UNITARY SPINORS IN THE LORENTZ GROUP[1]

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Matrix elements of the principal series of unitary representations of the Lorentz group are investigated. For this purpose, the notion of the finite-dimensional spinor is generalized to the unitary case. In this basis representations take a form analogous to the three-dimensional rotation group and are much simpler than those in angular momentum basis.

I. INTRODUCTION

The classification of all linear representations of the Lorentz group (LG) and the solution of related fundamental problems is now well established [1, 2, 3]. Yet in the investigations connected with relativistic expansions it has turned out that evaluation of the matrix elements of the unitary representations (UR) is far from trivial: the results obtained in the O(3) basis could be given in terms of multiple sums over complicated expressions. As a consequence of the fact that the bases O(2, 1) and E(2) can be obtained from the O(3) basis by means of a deformation procedure within the LG, it is not surprising that the UR do not assume a simple form in any of these bases either [4—8].

In the present paper, which is a modified version of a series of works cited in [9] it is shown that if the UR are derived in a basis that is a generalization of the finite-dimensional spinors to the unitary case [9—12], the matrix elements assume a considerably simpler form that in any of the above bases.

A simple derivation of representations in the unitary spinor basis is rendered possible by making use of the familiar form of the Lie algebra in which infinitesimal generators satisfy the commutation relations of two independent angular momenta. In this case parameters corresponding to the two angular momenta, are pairwise complex conjugate to each other, hence the eigenvalue equations of the two Casimir operators are independent, and therefore four

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linearly independent solutions of the differential equations can by built up immediately — in contrast to the case of the canonical parametrization and the angular momentum basis, where the above two equations are intimately linked.

By taking into account the regularity requirements imposed on the representations, a quantization condition is obtained for j_0, the first quantum number characterizing an irreducible representation. The other quantum number, σ, is continuous and takes real values for the principal series of unitary representations.

Throughout the paper only this series of the UR will be treated.

This is sufficient from the point of view of the expansions of square integrable functions on the LG, since according to a theorem by Gelfand and Neumark [3] any such function can be expanded in terms of the UR of the principal series only, and the supplementary series does not make any contribution.

II. EVALUATION OF THE MATRIX ELEMENTS OF UNITARY REPRESENTATIONS

Let us denote infinitesimal generators of rotations about and boosts along the k^{th} axis by M_k and N_k ($k = 1, 2, 3$). Then the linear combinations

$$\mathbf{J} = \tfrac{1}{2}(\mathbf{M} + \mathrm{i}\mathbf{N}), \qquad \mathbf{K} = \tfrac{1}{2}(\mathbf{M} - \mathrm{i}\mathbf{N}) \tag{1}$$

satisfy the Lie algebra of two independent angular momenta:

$$[J_i, J_k] = \mathrm{i}\varepsilon_{ikl} J_l, \quad [K_i, K_k] = \mathrm{i}\varepsilon_{ikl} K_l, \quad [J_i, K_k] = 0. \tag{2}$$

Since in Eq. (1) infinitesimal generators have been multiplied by complex numbers, the parameters corresponding to them fail to remain real. If the parameters and infinitesimal generators of a Lie group are denoted by ε^A and X_A, the required restriction on the parameters can be obtained from the well-known fact that the bilinear form $\varepsilon^A X_A$ is invariant under different parametrizations of the group as well as different choices of the basis in the algebra. The restriction obtained in this way is that parameters corresponding to J_k and K_k should be complex conjugate to each other. Furthermore, since for UR the infinitesimal generators conjugate to real angles have to be hermitian, the condition of unitarity is

$$J_k = K_k^+. \tag{3}$$

The form of the Lie algebra of the LG as given by Eq. (2) exhibits the isomorphism between the proper Lorentz group and the connected part of the three-dimensional complex rotation group SO(3, C) [13]. In view of this

isomorphism, it is natural to parametrize the LG by means of complex Euler angles as follows:

$$g = C_3(\varphi)C_2(\vartheta)C_3(\psi), \tag{4}$$

where a complex rotation about the k^{th} axis is defined as

$$C_k(\alpha) = \mathrm{e}^{-\mathrm{i}\alpha J_k}\mathrm{e}^{-\mathrm{i}\alpha^* K_k} = \mathrm{e}^{-\mathrm{i}(\alpha_1 M_k - \alpha_2 N_k)},$$

i.e. the real and imaginary parts of the complex angle $\alpha = \alpha_1 + \mathrm{i}\alpha_2$ describe a spatial and a hyperbolic rotation about the same axis.

