REALIZATIONS OF THE POINCARÉ GROUP ON HOMOGENEOUS SPACES[1]

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A classification of realizations of the Poincaré group on homogeneous spaces is given.

I. INTRODUCTION

In describing physical symmetries it is a fundamental problem to find out the reperesentations of the symmetry group. Even in the case when the group is already given as a transformation group in physics the knowledge of all its representations is important to obtain the full physical content of the symmetry within a physical theory. Besides the linear representations also non-linear representations of Lie groups have been frequently considered during the last years. Since each realization of a group (a linear or a non-linear representation) is a union of transitive realizations, it is a fundamental task to classify all transitive realizations. The problem to find out all nonequivalent transitive realizations of a Lie group is equivalent to the problem of classifying all non-conjugated closed subgroups of this Lie group.

With the subgroup structure of a symmetry group we know also the possible partial symmetries.

There is not a general and parcticable method to classify all Lie subgroups of a given Lie group. The known methods are related essentially to semisimple groups or only to special single groups. Therefore we have developed a new method for the non-semisimple Poincaré group. This method relies essentially on the fact that the Poincaré group is a semidirect product of the homogeneous Lorentz group and the 4-dimensional Abelian group of space-time translations. In such a case the representation defining the semidirect product is the key to the study of the structure of the group.

The connected Lie subgroups of the semisimple homogeneous Lorentz group were classified by several authors [1—5].

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II. REALIZATIONS AND SUBGROUPS

We note some known facts from the theory of homogeneous spaces [6, 7].

i. We speak of a realization of a topological group $\mathscr{G}$ on a Hausdorff space R if to each group element $g \in \mathscr{G}$ there corresponds a homoemorphism of R onto itself

$$p \to gp \quad p \in R$$

with the associativity

$$g_1(g_2p) = (g_1g_2)p$$

and with a continuity in g and p in the mapping $(g, p) \to (gp)$ of R onto R.

ii. We speak of a *transitive* realization if for any two elements p_1 and p_2 of R there exists a group element g with

$$gp_1 = p_2.$$

R is then called a homogeneous space.

iii. Each homogeneous space R is homeomorphic to a coset space $\mathscr{G}/\mathscr{H}$ and the realization is then given by the natural action of $\mathscr{G}$ on $\mathscr{G}/\mathscr{H}$. On the other hand each cosed space $\mathscr{G}/\mathscr{H}$ is a homogeneous space with respect to the natural action of the group elements $g \in \mathscr{G}$ on $\mathscr{G}/\mathscr{H}$. $\mathscr{H}$ is the group of stability of an element p_0 of R and therefore $\mathscr{H}$ is *closed.*

iv. For $\mathscr{H}' = g_0\mathscr{H}g_0^{-1}$, $g_0 \in \mathscr{G}$ the two homogeneous spaces $\mathscr{G}/\mathscr{H}'$ and $\mathscr{G}/\mathscr{H}$ are homeomorphic.

v. Each realization is a union of transitive realizations.

vi. Now we can say: For classifying all transitive realizations of a group we must classify all homogeneous spaces, that means all conjugacy classes of closed subgroups.

Now it is practically impossible to find out *all* closed subgroups of the Poincaré group including all discrete and disconnected subgroups. This problem was considered by Niederle and Mickelson [7] for the simple compact group SU(2) using a classification by Murnaghan [8], and already for this group there is no proof of completeness. Therefore we shall restrict ourselves to the Lie subgroups of the Poincaré group, more exactly to the connected Lie subgroups. The mean reason for this is the one-to-one correspondence between the class of connected Lie subgroups of a Lie group $\mathscr{G}$ and the class of subalgebras H of the Lie algebra of $\mathscr{G}$.

This correspondence is induced by the exponential mapping

$$\mathscr{H} = e^H$$

A weakness in this correspondence lies in the fact that the subalgebra does not decide in general whether the corresponding subgroup will be closed or not.

III. THE METHOD OF CLASSIFICATION

Our aim is to get a table of subalgebras of the Poincaré algebra P with one and only one representative subalgebra from each conjugacy class of subalgebras of P with respect to the proper Poincaré group $\mathscr{P}_+^\uparrow$. A table of subalgebras of the Lie algebra of the *homogeneous* Lorentz group with these properties related to the conjugation with respect to $\mathscr{L}_+^\uparrow$ were given by Winternitz and Friš [2], see Table 1. We shall give here only the idea of our method and not each of the steps which are a little involved [9].

