acta phys. slov. 24 (1974), No. 3

PADÉ'S APPROXIMANTS IN WEAK INTERACTIONS[1]

R. P. SAXENA*, Göteborg

Some work is described on higher order weak interactions employing Pade's approximants to the regularized perturbation theory. Basically the method is an application of a rigorous mathematical theorem proved by Villani et al [1] in the function theory. I shall describe some work that Dr. Allan, M. Din and myself have on done higher order weak interactions employing Padé technique.

Consider a function (which may be the invariant amplitude for any weak interaction process) $T(g, \lambda)$ which has an expansion of the type:

$$T(g, \lambda) = \sum_{n=0}^{\infty} g^n f_n(\lambda). \tag{1}$$

Here g may be a coupling constant and λ a cut off parameter. If T represents a weak interaction amplitude, then all the coefficients $f_n(\lambda)$ (except the first) diverge as λ approaches ∞. We want to put the following question: Does the limit of a sum of such terms

$$T(g) = \lambda \to \infty \quad T(g, \lambda) = \sum_{\lambda \to \infty} g^n f_n(\lambda) \tag{2}$$

exist and is there some way of approximating such a sum, merely by knowing the first few terms?

One constructs the Padé approximants $T^{(n, n+m)}(g, \lambda)$ (n and $n + m$ being the degree of polynomials in g in the denominator and numerator respectively). The combined limits of the sequence

$$\underset{\substack{n \to \infty \\ m \to \infty \\ \lambda \to \infty}}{} T^{(n, n+m)}(g, \lambda)$$

should yield, if the Padé approximants to such a term by term divergent series converge in some sense, the true answer $T(g)$, which one is interested in. This limiting procedure is the essence of the theorem of Villani et al.

For the sake of brevity, we shall discuss the cases $m = -1$ and 0 respectively. One can prove that the Padé approximants $T^{(n, n-1)}(g, \lambda)$ satisfy:

$$\underset{\lambda \to \infty}{} T^{(n, n-1)}(g, \lambda) \to 0 \tag{3}$$

---

[1] Talk given at the Triangle Meeting on Weak Interactions at SMOLENICE, June 4–6, 1973.

* Institute of Theoretical Physics, S-402, 20 GÖTEBORG 5, Sweden.

$$\lim_{\lambda \to 0} T^{(n,n-1)}(g, \lambda) \to 0.$$

Such a behaviour is obviously expected for a series of Stieltjes, which can be expressed by

$$F(z) = \int_0^\infty \frac{\Phi(t)}{1 + zt} \, dt \tag{4}$$

with a non-negative $\Phi(t)$. Obviously then $T^{(n,n-1)}(g, \lambda)$ must posses at least one maximum as a function of λ. Let the position of the maximum be at $\lambda = \lambda_n(g)$. Villani et al. then prove that

$$\text{i)}\ {}_{n\to\infty}\lambda(g) \to \infty$$

$$\text{and ii)}\ {}_{n\to\infty}T^{(n,n-1)}(g, \lambda_n(g)) \to T(g). \tag{5}$$

No such simple criterion exists for the Padé approximants $T^{(n,n)}(g, \lambda)$. However, they do have a stationary feature, in the sense that their derivatives possess maxima, i. e. $T^{(n,n)}(g, \lambda)$ have inflexion points as a function of λ. If the inflexion points are at $\lambda = \lambda_n(g)$, then one can prove that

$$\begin{aligned} &\text{i)}\ {}_{n\to\infty}\lambda_n(g) \to \infty \\ &\text{ii)}\ {}_{n\to\infty}T^{(n,n)}(g, \lambda_n(g) \to T(g). \end{aligned} \tag{6}$$

This completes the statement of the theorem of Villani et al. The proof of this theorem also exists [2] for the Born-series of the potential scattering amplitude with potentials more pronounced at the origin than r^{-2}. Unfortunately no such proofs exist in the quantum field theory. We shall, however, adopt the viewpoint that the theorem of Villani et al. is true and merely apply it in order to compute higher order weak interactions. In view of the fact that the new gauge theories of weak interactions [3, 4] are claimed to be renormalizable [5], it should be interesting to compare their results with ours and the experiments if any.

Our procedure is then straightforward. We compute all the Feynman diagrams up to the four order in the Fermi-coupling constant G, employing a Pauli-Villars regulator and construct the Padé approximants $T^{(1,1)}$, $T^{(2,2)}$, $T^{(1,2)}$ and $T^{(2,1)}$ (which is all that we are able to calculate) for the following processes:

i) $e - \nu_e$ (or $\mu - \nu_\mu$) elastic scattering
ii) $e - \bar{\nu}_e$ (or $\mu - \bar{\nu}_\mu$) elastic scattering
iii) $\mu \to e + \nu_e + \nu_\mu$ decay
iv) $K_L^0 \to \mu^+ + \mu^-$ decay.