For the LG the ranges of the complex Euler angles

$$\varphi = \varphi_1 + \mathrm{i}\varphi_2, \quad \vartheta = \vartheta_1 + \mathrm{i}\vartheta_2, \quad \psi = \psi_1 + \mathrm{i}\psi_2$$

are given by the inequalities

$$0 \leq \varphi_1, \quad \psi_1 < 2\pi, \quad 0 \leq \vartheta_1 < \pi, \quad -\infty < \varphi_2, \vartheta_2, \psi_2 < \infty.$$

For the SL(2, C) group the range of ψ_1 should be modified to $-2\pi \leq \psi_1 < 2\pi$.

It should be noted that parametrization of both the Lorentz and the LC(2,C) groups by means of complex Euler angles is valid in the group space almost everywhere, i.e. apart from a set of zero measure.

For two independent Casimir operators of the LG we can choose $\mathbf{j}^2$ and $\mathbf{K}^2$. The corresponding eigenvalues are conveniently written in the form

$$\mathbf{J}^2 \to j(j+1), \quad \mathbf{K}^2 \to k(k+1),$$

where

$$j = \tfrac{1}{2}(j_0 - 1 + \mathrm{i}\sigma), \quad k = -j^* - 1 = \tfrac{1}{2}(-j_0 - 1 + \mathrm{i}\sigma). \tag{5}$$

For the time being this is merely a definition, but it will turn out that as a consequence of the regularity requirements imposed on the representations, j_0 can take only integer or half-integer values.

Since the LG is treated here as a three-dimensional complex rotation group, it is natural to reduce irreducible representations according to those of the complex rotation group about the 3rd axis, O(2, C) = O(2) × O(1,1). This requires that the basis introduced be an eigenfunction of the generators J_3 and K_3:

$$J_3\,|m, m^*\rangle = m\,|m, m^*\rangle, \quad K_3\,|m, m^*\rangle = m^*\,|m, m^*\rangle. \tag{6}$$

If the eigenvalue m is decomposed according to Eq. (1) into real and imaginary parts

$$m = \tfrac{1}{2}(\mu + \mathrm{i}\nu), \quad m^* = \tfrac{1}{2}(\mu - \mathrm{i}\nu), \tag{7}$$

it is seen that μ will take integer (half-integer) values: for the LG(SL(2, C) group) at the same time ν ranges over a continuous spectrum. This basis is a straightforward generalization of the finite-dimensional spinors to the unitary case: the eigenvalues m and m^* correspond to undotted and dotted indices of spinors. Nevertheless, unlike the case of finite-dimensional spinors, matrix elements of the infinitesimal generators fail to remain classical functions but instead become distributions. This is a consequence of non-compactness of the subgroup O(1,1) generated by the N_3. For details we refer to [10].

Having introduced the parametrization and the basis, the task is now to represent the infinitesimal generators J_k and K_k in terms of complex Euler angles and afterwards to solve the eigenvalue equation for the operators $\boldsymbol{J}^2$ and $\boldsymbol{K}^2$. As a result of a straigtforward, but somewhat lengthy calculation we obtain the following eigenvalue equations for the matrix elements of unitary representations:

