mathfrak{B}(\mathscr{H})$, with a cyclic vector $\xi_0 \in \mathscr{H}$, i.e., $\overline{\mathfrak{A}\xi_0} = \mathscr{H}$. Let $\mathfrak{M} \subset \mathfrak{Z}$ ($\mathfrak{Z} \equiv \mathfrak{A}' \cap \mathfrak{A}''$) be a C^*-subalgebra of $\mathfrak{A}''$, $1_{\mathscr{H}} \in \mathfrak{M} (\mathfrak{A} \ni 1_{\mathscr{H}})$. Suppose that $\alpha_t \in \operatorname{aut} \mathfrak{A}''$ is such a one-parameter group that

(*a*) $\mathfrak{M}$ is α_t-invariant: $\alpha_t A \in \mathfrak{M}$, $\forall t \in \mathbf{R}$, $\forall A \in \mathfrak{M}$;

(*b*) the state $\omega(x) \equiv (\xi_0, x\xi_0)$ ($x \in \mathfrak{A}''$) is $\mathfrak{M}$-t-invariant, i.e. $\omega(\alpha_t A) = \omega(A)$ for all $A \in \mathfrak{M}$ and all $t \in \mathbf{R}$;

(*c*) the functions $t \to (x\xi_0, (\alpha_t A)x\xi_0)$ are continuous mappings of $\mathbf{R}$ to $\mathbf{C}$ for all $x \in \mathfrak{A}$ and all $A \in \mathfrak{M}$.

The selfadjoint operators from $\mathfrak{M}$ are interpreted as "macroscopic quantities" of the system (this definition is, in general, representation-dependent). The local perturbations of ω are the states $\omega_x(y) \equiv (x\xi_0, yx\xi_0)$ (by $\|x\xi_0\| = 1$), for arbitrary $x \in \mathfrak{A}$. Let P be the projector on the subspace $\overline{\mathfrak{M}\xi_0} \equiv P\mathscr{H} \subset \mathscr{H}$ and $[A, B] = AB - BA$ ($A, B \in \mathfrak{B}(\mathscr{H})$).

II. 1. Lemma. $[P, A] = 0$ for all $A \in \mathfrak{M}$

Proof. The function $\xi \to A\xi$ ($\xi \in \mathscr{H}$) is continuous in $\mathscr{H}$, $P\mathscr{H}$ is closed; $\mathfrak{M}\xi_0$ is dense in $P\mathscr{H}$ (all in the norm-topology of $\mathscr{H}$) and $A\mathfrak{M} \subset \mathfrak{M}$ for $A \in \mathfrak{M}$. From this we have $PAP\xi = AP\xi$, $\forall \xi \in \mathscr{H}$, i.e., $PAP = AP$. From this we get for $A^* = A$ by conjugation the result, q.e.d.

Hence $P\mathfrak{M}P = P\mathfrak{M}$ is a*-representation of $\mathfrak{M}$ in $\mathscr{H}_{\mathfrak{M}} \equiv P\mathscr{H}$ with the cyclic vector ξ_0. Since $\mathfrak{Z} = \mathfrak{Z}''$, we have $\mathfrak{M}'' \subset \mathfrak{Z}$ and $P \in \mathfrak{M}'$ implies $\mathfrak{M}'' \subset \{P\}'$. Clearly $\overline{\mathfrak{M}''\xi_0} = \mathscr{H}_{\mathfrak{M}}$. It follows from this that $P\mathfrak{M}''$ is a cyclic representation of $\mathfrak{M}''$ in $\mathscr{H}_{\mathfrak{M}}$. The next lemma implies α_t-invariance of $\mathfrak{M}'' : \alpha_t \in \operatorname{aut} \mathfrak{M}''$.

II. 2. Lemma. Let $\mathfrak{N}$ be a cyclic (in general noncommutative) C^*-subalgebra of $\mathfrak{B}(\mathscr{H})$, $\mathfrak{N} \ni 1_{\mathscr{H}}$, with the cyclic vector $\xi_0 \in \mathscr{H}$ and $\alpha \in \operatorname{aut} \mathfrak{N}$ such that $(\xi_0, \alpha x \xi_0) = (\xi_0, x\xi_0)$ for all $x \in \mathfrak{N}$. Then there is a unique σ-continuous *-isomorphism $\bar{\alpha}$ of $\mathfrak{N}''$ into $\mathfrak{B}(\mathscr{H})$, the restriction of which to $\mathfrak{N}$ is α, $\bar{\alpha} \in \operatorname{aut} \mathfrak{N}''$ and $\bar{\alpha}$ is unitarily implementable: $\bar{\alpha}x = U_\alpha^* x U_\alpha$ $(U_\alpha^* U_\alpha = U_\alpha U_\alpha^* = 1_{\mathscr{H}}$, $x \in \mathfrak{N}'')$. (The σ-continuity means the continuity in $\sigma(\mathfrak{B}(\mathscr{H}), \mathfrak{B}(\mathscr{H})_*)$ topology, see [5—8]).

Proof. The isometric mapping $U_\alpha^* : \mathfrak{N}\xi_0 \mapsto \mathfrak{N}\xi_0$ defined by $U_\alpha^* x\xi_0 \equiv \alpha x \xi_0$ is linear and its extension to $\mathscr{H}$ is unitary. Then $\alpha x = U_\alpha^* x U_\alpha (x \in \mathfrak{N})$ and $U_\alpha \xi_0 = \xi_0$. The operator U_α defines a σ-continuous *-automorphism of $\mathfrak{B}(\mathscr{H})$. If $\bar{\alpha}x \equiv U_\alpha^* x U_\alpha$ $(x \in \mathfrak{N}'')$, then $\bar{\alpha}\mathfrak{N}''$ is a W^*-algebra containing $\mathfrak{N}$, so that $\mathfrak{N}'' \subset \bar{\alpha}\mathfrak{N}'' = U_\alpha^* \mathfrak{N}'' U_\alpha$ and $U_\alpha \mathfrak{N}'' U_\alpha \subset \mathfrak{N}''$. Since $U_{\alpha^{-1}} = U_\alpha^*$ and the previous considerations are equally valid for $\bar{\alpha}^{-1}$ (instead of $\bar{\alpha}$), we have also $\mathfrak{N}'' \subset \bar{\alpha}^{-1}\mathfrak{N}'' = U_\alpha \mathfrak{N}'' U_\alpha^*$, hence $\bar{\alpha}\mathfrak{N}'' = \mathfrak{N}''$ and $\bar{\alpha} \in \operatorname{aut} \mathfrak{N}''$. The uniqueness of $\bar{\alpha}$ follows from the σ-continuity and from the fact that $\mathfrak{N}$ is σ-dense in $\mathfrak{N}''$, q.e.d.

The restriction of the state $\omega(x) \equiv (\xi_0, x\xi_0)$ to $\mathfrak{M}''$ is an α_t-invariant state $\in \mathscr{S}(\mathfrak{M}'')$. The assumption (c) above implies that in $\mathscr{H}_{\mathfrak{M}}$, $\alpha_t \in \operatorname{aut} \mathfrak{M}$ is unitarily implementable by a weakly (equivalently strongly) continuous group of unitary operators $U_{\mathfrak{M}}^t : P\alpha_t A = U_{\mathfrak{M}}^{-t} A U_{\mathfrak{M}}^t$.