We now look for the above mentioned stationary features in our Padé approximants and compute the values of the Padé approximants at these stationary points. Our results are:

i) $e - \nu_e$ scattering:
$T^{(1,1)}$ no stationary feature
$T^{(1,2)}(g, 38\text{ GeV}) = 0.832\,g$
$T^{(2,1)}(g, 42\text{ GeV}) = 0.923\,g$
$T^{(2,2)}(g, 47\text{ GeV}) = 0.975\,g$
$\sigma_{exp}(e + \nu_e)/\sigma_{V-A} < 40$

ii) $e - \bar{\nu}_e$ scattering
$T^{(1,1)}$ no stationary feature
$T^{(1,2)}(g, 34\text{ GeV}) = 1.334\,g$
$T^{(2,1)}(g, 41\text{ GeV}) = 1.487\,g$
$T^{(2,2)}(g, 48\text{ GeV}) = 1.632\,g$
$\sigma_{exp}(e + \bar{\nu}_e)/\sigma_{V-A} = 1.0 \pm 0.9$
$\sigma_{Weinberg}(e + \bar{\nu}_e) = (1.5)^2\sigma_{V-A}$

iii) μ-decay:

Here we assume that the weak radiative corrections generate an induced tensor term in the amplitude. (Scalar and pseudoscalar term turn out to be negligible). Denoting A as the $V-A$ amplitude and B as the induced tensor term in the μ-decay, we find:

$A^{(2,2)}(g, 21.4\text{ GeV}) = 0.995$
$B^{(2,2)}(g, 24\text{ GeV}) = 0.032$

The lower Padé approximants give similar results. One can now compute the μ-decay parameters ϱ, v and δ. (The parameter is zero within our approximations).

Present calculation	Experiment	$V-A$
$\varrho = 0.752$	0.745 ± 0.025	3/4
$\delta = 0.756$	0.78 ± 0.05	3/4
$\xi = 1.01$	0.97 ± 0.5	1

iv) $K_0^L \to \mu^+ + \mu^-$ decay:

The decay amplitude for the decay of a K^0 state to a $\mu^+\mu^-$ state is given by

$$A(K^0 \to \mu^+\mu^-) = \mathrm{i}\, u(p)\, (F_1 + \gamma_5 F_2) v(p'). \tag{7}$$

The decay rate for $K_L^0 \to \mu^+\mu^-$ is then given by

$$\Gamma(K_L^0 \to \mu^+\mu^-) = \frac{m_k}{8\pi}\left(1 - \frac{4m_\mu^2}{m_k^2}\right)^{1/2} |F_2|^2. \tag{8}$$

We compute all the contributions (up to the 4th order in g) to F_2 and calculate the Padé approximants as before. The experimental $K_L^0 \to \gamma\gamma$ and $K_L^0 \to \pi\mu\nu$ decay form-factor as well as the soft pion theorem for $K_L^0 \to \pi\pi\gamma$ decay are needed to compute numerical answers for F_2. All such form-factors are assumed to be momentum independent although they occur inside intermediate momentum integrations in higher order graphs. (The reason for this ad hoc assumption is simply the non availability of experimental data regarding the momentum dependence of these form-factors). Our results are

$F_2^{(1,1)}(g, 19.5\text{ GeV}) = 0.26 \times 10^{-11}$
$F_2^{(1,2)}(g, 7.5\text{ GeV}) = F_2^{(2,1)}(g, 7.5\text{ GeV}) = 0.31 \times 10^{-11}$
$F_2^{(2,2)}$ no stationary feature.

The relative branching ratio for $K_L^0 \to \mu^+\mu^-$ decay turns out to be

$$\frac{(K_L^0 \to \mu^+\mu^-)}{K_L^0 \to \text{all}} = \begin{matrix} 0.95 \times 10^{-8} \text{ from } F_2^{(1,1)} \\ 1.35 \times 10^{-8} \text{ from } F_2^{(1,2)} \text{ and } F_2^{(2,1)} \end{matrix}$$

The agreement between these two results and the recent experiments which give a value of $1-3 \times 10^{-8}$ for the above branching ratio is seen to be good.

I have described above an alternative procedure for regularizing perturbation series of non-renormalizable quantum field theories. Judging from the experimentally verifiable consequences above, the procedure would seem to be good.

REFERENCES

[1] Folgi G. L., Pellicoro M. F., Villani M., Nuovo Cimento, *6 A* (1971), 79.

[2] Villani M., 2nd *Colloquium on Advanced Computing Methods in Theoretical Physics*, Marseilles (1971).

[3] Weinberg S., Phys. Rev. Lett., *19* (1967), 1264.

[4] Lee B. W., *Rapporteur talk at the Batavia Conference* (1972).

[5] Hooft G't., Nucl. Phys., *B 35* (1971), 167.

Received December 4th, 1973.