$$\left[\frac{1}{\sin^2\vartheta}\left(\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial\psi^2}-2\cos\vartheta\frac{\partial^2}{\partial\varphi\partial\psi}\right)+\frac{\partial^2}{\partial\vartheta^2}+\cot\vartheta\frac{\partial}{\partial\vartheta}+j(j+1)\right]\times$$
$$\times\, T^{jj^*}_{mm^*;nn^*}(\varphi,\vartheta,\psi)=0 \tag{8}$$
$$\left[\frac{1}{\sin^2\vartheta^*}\left(\frac{\partial^2}{\partial\varphi^{*2}}+\frac{\partial^2}{\partial\psi^{*2}}-2\cos\vartheta^*\frac{\partial^2}{\partial\varphi^*\partial\psi^*}\right)+\frac{\partial^2}{\partial\vartheta^{*2}}+\cot\vartheta^*\frac{\partial}{\partial\vartheta^*}+j^*(j^*+1)\right]\times$$
$$\times\, T^{jj^*}_{mm^*;nn^*}(\varphi,\vartheta,\psi)=0.$$

For real variables these equations are easily recognizable as differential equations of the D-functions of the real rotation group.

The general solution of (8) can be written in the form

$$T^{jj^*}_{mm^*;nn^*}(\varphi,\vartheta,\psi)=e^{-i(m\varphi+m^*\varphi^*+n\psi+n^*\psi^*)}R^{jj^*}_{mm^*;nn^*}(\vartheta,\vartheta^*), \tag{9}$$

where

$$R^{jj^*}_{mm^*;nn^*}(\vartheta,\vartheta^*)=c_1d^*g+c_2dg^*+c_3d^*d+c_4g^*g. \tag{10}$$

Here $d\equiv d^j_{mn}(\cos\vartheta)$ is well known from the theory of real rotation group. For $\mathrm{Re}(m\pm n)\geqslant 0$ we define these by means of the same formula as usual for real values of m and n:

$$d^j_{mn}(z)=n^j_{mn}\left(\frac{1-z}{2}\right)^{\frac{m-n}{2}}\left(\frac{1+z}{2}\right)^{\frac{m+n}{2}}{}_2F_1\left(-j+m,j+m+1;m-n+1;\frac{1-z}{2}\right) \tag{11}$$

where $z=\cos\vartheta$ and

$$n^j_{mn}=\frac{1}{\Gamma(m-n+1)}\sqrt{\frac{\Gamma(j+m+1)\,\Gamma(j-n+1)}{\Gamma(j-m+1)\,\Gamma(j+n+1)}},\ (\mathrm{Re}(m\pm n)\geqslant 0).$$

We shall return later to their definition in the general case.

As the second kind function we introduce

$$g\equiv g^j_{mn}(z)=\frac{\sin\pi(j-m)}{\sin\pi(j-n)}d^j_{nm}(z). \tag{12}$$

If the above d-functions are used as the first kind functions, then the second kind functions e^j_{mn} introduced in [14] are inconvenient for our purposes, which is why in Eq. (11) we introduced g^j_{mn} proportional to d^j_{nm}. If m and n take simultaneously integer or half-integer values, g is not linearly independent from d. In the complex case, however, their Wronskian takes the form:

$$W[d,g]=\frac{2}{1-z^2}\,\frac{\sin\pi(j-m)}{\sin\pi(j-n)}\,\frac{1}{\Gamma(m-n+1)}\,\frac{1}{\Gamma(n-m)},\quad(\mathrm{Re}(m\pm n)\geqslant 0),$$

i.e. d and g are linearly independent, apart from the singular points $n-m==0,-1,-2,\ldots$.

The constants c_1, c_2, c_3, c_4 in Eq. (10) should be determined from the requirement of finiteness of the R-function at the singular points $z=\cos\vartheta=\pm 1,\infty$. A further requirement is that the matrix elements should approach the unit matrix as $\varphi\to 0$, $\vartheta\to 0$, $\psi\to 0$. In the present case the unit matrix contains a Dirac delta as well, due to the continuous spectrum of ν. This requirement is not trivial at all, and it forces $c_3=0$.

The regularity at $z=1$ rules out the term g^*g as well i.e. $c_4=0$.