II. 3. Lemma. Let $\mathfrak{N}$ be a cyclic C^*-algebra in $\mathscr{H}$ with the cyclic vector $\xi_0 \in \mathscr{H}$ and $\alpha_t \in \operatorname{aut} \mathfrak{N}$ is such a one-parameter group that the functions $t \mapsto (x\xi_0, (\alpha_t y)x\xi_0)$, for all $x, y \in \mathfrak{N}$, are continuous and the state $\omega_{\xi_0}(x) \equiv (\xi_0, x\xi_0)$ is α_t-invariant on $\mathfrak{N}$. Then $\bar{\alpha}_t x = U^{-t} x U^t$ for all $x \in \mathfrak{N}''$, where $\bar{\alpha}_t$ is the extension of α_t to $\mathfrak{N}''$ according to II. 2. and U^t is weakly continuous: $U^t = \exp(-itH)$, $H^* = H \in \mathfrak{L}(\mathscr{H})$ ($\equiv$ the set of all linear operators in $\mathscr{H}$).

Proof. It suffices to prove the continuity of U^t. The functions $t \mapsto \|(U^{-t} - 1)x\xi_0\|^2 = \|(U^{-t}xU^t - x)\xi_0\|^2 = (\xi_0, \alpha_t(x^*x)\xi_0) - (\xi_0, (\alpha_t x^*)x\xi_0) + (\xi_0, x^*x\xi_0) - (\xi_0, x^*\alpha_t x\xi_0)$ are continuous (by polarization) for all $x \in \mathfrak{N}$, and $\mathfrak{N}\xi_0$ is dense in $\mathscr{H}$. Since $\|U^{-t} - 1\| \leqslant 2$, all functions $t \mapsto \|(U^t - 1)\xi\|$ (for all $\xi \in \mathscr{H}$) are continuous, and this means that U^t is strongly continuous, q.e.d.

Applying II. 3. to the algebra $P\mathfrak{M}$ in $\mathscr{H}_{\mathfrak{M}}$ we see that

$$\omega_x(\alpha_t A) = (x^*x\xi_0, U_{\mathfrak{M}}^{-t} A \xi_0) \qquad \text{for all } x \in \mathfrak{A} \text{ and all } A \in \mathfrak{M}'' \tag{7}$$

are continuous functions of t, $U_{\mathfrak{M}}^{-t} = \exp(itH_{\mathfrak{M}})$ and $H_{\mathfrak{M}}$ is a selfadjoint operator in $\mathscr{H}_{\mathfrak{M}}$. The group $\alpha_t \in \operatorname{aut} \mathfrak{A}''$ need not have such good continuity

properties $(\mathfrak{A}'' \in \mathfrak{B}(\mathscr{H}), \mathscr{H} \supset \mathscr{H}_{\mathfrak{M}})$. Since $U_{\mathfrak{M}}^t(1 - P) = 0$, $U_{\mathfrak{M}}^t$ is a group of partial isometries in $\mathscr{H}$, which is unitary in $\mathscr{H}_{\mathfrak{M}}$.

II. 4. Remark. Suppose $\alpha_t x = \mathfrak{U}^{-t} x \mathfrak{U}^t$ for all $x \in \mathfrak{A}''$, $t \in \mathbf{R}$. We have $P\mathfrak{U}^{-t} A \mathfrak{U}^t P = U_{\mathfrak{M}}^{-t} A U_{\mathfrak{M}}^t$ for all $A \in \mathfrak{M}''$. α_t-invariance of $\mathfrak{M}''$ implies $[P, \mathfrak{U}^t] = 0$, $\forall t \in \mathbf{R}$. Then $PA = \mathfrak{U}^t U_{\mathfrak{M}}^{-t} A U_{\mathfrak{M}}^t \mathfrak{U}^{-t}$. Set $V^t \equiv \mathfrak{U}^t U_{\mathfrak{M}}^{-t}$. Clearly $V^t \in \mathfrak{M}'$, $P\mathfrak{U}^t = V^t U_{\mathfrak{M}}^t$. Unitary operators V^t do not form (in general) a one-parameter group; V^t form a group in the case of $[\mathfrak{U}^{t_1}, U_{\mathfrak{M}}^{t_2}] = 0$ for all $t_1, t_2 \in \mathbf{R}$. The time evolution of the vector states ω_ξ $(\xi \in \mathscr{H}_{\mathfrak{M}})$ on $\mathfrak{A}''$ can be expressed by $\omega_\xi(\alpha_t x) = (\xi, U_{\mathfrak{M}}^{-t}(V^t)^* x V^t U_{\mathfrak{M}}^t \xi)$, $x \in \mathfrak{A}''$. Since V^t commutes with $\mathfrak{M}''$, the formula

$$P\mathfrak{U}^t = V^t U_{\mathfrak{M}}^t (\mathscr{H}_{\mathfrak{M}} \equiv P\mathscr{H})$$

can be understood as a separation of the "macroscopic time development" from the total (i.e. "microscopic") one. If ω_{ξ_0} is not the α_t-invariant state on $\in \mathscr{S}(\mathfrak{A}'')$, then $V^t\xi_0 \not\equiv \xi_0$.

The continuity of all the functions in (7) implies the seemingly stronger property, namely the continuity of all functions

$$t \mapsto \varphi_0\alpha_t(A) \equiv \varphi(\alpha_t A), \qquad \text{for all } \varphi \in \mathscr{S}(\mathfrak{M}'')_* \; A \in \mathfrak{M}'' \tag{8}$$

where $\mathscr{S}(\mathfrak{M}'')_* = \overline{\operatorname{co}\{\omega_\xi \mid \xi \in \mathscr{H}_{\mathfrak{M}}\}}$ ($\equiv$ the norm closure of the convex hull of $\{\ldots\}$ in the dual space $(\mathfrak{M}'')^*$ of $\mathfrak{M}''$) is the set of all normal states on the W^*-algebra $\mathfrak{M}''$ (compare [5, 6, 11]). One might be interested in knowing whether it is possible to change $\mathscr{S}(\mathfrak{M}'')_*$ by $\mathscr{S}(\mathfrak{M}'')$ in (8). The answer is contained in

II. 5. Proposition. Let $\mathfrak{M}$ be a commutative W^*-algebra and let $\alpha_t \in \operatorname{aut} \mathfrak{M}$ be a one-parameter group. If functions $t \mapsto \varphi(\alpha_t A)$ are continuous for all $A \in \mathfrak{M}$ and all pure states φ on $\mathfrak{M}$, then $\alpha_t = 1$ for all $t \in \mathbf{R}$ (i.e., α_t is trivial).

Proof. A commutative W^*-algebra $\mathfrak{M} = C(\mathfrak{X}) \equiv$ the space of all continous functions on a Stonean space $\mathfrak{X}$, [5] (a Stonean space is a compact Hausdorff space in which the closure of every open set is open). States on $\mathfrak{M}$ are determined by probabilistic Radon measures on $\mathfrak{X}$:

$$\omega(x) = \int_{\mathfrak{X}} x(\iota)\, d\mu_\omega(\iota), \qquad \omega \in \mathscr{S}(\mathfrak{M}),\; x \in C(\mathfrak{X}).$$

The atomic (or Dirac) measures δ_ι correspond to pure states, i.e. pure states are in a one to one correspondence with points of $\mathfrak{X}$ and for a pure state $\omega_\iota(\iota \in \mathfrak{X})$ we have $\omega_\iota(x) = x(\iota)$ for all $x \in \mathfrak{M}(x(\iota)$ is the function from $C(\mathfrak{X})$ which corresponds to $x \in \mathfrak{M})$. An automorphism $\alpha \in \operatorname{aut} \mathfrak{M}$ determines the transformation α^* of the "spectrum space" $\mathfrak{X}$ onto itself by

$$\alpha x(\iota) \equiv x(\alpha^*\iota).$$

The α^* is a homeomorphism of $\mathfrak{X}$ on $\mathfrak{X}$. Since the functions $t \mapsto x(\alpha_t^* \iota)$ are supposed to be continuous for all $x \in \mathfrak{M}$ and all $\iota \in \mathfrak{X}$, the functions $t \mapsto \alpha_t^* \iota$ are continuous from $\mathbf{R}$ to $\mathfrak{X}$ (for all $\iota \in \mathfrak{X}$). Hence the manifold $\{\alpha_t^* \iota \mid t \in \mathbf{R}\}$ is contained in the connected component of $\iota \in \mathfrak{X}$. Since $\mathfrak{X}$ is Stonean, the connected component of ι reduces to the one-point set $\{\iota\} \subset \mathfrak{X}$ and $\alpha_t^* \iota = \iota$ for all $t \in \mathbf{R}$. This implies $\alpha_t x(\iota) = x(\iota)$, $\alpha_t x = x$ for all $x \in \mathfrak{M}$ and all $t \in \mathbf{R}$, q.e.d.