At the infinity we have two dangerous terms, which vanish if

$$e^{i\pi(m^*-n^*)}\frac{\sin\pi(j^*+n^*)}{\sin\pi(j^*+m^*)}c_1+e^{-i\pi(m-n)}\frac{\sin\pi(j+n)}{\sin\pi(j+m)}c_2=0,$$
$$e^{i\pi(m^*-n^*)}\frac{\sin\pi(j^*-n^*)}{\sin\pi(j^*+m^*)}c_1+e^{-i\pi(m-n)}\frac{\sin\pi(j-n)}{\sin\pi(j-m)}c_2=0.$$

These equations have a non trivial solution if

$$\frac{\sin\pi(j+m)\sin\pi(j-n)}{\sin\pi(j-m)\sin\pi(j+n)}=\frac{\sin\pi(j^*+m^*)\sin\pi(j^*-n^*)}{\sin\pi(j^*-m^*)\sin\pi(j^*+n^*)}. \tag{13}$$

The crucial point is that this is satisfied if j_0 defined by Eq. (5) takes integer or half-integer values along with μ, the eigenvalue of M_3. It is obvious that half-integer values of μ and j_0 are allowed only for the SL(2, C) group.

By choosing an appropriate normalization factor, the R-function assumes the form

$$R^{jj^*}_{mm^*;nn^*}(\vartheta, \vartheta^*) = \frac{N^j_{mn}}{4\mathrm{i}\sqrt{\sin\pi(m-n)\sin\pi(m^*-n^*)}} \times$$
$$\times \left[d^j_{mn}(z)^* g^j_{mn}(z) - d^j_{mn}(z) g^j_{mn}(z)^*\right] \tag{14}$$

where

$$N^j_{mn} = \sqrt{\frac{\sin\pi(j-n)\sin\pi(j^*-n^*)}{\sin\pi(j-m)\sin\pi(j^*-m^*)}}\,. \tag{15}$$

It can be shown in a simple but not trivial way that as $z = \cos\vartheta \to 1$ the representations approach the unit matrix, i.e.

$$\lim_{z\to 1} R^{jj^*}_{mm^*;nn^*}(\vartheta, \vartheta^*) = \delta_{\mu\varkappa}\delta(\nu - \lambda),$$

where

$$m = \tfrac{1}{2}(\mu + \mathrm{i}\nu), \quad n = \tfrac{1}{2}(\varkappa + \mathrm{i}\lambda).$$

For details we refer to [10].

Up to now the representations have been given under the restriction $\mathrm{Re}(m \pm n) \geqslant 0$, but the generalization for arbitrary values of m and n is straightforward enough. The definitions of d^j_{mn}, g^j_{mn} and $R^{jj^*}_{mm^*;nn^*}$ functions given by Eqs. (11), (12) and (14) should be extended by substituting:

$$m \to M = \tfrac{1}{2}(||m+n|| + ||m-n||), \quad n \to N = \tfrac{1}{2}(||m+n|| - ||m-n||) \tag{16}$$

i. e.

$$d^j_{mn} \to d^j_{MN}, \quad g^j_{mn} \to g^j_{MN}, \quad R^{jj^*}_{mm^*;nn^*} \to R^{jj^*}_{MM^*;NN^*},$$

where the symbol $||u||$ is defined for a complex u as

$$||u|| = u \,\mathrm{sign}\, \mathrm{Re}u = \begin{cases} u & \text{if} \quad \mathrm{Re}u \geqslant 0 \\ -u & \text{if} \quad \mathrm{Re}u < 0. \end{cases}$$

We would like to stress the importance of this point. In the real case, i. e. when both m and n take integer or half-integer values, the d^j_{mn} and e^j_{mn} functions (see [14]) are usually defined first for $m + n \geqslant 0$, $m - n \geqslant 0$ and afterwards, by making use of the symmetry properties for these functions, their definition is extended to the remaining values of m and n. However, these symmetry properties fail to hold for complex values of m and n. Therefore, having ascertained that the solutions of (11), (12) and (14) are valid only for $\mathrm{Re}(m+n) \geqslant 0$, $\mathrm{Re}(m-n) \geqslant 0$ by means of the regularity requirements imposed on $R^{jj^*}_{mm^*;nn^*}$ at the singular points, for the remaining values of m and n we have to see another pair of solutions which satisfies the regularity

requirements in this case as well. The result of this procedure is summarized in the above substitution.