The strong continuity of α_t, i.e., $\lim_{t \to 0} \| \alpha_t x - x \| = 0$ for all $x \in \mathfrak{M}$ implies the continuity of all $\omega(\alpha_t x)$ in $t \in \mathbf{R}$ ($\omega \in \mathscr{S}(\mathfrak{M})$, $x \in \mathfrak{M}$):

$$|\omega(\alpha_{t'} x) - \omega(\alpha_t x)| \leqslant \|\alpha_{t'-t} x - x\| \to 0 \text{ for } t' \to t.$$

As a consequence of II. 5. we see that the nontrivial group $\alpha_t (t \in \mathbf{R})$ of automorphisms of a commutative W^*-algebra is strongly discontinuous.

The existence of a nontrivial group $\alpha_t \in \operatorname{aut} \mathfrak{M}''$ which is weakly continuous on all normal states of a commutative W^*-algebra $\mathfrak{M}''$ in the sense of the continuity in (8) follows from the existence of nontrivial motion in classical mechanics. In the next simple example this is proved in details.

II. 6. **An example.** The classical linear harmonic oscillator with the Hamiltonian $H(q, p) = p^2 + q^2$ (frequency $\omega = 2$) is described by a C^*-algebra $\mathfrak{M}$ of all bounded continuous complex-valued functions $x(z)$ on the complex plane $\mathbf{C} (\ni z)$ tending to a finite limit for $z \to \infty$; we identify here $z \equiv q - ip$ ($\mathfrak{M}$ is the algebra of all continuous functions on the compact space $\bar{\mathbf{C}} \equiv \mathbf{C} \cup (\infty)$, i.e., on the complex plane compactified by adjoining a one-point in the usual manner). The Hamiltonian $H(q \cdot p) = |z|^2$ is not an observable here (since $H \notin \mathfrak{M}$), but e.g., $\tanh H \in \mathfrak{M}$. The time development is described by the group

$$\alpha_t x(z) = x(e^{i2t} z), \qquad x \in \mathfrak{M},\ z \in \mathbf{C}.$$

Probabilistic Radon measures $\mu_\omega(z)$ on $\bar{\mathbf{C}}$ determine the states $\omega \in \mathscr{S}(\mathfrak{M})$, and α_t-invariant measures μ_ω (i.e. measures invariant with respect to all rotations of $\mathbf{C}$ around the point $z = 0$) correspond to α_t-invariant states $\omega = \omega \circ \alpha_t$. Thus e.g., the Gibbs state ω_1 with the temperature $kT = 1$ corresponds to the α_t-invariant measure μ_1, $d\mu_1(z) = 1/\pi\, e^{-|z|^2} dz$ (here $dz \equiv dq\, dp$). The Gibbs state is, moreover, a faithful state: $\omega_1(x) = 0$ ($x \geqslant 0$) implies $x = 0$ ($x \in \mathfrak{M}$). The *GNS*-representation corresponding to this state is a faithful one. The Hilbert space $\mathscr{H}_1$ of this representation is $\mathscr{H}_1 = L^2(\mathbf{C}, \mu_1)$ and the cyclic vector ξ_1 is described by the constant function $\xi_1(z) \equiv 1$. For $x, y \in \mathfrak{M}$ the function

$$t \mapsto \omega_1(x \alpha_t y) = \int_{\mathbf{C}} x(z) y(e^{i2t} z)\, d\mu_1(z)$$

are continuous and the algebra $\pi_1(\mathfrak{M})$ fulfils the above mentioned conditions (a)—(c). Hence α_t can be extended to $\bar{\alpha}_t \in \operatorname{aut} \pi_1(\mathfrak{M})''$ and for all $\varphi \in \mathscr{S}(\pi_1(\mathfrak{M})'')_*$ the functions $\varphi(\bar{\alpha}_t x)$ ($x \in \pi_1(\mathfrak{M})''$) are continuous functions of $t \in \mathbf{R}$. It is clear that $\bar{\alpha}_t \not\equiv 1$. One can also prove $\pi_1(\mathfrak{M})'' = L^\infty(\mathbf{C}, \mu_1)$, since L^∞ is clearly a W^*-algebra containing $\pi_1(\mathfrak{M})$ and the cyclicity of π_1 implies that $\pi_1(\mathfrak{M})''$ is maximal commutative in $\mathfrak{B}(\mathscr{H}_1)$, [5] (2.9.4.).

Let $H_{\mathfrak{M}}$ be the generator of α_t in the representation $P\mathfrak{M}$ in $\mathscr{H}_{\mathfrak{M}}$, $P\alpha_t A = U_{\mathfrak{M}}^{-t} A U_{\mathfrak{M}}^t$, $U_{\mathfrak{M}}^t \equiv \exp(-itH_{\mathfrak{M}})$, $H_{\mathfrak{M}}^* = H_{\mathfrak{M}}$. The properties of functions $t \mapsto \omega_x(\alpha_t A)$ are dependent on the spectral properties of $H_{\mathfrak{M}}$ (compare also (7)). The t-invariance of $\omega \equiv \omega_{\xi_0} \in \mathscr{S}(\mathfrak{M})$ implies $H_{\mathfrak{M}} \xi_0 = 0$, hence the point spectrum of $H_{\mathfrak{M}}$ is not empty. Let P_p be the projector in $H_{\mathfrak{M}}$ on the subspace $\mathscr{H}_p = P_p \mathscr{H}_{\mathfrak{M}}$ generated by all eigenvectors of $H_{\mathfrak{M}}$ and let P_0 ($\leqslant P_p$) be the projector on the subspace of all eigenvectors of $H_{\mathfrak{M}}$ with the eigenvalue equal to zero. Then $[P_p, U_{\mathfrak{M}}^t] = [P_0, U_{\mathfrak{M}}^t] = 0$ for all $t \in \mathbf{R}$. Let $E(\lambda) \equiv E((-\infty, \lambda\rangle)$ ($\lambda \in \mathbf{R}$) be the spectral measure of $H_{\mathfrak{M}}$:

$$H_{\mathfrak{M}} = \int_{\mathbf{R}} \lambda\, dE(\lambda),\ E(\lambda) \equiv E(\lambda + 0). \tag{10}$$

The limits in (10) are understood in the strong operator topology in $\mathfrak{B}(\mathscr{H}_{\mathfrak{M}})$. The point spectrum $\dot{sp}(P_p H_{\mathfrak{M}})$ of $H_{\mathfrak{M}}$ consists of all points $\lambda \in \mathbf{R}$ in which $E(\lambda)$ is discontinuous. In a general case $\dot{sp}(P_p H_{\mathfrak{M}})$ is an arbitrary part of $\mathbf{R}$, e.g., it might be dense in $\mathbf{R}$, or (in the nonseparable $\mathscr{H}_{\mathfrak{M}}$) it might contain intervals or even it might coincide with $\mathbf{R}$. The projector $P_c \equiv 1_{\mathfrak{M}} - P_c$ commutes with $U_{\mathfrak{M}}^t$ and the spectral mesaure $P_c E(\lambda)$ of $P_c H_{\mathfrak{M}}$ is continuous on $\mathbf{R}$. This continuous part of $E(\lambda)$ can be decomposed into the absolutely continuous part (with the projector P_{ac}) and the singular continuous one (with the projector P_{sc}). This decomposition is characterized by the continuity properties of the measure $(\xi, P_{ac} E(\lambda) \xi)$ (resp. $(\xi, P_{sc} E(\lambda) \xi) \not\equiv 0$). For an arbitrary $\xi \in \mathscr{H}$ this measure is absolutely continuous (resp. singularly continuous) on $\mathbf{R}$ (with respect to the Lebesgue measure m). Moreover, $[P_{sc}, E(\lambda)] = [P_{ac}, E(\lambda)] = 0$, $P_{ac} + P_{sc} = P_c$ and the Hilbert space is the orthogonal sum of the three subspaces: $\mathscr{H}_{\mathfrak{M}} = \mathscr{H}_p \oplus \mathscr{H}_{sc} \oplus \mathscr{H}_{ac}$ (where $\mathscr{H}_n \equiv P_n \mathscr{H}_{\mathfrak{M}}$ for $n = p, sc, ac$), [10] (X.§1.2.).

Suppose that $\lim_{t \to +\infty} \omega_x(\alpha_t A)$ exists for all $A \in \mathfrak{M}$. This means, according to (7) and $\mathscr{H}_{\mathfrak{M}} = \overline{\mathfrak{M}\xi_0}$, that

$$w - \lim_{t \to \infty} U_{\mathfrak{M}}^t x^* x \xi_0 = \varphi_x^+ \in \mathscr{H}_{\mathfrak{M}} \tag{11}$$

exists, since the norm $\|U_{\mathfrak{M}}^t x^* x \xi_0\| = \|P x^* x \xi_0\|$ is uniformly bounded in $t \in \mathbf{R}$. Put $\varphi_x \equiv P x^* x \xi_0$. The vector φ_x^+ in (11) is $U_{\mathfrak{M}}^t$-invariant, i.e. $\varphi_x^+ \in P_0 \mathscr{H}_{\mathfrak{M}}$.

II. 7. Lemma. The necessary and sufficient conditions for the existence of the

$$w - \lim_{t\to+\infty} U^t_{\mathfrak{M}}\varphi = \varphi^+ (\varphi \in \mathscr{H})$$

is

$$w - \lim_{t\to+\infty} U^t_{\mathfrak{M}}(P - P_0)\varphi = 0.$$

If the last condition is fulfilled, then $\varphi^+ = P_0\varphi$.

Proof. $U^t_{\mathfrak{M}}\varphi = U^t_{\mathfrak{M}}P\varphi = U^t_{\mathfrak{M}}P_0\varphi + U^t_{\mathfrak{M}}(P - P_0)\varphi = P_0\varphi + U^t_{\mathfrak{M}}(P - P_0)\varphi$, hence the necessary and sufficient condition for the existence of $w - \lim_{t\to+\infty} U^t_{\mathfrak{M}}\varphi$ is the existence of $w - \lim_{t\to+\infty} U^t_{\mathfrak{M}}(P - P_0)\varphi \equiv \varphi_1^+$. Since the subspace $(P - P_0)\mathscr{H}$ is $U^t_{\mathfrak{M}}$ — invariant and closed in the weak topology in $\mathscr{H}$, $\varphi_1^+ \in (P - P_0)\mathscr{H}$. However, $U^t_{\mathfrak{M}}\varphi_1^+ = \varphi_1^+$ implies $\varphi_1^+ \in P_0\mathscr{H}$, hence $\varphi_1^+ = 0$ if φ_1^+ exists, q.e.d.

The lemma shows that $\varphi_x^+ = P_0\varphi_x$ in (11). If $(P_p - P_0)\xi \neq 0$ for a $\xi \in \mathscr{H}$, then there is some $\xi_n \in \mathscr{H}_{\mathfrak{M}} : H_{\mathfrak{M}}\xi_n = \lambda_n\xi_n(\lambda_n \neq 0)$ such that $(\xi_n, (P_p - P_0)\xi) \neq 0$ and $(\xi_n, U^t_{\mathfrak{M}}(P - P_0)\xi) = e^{-i\lambda_n t}(\xi_n, (P - P_0)\xi)$ and $U^t_{\mathfrak{M}}\xi$ does not converge for $t \to \infty$. Thus we have

II. 8. Lemma. Tme necessary condition for the existence of

$$\xi^+ \equiv w - \lim_{t\to+\infty} U^t_{\mathfrak{M}}\xi \quad (\xi \in \mathscr{H}) \text{ is } (P_p - P_0)\xi = 0.$$

Combining II. 7. and II. 8. we get

II. 9. Proposition. The limit $w - \lim_{t\to+\infty} U^t_{\mathfrak{M}}\xi$ $(= P_0\xi,\ \xi \in \mathscr{H})$ exists if the conditions (*i*) $(P_p - P_0)\xi = 0$, (*ii*) $w\text{-}\lim_{t\to\infty} U^t P_c\xi = 0$ are fulfilled simultaneously. The vectors $\xi \in \mathscr{H}$ satisfying (*i*) and (*ii*) form a $U^t_{\mathfrak{M}}$ — invariant subspace of $\mathscr{H}$ which is the same for $t \to +\infty$ as that for $t \to -\infty$.

Proof. The first part of II. 9. has been proved above. Since the operations in (*i*) and (*ii*) are norm-continuous and linear, vectors satisfying both conditions for $t \to +\infty$ (resp. for $t \to -\infty$) form the closed subspace $P_+\mathscr{H}$ (resp. $P_-\mathscr{H}$) of $\mathscr{H}$. For $\xi \in P_+\mathscr{H}$ and $\eta \in \mathscr{H}$ the limit $\lim_{t\to-\infty} (\eta, U^t_{\mathfrak{M}}\xi) = \lim_{t\to+\infty} \overline{(\xi, U^t_{\mathfrak{M}}\eta)} = \lim_{t\to+\infty} \overline{(P_+\xi, U^t_{\mathfrak{M}}\eta)} = \lim_{t\to+\infty} \overline{(\xi, U^t_{\mathfrak{M}}P_+\eta)}$ exists. Hence also $\xi \in P_-\mathscr{H}$. Changing the role of $+$ and $-$ we get $P_+ = P_-$, q.e.d.

This proposition implies $\lim_{t\to+\infty} \omega_x(\alpha_t A) = \lim_{t\to-\infty} \omega_x(\alpha_t A) \equiv \bar{\omega}_x(A)$ if one of these limits exists. If P_0 is one-dimensional and x is such an element of A that

$\varphi_x \equiv Px^*x\xi_0 \in P_+\mathscr{H}$, then $\bar{\omega}_x = \omega_{\xi_0} \equiv \omega \in \mathscr{S}(\mathfrak{M})$ and the state ω is stable with respect to such preturbations. The state ω is ergodic (i.e. extremal α_t-invariant) in the case $\dim P_0 = 1$, the only constants of motion in $\mathfrak{M}''$ are the elements $\lambda 1_{\mathfrak{M}} (\lambda \in \mathbf{C})$ and the mean $\lim_{T\to\infty} 1/T \int_0^T \omega(A\alpha_t B)\, dt = \omega(A)\omega(B)$ for all $A, B \in \mathfrak{M}$ (compare e.g. [11] (Theorem II. 2. 8)).