Thus, the final form of UR in the unitary spinor basis is

$$T^{jj^*}_{mm^*;nn^*}(\varphi, \vartheta, \psi) = \mathrm{e}^{-\mathrm{i}(m\varphi + m^*\varphi^* + n\psi + n^*\psi^*)} R^{jj^*}_{MM^*;NN^*}(\vartheta, \vartheta^*). \tag{17}$$

The resemblance to the representations of the rotation group is striking and is due to the earlier mentioned isomorphism between the proper Lorentz group and SO(3, C).

The hypergeometric function which defines the d^j_{mn} and g^j_{mn} function in Eqs. (11) and (12) converges within the circle $|\sin^2\vartheta/2| = |(1-z)/2| < 1$, but its analytic continuation is well known. For instance, if the behaviour of the representations at the infinity is required, it is convenient to use the second kind functions introduced in [14] for integer (half-integer) values of m and n as

$$e^j_{mn}(z) = A^j_{mn}\left(\frac{1+z}{2}\right)^{\frac{m+n}{2}}\left(\frac{1-z}{2}\right)^{\frac{-(m-n)}{2}}\left(\frac{z-1}{2}\right)^{-j-n-1} \times$$
$$\times\, {}_2F_1\left(j+m+1, j+n+1, 2j+2; \frac{2}{1-z}\right)$$

with

$$A^j_{mn} = \frac{1}{2}\frac{1}{\Gamma(2j+2)}\left[\Gamma(j+m+1)\Gamma(j-m+1)\Gamma(j+n+1)\Gamma(j-n+1)\right]^{1/2}.$$

The same form will be accepted as the definition of the functions for complex values of m and n, if $\mathrm{Re}(m \pm n) \geqslant 0$. In the remaining cases we extend this definition by substituting $m \to M$, $n \to N$, where M and N are related to m and n again through Eq. (15). We shall need also a function f^j_{MN}, which is defined as

$$f^j_{MN}(z) = e^{-j-1}_{MN}(z).$$

The analytic continuation of the hypergeometric function ${}_2F_1$ relates the d^j_{MN} and g^j_{MN} functions to those of e^j_{MN} and f^j_{MN}:

$$d^j_{MN}(z) = \mathrm{e}^{\mp\mathrm{i}\pi(M-N)}\,\frac{2}{\pi}\,\frac{\sin\pi(j-M)\sin\pi(j+N)}{\sin 2\pi j}\left[e^j_{MN}(z) - \xi^j_{MN} f^j_{MN}(z)\right]$$

$$g^j_{MN}(z) = \frac{2}{\pi}\,\frac{\sin\pi(j+M)\sin\pi(j-M)}{\sin 2\pi j}\left[e^j_{MN}(z) - \xi^j_{MN} f^j_{MN}(z)\right],$$

$$\left(\mathrm{Im}\left(\frac{z-1}{2}\right) \gtrless 0\right),$$

where

$$\xi^j_{MN} = \left[\frac{\sin\pi(j+M)\sin\pi(j-N)}{\sin\pi(j-M)\sin\pi(j+N)}\right]^{1/2}.$$

Taking into account the quantization (13), the R-function can be expressed in terms of e and f as

$$R^{jj^*}_{MM^*;NN^*}(\vartheta,\vartheta^*) = S^{jj^*}_{MM^*;NN^*}\left(e^j_{MN}(z)^* f^j_{MN}(z) - e^j_{MN}(z) f^j_{MN}(z)^*\right),$$

$$S^{jj^*}_{MM^*;NN^*} = \frac{1}{i\pi^2} e^{\mp i\pi(M-N)} \frac{\sin\pi(j^*+M^*)}{\sin 2\pi j^*} \times$$