In the case of $\dim P_0 \geqslant 2$ local perturbations ω_x of the state ω might tend in the limit $t \to \infty$ to $\bar{\omega}_x \neq \omega$ for some $x \in \mathfrak{A}$.

III. SOME RESTRICTIONS ON THE TIME DEVELOPMENT

The intuitively acceptable condition of the continuity of functions $t \to \omega(\alpha_t A)$ for all $\omega \in \mathscr{S}(\mathfrak{M})$ and all elements A of a commutative W^*-algebra $\mathfrak{M}$ leads to the trivial group $\alpha_t \equiv 1 (t \in \mathbf{R})$ (compare II. 5.). We shall give now some further necessary conditions of non-triviality of the group $\alpha_t \in \operatorname{aut} \mathfrak{M}$ in the case of a commutative C^*-algebra $\mathfrak{M}$.

III. 1. Proposition. Let $\mathfrak{M}$ be a commutative W^*-algebra of operators in a Hilbert space $\mathscr{H}$, $\mathfrak{M} \subset \mathfrak{B}(\mathscr{H})$ and let $\alpha_t x \equiv \exp(ith)x \exp(-ith)$ be a one parameter group of automorphisms of $\mathfrak{M}(h^* = h \in \mathfrak{L}(\mathscr{H}),\ t \in \mathbf{R})$. If the spectrum of h is (at least) one-sidedly bounded, then $\alpha_t \equiv 1 (t \in \mathbf{R})$.

Proof. According to [5] (4. 1. 15) (the Borchers theorem) if h is lower bounded, then α_t is a group of inner automorphisms: $\alpha_t x = u_t^* x u_t$ $(u_t \in \mathfrak{M}, u_t^* u_t = 1$ for all $t \in \mathbf{R})$. In the case of commutative $\mathfrak{M}$ this implies $\alpha_t \equiv 1$, q.e.d.

We have seen above that a unitarily implementable one-parameter group of automorphisms of a C^*-algebra $\mathfrak{M}(\ni 1_{\mathscr{H}})$ in $\mathfrak{B}(\mathscr{H})$ is extendable to the unitarily implementable group $\bar{\alpha}_t \in \operatorname{aut} \mathfrak{M}''$ of the weak closure of $\mathfrak{M}$. Since $\mathfrak{M}''$ is commutative if $\mathfrak{M}$ is, the nontrivial automorphic group α_t of a commutative C^*-subalgebra $\mathfrak{M}$ of $\mathfrak{B}(\mathscr{H})$, which is unitarily implemented by a weakly continuous group of unitary operators $\exp(-ith)$ cannot have a generator h bounded from any side. Hence, the connection of h and the Hamiltonian ($\equiv$ energy operator) is more complicated in general as it is in conventional quantum mechanics. The next example of a classical system with one degree of freedom illustrates the situation.

III. 2. An example. Let ω be an α_t-invariant state of the system from II. 6. The Hamiltonian is $H(q, p) = q^2 + p^2 = |z|^2$ $(z \equiv q - ip)$. An α_t-invariant measure μ_ω on $\bar{\mathbf{C}}$ corresponds to the state ω. $\pi_\omega(\alpha_t x) = \exp(ith)\pi_\omega(x)\exp(-ith)$ in the *GNS*-representation $(\mathscr{H}_\omega, \pi_\omega(\mathfrak{M}), \xi_\omega)$ as a consequence of the continuity properties of α_t and the α_t-invariance of $\omega \in \mathscr{S}(\mathfrak{M})$. Here $\mathfrak{M} \equiv C(\bar{\mathbf{C}})$, $\mathscr{H}_\omega \equiv L^2(\bar{\mathbf{C}}, \mu_\omega)$ and $\xi_\omega(z) \equiv 1 \in L^2(\bar{\mathbf{C}}, \mu_\omega)$. If we write $z \equiv |z|e^{i\varphi}$, then $\pi_\omega(x) \equiv$

$\equiv x_\omega \in L^\infty(\mathbf{C}, \mu_\omega)$ is a function $x_\omega(|z|, \varphi)$ $(x \in \mathfrak{M})$ and $(\alpha_t x)_\omega(|z|, \varphi) = x_\omega(|z|, \varphi + 2t)$. The generator h can be written in the form

$$h = -\mathrm{i}2\frac{\partial}{\partial\varphi} = \mathrm{i}\left(\frac{\partial H}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\right),$$

which is the Liouville operator of our system. It is known that the spectrum of h consists of the isolated points $\lambda_n = 2n$, $n = 0, \pm 1, \pm 2, \ldots$ Hence the generator h is unbounded (from both sides). The time dependence of the perturbed states ω_x is periodic with a period $\Delta t = \pi$.

The generator of time development is the Liouville operator also in some noncommutative cases:

III. 3. Remark. Let $\mathfrak{A}$ be a W^*-algebra in a Hilbert space $\mathscr{H}$, $\mathfrak{A} = \mathfrak{A}''$ in $\mathfrak{B}(\mathscr{H})$. Let $\alpha_t x = \exp(\mathrm{i}tH)x\exp(-\mathrm{i}tH)$ be a one parameter group of automorphisms of $\mathfrak{A}(x \in \mathfrak{A}, H^* = H \in \mathfrak{L}(\mathscr{H}))$. Let $\omega \in \mathscr{S}(\mathfrak{A})_*$ be a faithful normal α_t-invariant state, i.e., $\omega(x^*x) = 0$ implies $x = 0$ $(x \in \mathfrak{A})$, $\omega \circ \alpha_t = \omega$ and $\omega(x) = Tr_{\mathscr{H}}(\varrho^{(\omega)}x)$ $(x \in \mathfrak{A}$, $\varrho^{(\omega)}$ is the corresponding density matrix). The *GNS*-representation $(\mathscr{H}_\omega, \pi_\omega(\mathfrak{A}), \xi_\omega)$ is a faithful representation with the joint cyclic and separating vector ξ_ω. Let us denote $\pi_\omega(x)\xi_\omega \equiv x_\omega(x \in \mathfrak{A})$, $1_\omega = \xi_\omega$. In this representation $\pi_\omega(\alpha_t x) = \exp(\mathrm{i}th_{(\omega)})\pi_\omega(x)\exp(-\mathrm{i}th_{(\omega)})$, $h^*_{(\omega)} = h_{(\omega)} \in \mathfrak{L}(\mathscr{H}_\omega)$, $\exp(\mathrm{i}th_{(\omega)})1_\omega = 1_\omega$. Then $(x_\omega, \exp(\mathrm{i}th_{(\omega)})y_\omega) = (x_\omega, (\alpha_t y)_\omega) = Tr_{\mathscr{H}}(\varrho_{(\omega)}x^*e^{\mathrm{i}tH}ye^{-\mathrm{i}tH})$. The algebra $\mathfrak{A}$ is embedded by the linear injective mapping $x \to x_\omega$ in the Hilbert space $\mathscr{H}_\omega$ and the operator $h_{(\omega)}$ is in fact the Liouville operator L defined usually on the Hilbert space of Hilbert Schmidt operators by $e^{\mathrm{i}tL}x \equiv e^{\mathrm{i}tH}xe^{-\mathrm{i}tH}$ ($x \in$ Hilbert Schmidt ideal in $\mathfrak{B}(\mathscr{H})$). If H is one-sidedly bounded, then according to the Borchers theorem (see e.g. [5] (4. 1. 15)) we have $\exp(\mathrm{i}tH) \in \mathfrak{A}$ and

$$(x_\omega, \exp(\mathrm{i}th_{(\omega)})y_\omega) = \omega(x^*e^{\mathrm{i}tH}ye^{-\mathrm{i}tH}).$$

In proving selfadjointness of $h_{(\omega)}$ one uses α_t-invariance of ω. If the state ω is tracial (i.e. $\omega(xy) \equiv \omega(yx)$) and the group of automorphisms is inner, it is not necessary to suppose the α_t-invariance of ω:

$$((\alpha_t x)_\omega, (\alpha_t y)_\omega) = \omega(e^{\mathrm{i}tH}x^*ye^{-\mathrm{i}tH}) = \omega(x^*y) = (x\omega, y\omega)$$

and the α_t-invariance of ω follows.