$$\times \left[\frac{\sin\pi(j+N)\sin\pi(j^*-N^*)\sin\pi(j^*-M^*)\sin\pi(M-N)}{\sin\pi(j+M)\sin\pi(M^*-N^*)}\right]^{1/2},$$

$$\left(\mathrm{Im}\left(\frac{z-1}{2}\right) \gtrless 0\right).$$

This form exhibits the behaviour of the representations at the infinity $\cos\vartheta = \infty$. There remains one more singular point: $z = \cos\vartheta = -1$. Again, the analytic continuation of the hypergeometric function provides linear ralations between $d^j_{MN}(z)$, $g^j_{MN}(z)$ and $d^j_{M,-N}(-z)$, $g^j_{M,-N}(-z)$:

$$d^j_{MN}(z) = \frac{\sin\pi(j+N)}{\sin\pi(M+N)}\left[d^j_{M,-N}(-z) - g^j_{M,-N}(-z)\right],$$

$$g^j_{MN}(z) = \frac{\sin\pi(j+M)\sin\pi(j-M)}{\sin\pi(j-N)\sin\pi(M+N)} d^j_{M,-N}(-z) - \frac{\sin\pi(j+N)}{\sin\pi(M+N)} g^j_{M,-N}(-z).$$

In terms of these functions the R-function read

$$R^{jj^*}_{MM^*;NN^*}(\vartheta,\vartheta^*) = \frac{i}{4} N^j_{MN} \frac{\sin\pi(j^*+N^*)}{\sin\pi(j-N)\sin\pi(M^*+N^*)} \times$$

$$\times \left[\frac{\sin\pi(M-N)}{\sin\pi(M^*-N^*)}\right]^{1/2} \left[d^j_{M,-N}(-z)^* g^j_{M,-N}(-z) - d^j_{M,-N}(-z)\, g^j_{M,-N}(-z)^*\right],$$

where N^j_{MN} was given by Eq. (15).

This form of R-function exhibits the behaviour of the representations around $\cos\vartheta = -1$. The representations obtained form an orthogonal complete set over the square integrable functions on the Lorentz group. The orthogonality reads

$$\int dg\, T^{j'j'^*}_{m'm'^*n'n'^*}(g)^*\, T^{jj^*}_{mm^*;nn^*}(g) = \frac{32\pi^4}{(2j+1)(2j^*+1)} \times$$

$$\times\ \delta_{\mu'\mu}\delta_{\varkappa'\varkappa}\delta(\nu'-\nu)\delta(\lambda'-\lambda)\delta_{j'_0j_0}\delta(\sigma'-\sigma)\ (\sigma,\sigma' \geqslant 0)$$

where dg is the Haar measure on the LG:

$$dg = \left(\frac{i}{2}\right)^3 \sin\vartheta^* \sin\vartheta\, d\varphi\, d\varphi^*\, d\vartheta\, d\vartheta^*\, d\psi\, d\psi^*$$

and

$$j = \tfrac{1}{2}(j_0 - 1 + i\sigma),\quad m = \tfrac{1}{2}(\mu + i\nu),\quad n = \tfrac{1}{2}(\varkappa + i\lambda)$$

$$j' = \tfrac{1}{2}(j'_0 - 1 + i\sigma'),\quad m' = \tfrac{1}{2}(\mu' + i\nu'),\quad n' = \tfrac{1}{2}(\varkappa' + i\lambda)'.$$

The factor $(2j+1)(2j^*+1)$ is the Plancherel measure, which is the complex counterpart to that of the real rotation group.

The condition of completeness can be written as

$$\frac{1}{32\pi^4}\sum_{j_0=-\infty}^{\infty}\sum_{\mu=-\infty}^{\infty}\sum_{\varkappa=-\infty}^{\infty}\int_0^{\infty} d\sigma \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\lambda (2j+1)(2j^*+1) T^{jj^*}_{mm^*;nn^*}(g') \times$$

$$\times\ T^{jj^*}_{mm^*;nn^*}(g) = \delta(g'-g).$$

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