The situation is particularly simple in the case of the bounded Hamiltonian, $H \in \mathfrak{B}(\mathscr{H}) : H \in \mathfrak{A}(\alpha_t$ is inner) and $Lx = [H, x]$, since

$$e^{\mathrm{i}tL}x = \sum_{n=0}^{\infty}\frac{(\mathrm{i}t)^n}{n!}L^n x = \sum_{n=0}^{\infty}\frac{(\mathrm{i}t)^n}{n!}[H, x]^{(n)} = e^{\mathrm{i}tH}xe^{-\mathrm{i}tH},$$

where both the series converge in the norm-topology of $\mathfrak{B}(\mathscr{H})$, and $[A, B]^{(n)} \equiv [A, [A, B]^{(n-1)}]$, $[A, B]^{(0)} \equiv B$.

The above mentioned case of the tracial state ω leads to the time evolution of the perturbed states ω_x expressed by

$$\omega_x(\alpha_t y) = \omega(x^*(\alpha_t y)x) = \omega(xx^*(\alpha_t y)) = ((x^*x)_\omega, e^{\mathrm{i}th_{(\omega)}}y_\omega).$$

The analysis of the convergence of $\omega_x(\alpha_t y)$ for $t \to \infty$ can be carried out in the same way as in the commutative case.

We have seen in II. 5. that the group α_t can not be "very continuous" to be nontrivial. The next proposition gives further restrictions on the continuity properties of a nontrivial group of automorphisms of a commutative C^*-algebra.

III. 4. Proposition. Let $\mathfrak{M}$ be a commutative C^*-algebra and $\alpha \in \operatorname{aut}\mathfrak{M}$. If $\|\alpha - 1\| < 2$, then $\alpha = 1$.

Proof. The assertion is contained in [12] (Lemma 4): If $\mathfrak{M} \in \mathfrak{B}(\mathscr{H})$ is a C^*-algebra and $\|\alpha - 1\| < 2$, then there is an extension $\bar{\alpha} \in \operatorname{aut}\mathfrak{M}''$ of α leaving all elements of the centre of $\mathfrak{M}''$ fixed. In the case of commutative $\mathfrak{M}$, $\mathfrak{M}$ is contained in the centre of $\mathfrak{M}''$ and $\alpha = 1$, q.e.d.

If $\alpha_t \in \operatorname{aut}\mathfrak{M}$ $(t \in \mathbf{R})$ is a group, then $\|\alpha_{t_1} - \alpha_{t_2}\| = \|\alpha_{t_1 - t_2} - 1\|$ and $\|\alpha_{t_0} - 1\| < 2$ implies $\|\alpha_{t+t_0} - \alpha_t\| = \|\alpha_{t_0} - 1\| = 0$, i.e., the periodicity of $\alpha_t : \alpha_{t+t_0} = \alpha_t$ for all $t \in \mathbf{R}$. Phrasing this in a different way we have

III. 5. Corollary. If $t_0 \in \mathbf{R}$ is not a period of a one-parameter group α_t of $*$-automorphisms of a commutative C^*-algebra, then $\|\alpha_{t_0} - 1\| = 2$.

An immediate consequence of the preceding considerations is the norm-discontinuity of a nontrivial group α_t on a commutative C^*-algebra. According to II. 5. a nontrivial group α_t $(t \in \mathbf{R})$ of automorphisms of a commutative W^*-algebra is even strongly discontinuous.

IV. THE RATE OF DECAY AND THE SPECTRAL PROPERTIES OF GENERATORS

We shall use the notation from Sec. II. The properties (a)—(c) of α_t are supposed to be valid. Since only the component $(P_c + P_0)\varphi_x$ contributes to the convergent $\omega_x(\alpha_t A) = (\varphi_x, U^{-t}_{\mathfrak{M}}A\xi_0)$ $(A \in \mathfrak{M}, t \to \infty)$ and $U^t P_0\varphi_x = P_0\varphi_x$, we can restrict our subsequent considerations to the subspace $\mathscr{H}_c \equiv P_c\mathscr{H} \subset \mathscr{H}_{\mathfrak{M}}$. Let us denote $U^t_c \equiv P_c U^t_{\mathfrak{M}} \equiv \exp(-\mathrm{i}tH_c)$, $H_c \equiv \int_{\mathfrak{R}} \lambda\, \mathrm{d}E_c(\lambda)$ and $E_c(\lambda) = P_c E(\lambda) \in \mathfrak{B}(\mathscr{H}_c)$. Thus $E_c(\lambda + 0) = E_c(\lambda - 0)$ for all $\lambda \in \mathbf{R}$. Writing $\mathscr{H}_c = \mathscr{H}_{ac} \oplus \mathscr{H}_{sc}$ the measure $\mu_\varphi(\lambda) \equiv (\varphi, E(\lambda)\varphi)$ on $\mathbf{R}$ with $\varphi \in \mathscr{H}_{ac}$ (resp. $\varphi \in \mathscr{H}_{sc}$) is absolutely continuous (resp. singular continuous) with respect to the Lebesgue measure m. (A finite measure μ on $\mathbf{R}$ is singular iff there is $M \subset$

$\subset \mathbf{R}$, $m(M) = 0$ and $\mu(M) = \mu(\mathbf{R}) \neq 0$). We shall show how smoothness properties of $\mu_\varphi(\lambda)$ determine the behaviour of $\hat{\mu}_\varphi(t) \equiv \int_{\mathfrak{R}} e^{it\lambda}\, d\mu_\varphi(\lambda)$ for $|t| \to \infty$. Let us start with the reversed connection.

IV. 1. Proposition. Let $\hat{\mu}(t) \equiv \int_{\mathfrak{R}} e^{it\lambda}\, d\mu(\lambda)$, where μ is a probabilistic Radon measure on $\mathbf{R}$. If

$$|\mu(t)| = 0(|t|^{-\gamma}) \text{ for } |t| \to \infty,\ \gamma > p \geqslant 1 \ (p \text{ integer}),$$

then $\mu(\lambda) \in C^p(\mathbf{R})$ ($\equiv$ functions with the bounded continuous p-th derivative on $\mathbf{R}$) and $\lim\limits_{|\lambda|\to\infty} \dfrac{d^q\mu(\lambda)}{d\lambda^q} = 0$ for $q = 1, 2, \ldots, p$.

Proof. $\hat{\mu}(t) \to 0$ $(t \to \infty)$ implies continuity of $\mu(\lambda)$. The inverse Fourier transform of $\hat{\mu}$ gives [13] (p. 27. pp. 85—87)

$$\mu(\lambda) - \mu(0) = \frac{1}{2\pi} \int_{\mathbf{R}} \hat{\mu}(t)\, \frac{1 - e^{-i\lambda t}}{it}\, dt\,.$$

The continuity and the behaviour for $t \to \infty$ of $\hat{\mu}$ leads to $\int_{\mathbf{R}} |t^{p-1}\hat{\mu}(t)\, dt < \infty$, which implies the existence of derivatives $\mu^{(q)}(\lambda) \equiv \dfrac{d^q}{d\lambda^q}\, \mu(\lambda) = \dfrac{(-i)^{q-1}}{2\pi} \displaystyle\int_{\mathbf{R}} t^{q-1}\hat{\mu}(t) e^{-it\lambda}\, dt$ for $q = 1, 2, \ldots p$.

The continuity and the convergence to zero (for $|\lambda| \to \infty$) of $\mu^{(q)}$ follow from the known properties of the Fourier transforms of integrable functions, q.e.d.

The derivative of a singular function is not continuous and the convergence (if any) of $\hat{\mu}_\varphi(t)$ $(\varphi \in \mathscr{H}_{sc})$ for $t \to \infty$ is very slow. For $\varphi \in \mathscr{H}_{ac}$ we have

$$\hat{\mu}_\varphi(t) = \int_{\mathbf{R}} e^{it\lambda}\, \frac{d\mu_\varphi(\lambda)}{d\lambda}\, d\lambda\,. \tag{12}$$

The last formula implies $\hat{\mu}_\varphi(t) \to 0$ for $|t| \to \infty$ for all $\varphi \in \mathscr{H}_{ac}$ and by polarization we get

$$\lim_{|t|\to\infty} (\varphi, U^t_{\mathfrak{M}}\psi) = 0 \text{ for } \varphi \in \mathscr{H}_{ac},\ \psi \in \mathscr{H}.$$

Hence $P_{ac} \leqslant P_+ (= P_-)$. We are interested in the speed of the convergence of $(\varphi, U^t_c\psi)$ $(\varphi \in \mathscr{H}_{ac})$ for $t \to \infty$. For the polynomial decrease we have

IV. 2. Proposition. If the probabilistic Radon measure $\mu(\lambda)$ on $\mathbf{R}$ has the m-integrable continuous p-th derivative $\mu^{(p)}$ on $\mathbf{R}$ $(p \geqslant 1$, integer) and $\lim\limits_{|\lambda|\to\infty} \mu^{(q)}(\lambda) = 0$ for $q = 1, 2, \ldots, p - 1$, then $|\hat{\mu}(t)| = o(|t|^{-p+1})$ for $|t| \to \infty$.

Proof. Put $J_q(a, b; t) \equiv \int_a^b e^{i\lambda t}\mu^{(q)}(\lambda)\, d\lambda$. For $|a| + |b| < \infty$ and an integer $q \leqslant p$ $(q \geqslant 0)$, J_q is a continuous function of t and $\lim\limits_{|t|\to\infty} J_q = 0$. Put $\mathcal{J}_q(t) \equiv \lim\limits_{\min(-a,b)\to+\infty} J_q(a, b; t)$, if the limit exists. Clearly $\mathcal{J}_1(t) = \hat{\mu}(t)$. Integration per partes in J_q gives

$$J_q(a, b; t) = \frac{1}{it}\,[e^{itb}\mu^{(q)}(b) - e^{iat}\mu^{(q)}(a)] + \frac{i}{t}\, J_{q+1}(a, b; t) \text{ for } q = 1, 2, \ldots, p - 1.$$

In the limit $b \to +\infty$, $a \to -\infty$ we get

$$\mathcal{J}_q(t) = \frac{i}{t}\mathcal{J}_{q+1}(t) \text{ for } q = 1, 2, \ldots, p - 1, \tag{13}$$

since $\mu^{(q)}(\pm\infty) = 0$ and $\mathcal{J}_1(t)$ exists. By repeated using of (13) we get

$$\hat{\mu}(t) \equiv \mathcal{J}_1(t) = \left(\frac{t}{i}\right)^{p-1} \mathcal{J}_p(t)\,. \tag{14}$$

Since $\mu^{(p)} \in L^1(\mathbf{R}, m)$, $\mathcal{J}_p(t) \to 0$ for $|t| \to \infty$ and (14) gives the wanted result, q.e.d.

The better the analytic properties of $\mu(\lambda)$ are, the faster is the convergence $\hat{\mu}(t) \to 0$ $(|t| \to \infty)$. For the exponential "decay law" we have

IV. 3. Proposition. Let $\mu(\lambda)$ be a complex Radon measure on $\mathbf{R}$ and $d\mu/d\lambda = F(\lambda)$ for $\lambda \in \mathbf{R}$. Let $F(z)$ be an analytic function in the region $-\nu_1 \leqslant \operatorname{Im} z \leqslant \nu_2$ $(\nu_i \geqslant 0)$ and let

$$\int_{\mathbf{R}} |F(\lambda + i\sigma)|^2\, d\lambda \leqslant C < \infty \text{ for } -\nu_1 \leqslant \sigma \leqslant \nu_2\,.$$

Then

$$(i)\ \ |\hat{\mu}(t)| = o(e^{+\nu_1}) \text{ for } t \to -\infty,$$

$$(ii)\ \ |\hat{\mu}(t)| = o(e^{-\nu_2 t}) \text{ for } t \to +\infty.$$

Proof. According to [14] (Theorem IV.) there is a function $g(t)$ which is a Fourier transform of $F(\lambda)$ and satisfies (i) and (ii) (after the replacement of $\hat{\mu}$ by g). From the continuity of $d\mu/d\lambda$, $|\mu(\mathbf{R})| < \infty$ and the Plancherel theorem we have $\hat{\mu} = g$ (m-a.e.). The result is then a consequence of the continuity of $\hat{\mu}(t)$, q.e.d.

It is clear from the proofs of the propositions IV. 1.—IV. 2. that they are also valid for complex measures μ with a finite total variation. Hence, we can apply them to the measures of the form $\mu(\lambda) \equiv (\varphi, E(\lambda)\psi)$, $\varphi, \psi \in \mathscr{H}$. In this case $\hat{\mu}(t) \equiv (\varphi, U_M^{-t}\psi)$. Propositions give the wanted connection between the "smoothness properties" of $(\varphi_x, E(\lambda)A\xi_0)$ and the speed of the convergence of $\omega_x(\alpha_t A)$ for $t \to \pm\infty$ $(x \in \mathfrak{A}, A \in \mathfrak{M} \subset \mathfrak{Z}(\mathfrak{A}'))$.

The next simple example of a classical system illustrates the transition from the perturbed t-invariant state ω_x to another t-invariant state $\bar{\omega}_x = \lim_{t\to\infty} \omega_x \circ \alpha_t$ ($\neq \omega$, in general).

IV. 4. An example. Let the physical system we want to describe here be a classical freely moving point particle on the unit circle. The Hamiltonian $H(q, p) \equiv \frac{1}{2} p^2$ ($q \in S \equiv$ the unit circle, $p \in \mathbf{R}$). Let $\bar{\mathbf{R}}$ be a compactified real axis obtained from $\mathbf{R}$ by adjoining the point (∞) in a usual manner. Then $S \times \bar{\mathbf{R}}$ is a compact space. Let $\mathfrak{M} \equiv C(S \times \bar{\mathbf{R}})$ be the algebra of observables of the described system. States on $\mathfrak{M}$ are determined by probabilistic Radon measures on $S \times \bar{\mathbf{R}}$. The time development is described by a group $\alpha_t \in \operatorname{aut} \mathfrak{M}$,

$$[\alpha_t x](q, p) = x(q + pt, p) \; (x \in \mathfrak{M}, x(q + 2\pi, p) = x(q, p)) .$$

For $\omega \in \mathscr{S}(\mathfrak{M})$ we have

$$\omega(x) = \int_{S\times\bar{\mathbf{R}}} x(\xi)\, d\mu_\omega(\xi), \qquad \xi \equiv (q, p) .$$

Continuity of $x(\xi)$ implies continuity of $t \mapsto \omega(\alpha_t x)$ for all $\omega \in \mathscr{S}(\mathfrak{M})$ and all $x \in \mathfrak{M}$. If the support of an α_t-invariant measure μ_0 is $S \times \{p_0\}$ $(p_0 \in \mathbf{R})$, then the perturbed states depend periodically on t with the period $\Delta t = 2\pi/p_0$. In this case the spectrum of the generator coincides with its point spectrum (compare the generator in III. 2). Although, in the general case of an α_t-invariant state ω we can get an aperiodical time development. Let μ_ω be an m-a.c. $(dm \equiv dq\, dp)$; it follows

$$\omega(x) = \int_{S\times\bar{\mathbf{R}}} x(q, p)\mu'_\omega(q, p)\, dq\, dp, \; \mu'_\omega \in L^1(S \times \mathbf{R}, m) . \tag{15}$$

For $\omega\circ\alpha_t = \omega$ we have $\mu'_\omega(q - pt, p) = \mu'_\omega(q, p)$ (m-a.e.) and we can choose μ'_ω independent on q. Then

$$\omega_x(\alpha_t y) = \int [x^* x](q, p) y(q + pt, p)\mu'_\omega(p)\, dq\, dp = \\ = \int_{S\times\bar{\mathbf{R}}} |x(q - pt, p)|^2 y(q, p)\mu'_\omega(p)\, dq\, dp . \tag{16}$$

We can write

$$|x(q, p)|^2 = \sum_n c_n(x; p) e^{inq} \; (m\text{-a.e. in } S \times \bar{\mathbf{R}}) . \tag{17}$$

In the case of an "appropriate choice" of $x \in \mathfrak{M}$ (e.g. if $\sum_n |c_n(x; p)| \in L^1(\mathbf{R}, |\mu'_\omega(p)|\, dp)$; we shall refer to this condition as to the condition "X")

we can interchange the summation and integration and according to the Fubini theorem we have

$$\omega_x(\alpha_t y) = \sum_n \int_{\mathbf{R}} \mu'_\omega(p)\, dp\, e^{-intp} c_n(x; p) \int_0^{2\pi} e^{inq} y(q, p)\, dq . \tag{18}$$

Each member of the sum in (18) with $n \neq 0$ tends for $|t| \to \infty$ to zero. Interchanging $\lim_{|t|\to\infty}$ and $\sum_n$ (if possible, e.g., if the condition "X" is fulfilled) we get

$$\bar{\omega}_x(y) \equiv \lim_{|t|\to\infty} \omega_x(\alpha_t y) = \int_{S\times\bar{\mathbf{R}}} c_0(x; p) y(q, p)\, d\mu_\omega(q, p) . \tag{19}$$

If $c_0(x; p)$ is independent on p, we have $\bar{\omega}_x \equiv \lim_{|t|\to\infty} \omega_x \circ \alpha_t = \omega$ and the system comes back to the original unperturbed state ω. For a general $x \in \mathfrak{M}$ we can get $\bar{\omega}_x \neq \omega$.

It might be superfluous to note that the existence of limits $\omega_x \circ \alpha_t \to \bar{\omega}_x$ in the previous example is of another nature than the existence of similar limits in the classical ergodic theory due to the "mixing property" of some ergodic states [15]. The occurrence of the "mixing" in the ergodic theory implies ergodicity of the state (or equivalently the ergodicity of the measure). In the example IV. 4. the ergodic measures are concentrated on the manifolds $S \times \{p_0\}$ $(p_0 \in \mathbf{R})$ and no such measure has the mixing property defined by

$$\lim_{t\to\infty} \mu(N \cap M_t) = \mu(M)\mu(N) \text{ for all } M, N \subset \mathfrak{X} , \tag{20}$$

where the manifold M_t is defined by the transformation of points in the "spectrum space" $\mathfrak{X}$ corresponding to the group of time transformations of the system.

V. AN APPLICATION TO THE QUANTUM THEORY OF MEASUREMENT

It is shown in [3] that the old problem of the description of the "reduction of a wave packet" without any special postulate (i.e. without postulating the occurrence of the "processes of the first kind" in the process of measurement, in the terminology of von Neumann [16]) can be expected to be solvable within the framework of the quantum mechanics of infinite systems. The decisive feature for such a solution is the nontriviality of the centre of the algebra of observables; this fact is stressed also in [17]. The relevance of the formalism explained in our Sec. II. for the "problem of measurement" might be seen from the following example.

V. 1. Example. With the notation of Sec. II. let $P_c = P_{ac}$, $P_p = P_0$,

$\dim P_0 \geqslant 2$ and an integer $N \leqslant \dim P_0$ (it could be also $N = \infty$). Let $x_i \in \mathfrak{A}$, $\omega(x_i^* x_i) = \|x_i \xi_0\|^2 = 1$ and $P_0 x_j^* x_i \xi_0 = 0$ (for $j \neq i$) $(i, j = 1, 2, \ldots, N; \omega \in \mathscr{S}(\mathfrak{A})$ is $\mathfrak{M}$-t-invariant). Put $x \equiv \sum_{i=1}^{N} c_i x_i (\sum_{i=1}^{N} |c_i|^2 = 1)$, $c_i \in \mathbf{C}$. By these assumptions we have

$$\bar{\omega}_x(A) = \lim_{t\to\infty} \omega_x(\alpha_t A) = \sum_{i=1}^{N} |c_i|^2 (x_i^* x_i \xi_0 , P_0 A \xi_0) \text{ for all } A \in \mathfrak{M} . \tag{21}$$

The state $\omega_x(A) = \sum_{ij} c_i^* c_j (x_j^* x_i \xi_0 , A\xi_0)$ is in general a coherent superposition of the states $\omega_{xi}(A) \equiv (x_i \xi_0 , A x_i \xi_0)$ and the state $\bar{\omega}_x = \sum_i |c_i|^2 \bar{\omega}_{xi}$ is a mixture (on the algebra $\mathfrak{M}$). Hence, in the state $\bar{\omega}_x$ "after a measurement" there is no macroscopic interference between different "pointer positions". According to [19] each ($\mathfrak{M}$-) t-invariant state on $\mathfrak{M}$ can be extended to an invariant state on $\mathfrak{A}$. After such an extension of all $\bar{\omega}_{xi}$ we obtain

$$\bar{\omega}_x(y) = \sum_{i=1}^{N} |c_i|^2 \bar{\omega}_{xi}(y) \text{ for all } y \in \mathfrak{A} . \tag{22}$$

This is the state of the wanted form "after the reduction". Such a state on $\mathbf{U}$ can be obtained by taking the time average of $\omega_x(\alpha_t y)$.

We shall give a more detailed analysis of the process of measurement in quantum theory in the frame of the quantum mechanics of infinite systems in another paper. The question of the existence of a model fulfilling all the conditions assumed in V. 1. is left open here.

ACKNOWLEDGEMENT

The author is indebted to Dr. J. Pišút for some corrections in the English version of this paper.

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Received November 5th, 1973.