In analogy to the group elements $(l \mid d) \in \mathscr{P}$, $l \in \mathscr{L}$, $d \in \mathscr{T}_4$

$$(l_1 \mid d_1)(l_2 \mid d_2) = (l_1l_2 \mid l_1d_2 + d_1)$$

$$(l \mid d)^{-1} = (l^{-1} \mid -l^{-1}d)$$

Table 1

Subalgebras of the Lorentz Algebra (Friš and Winternitz [2])

1-dimensional subalgebras

a. $C = \cos\alpha\, A_1 + \sin\alpha\, B_1$ $\qquad 0 \le \alpha < \pi$
b. $A = A_1 + B_2$

2-dimensional subalgebras

a. A_1, B_1
b. $A_1 + B_2, A_2 - B_1$
c. $A_1 + B_2, -B_3$

3-dimensional subalgebras

a. A_1, A_2, A_3 — semisimple, Lie algebra of SO (3)
b. B_1, B_2, A_3 — semisimple, Lie algebra of SO (2, 1)
c. $A = A_1 + B_2$, $B = A_2 - B_1$,
$C_\alpha = \cos\alpha\, A_3 + \sin\alpha\, B_3$, $\qquad 0 \le \alpha < \pi$
$[A, B] = 0$ — This algebra is for $\alpha = 0$ isomorphic to the Lie algebra of E (2).
$[B, C_\alpha] = \cos\alpha \,.\, A - \sin\alpha \,.\, B$
$[C_\alpha, A] = \cos\alpha \,.\, B + \sin\alpha \,.\, A$

4-dimensional subalgebra

$A_1, B_1, C = A_3 + B_2, D = A_2 - B_3$
$[A_1, B_1] = 0, [A_1, C] = -D, [B_1, C] = -C$
$[C, D] = 0, [A_1, D] = C, [B_1, D] = -D$

The Lorentz algebra has no 5-dimensional subalgebra.

we can write each element of the Poincaré algebra in the form $\langle A \mid t\rangle = A + t$, $A \in L$, $t \in T_4$, which points at the semidirect sum

$$P = L \oplus_s T_4.$$

We have then

$$[\langle A' \mid t'\rangle, \langle A'' \mid t''\rangle] = \langle [A', A''] \mid [A', t''] - [A'', t']\rangle. \tag{1}$$

The representation defining the semidirect sum is here the adjoint representation of the elements of the Lorentz algebra L in the space of the translation elements. We have no mixing of homogeneous and translation elements in the homogeneous part.

For a subalgebra H of P we can choose a basis

$$\langle A^1 \mid t^1\rangle, \langle A^2 \mid t^2\rangle, \ldots, \langle A^{n_0} \mid t^{n_0}\rangle, \langle 0 \mid t^{n_0+1}\rangle, \ldots, \langle 0 \mid t^n\rangle$$

where $A^1, \ldots, A^{n_0}$ give a basis of a *subalgebra* H_0 of the Lorentz algebra L. H_0 is called the homogeneous part of H.

We remark that in general $t^1, \ldots, t^{n_0}$ cannot be made zero by basis transformation. This comes from the fact that among the subalgebras H_0 of the semisimple Lorentz algebra there occur non-semisimple Lie algebras.

Using the relation

$$g e^X g^{-1} = e^{gXg^{-1}},$$

it is clear that conjugated subgroups

$$\mathscr{H}' = g\mathscr{H}g^{-1},$$

have conjugated Lie algebras

$$H' = gHg^{-1}.$$

We have

$$(l \mid d)\langle A \mid t\rangle(l \mid d)^{-1} = \langle lAl^{-1} \mid -lAl^{-1}d + lt\rangle. \tag{2}$$

This relation can be checked easily in the 5×5-matrix representation

$$(l \mid d) \leftrightarrow \left(\begin{array}{c|c} l_{\mu\nu} & \begin{matrix} d_1 \\ \cdot \\ \cdot \\ d_4 \end{matrix} \\ \hline 0\ 0\ 0\ 0 & 1 \end{array}\right), \quad \langle A \mid t\rangle \leftrightarrow \left(\begin{array}{c|c} A_{\mu\nu} & \begin{matrix} t_1 \\ \cdot \\ \cdot \\ t_4 \end{matrix} \\ \hline 0\ 0\ 0\ 0 & 0 \end{array}\right),$$

which gives a faithful representation of the Poincaré group and the Poincaré algebra.

From equation (2) we see that for H' conjugated to H with respect to $\mathscr{P}_+^{\uparrow}$ it follows that H_0' is conjugated to H_0 with respect to $\mathscr{L}_+^{\uparrow}$. Therefore the homogeneous part H_0 of H and its basis elements $A^1, \ldots, A^{n_0}$ can be thought of as taken from Table 1.

This is the starting point for our method. At first we set up the manifold

$$\langle A^1 \mid \alpha_i^1 t_i\rangle, \ldots, \langle A^{n_0} \mid \alpha_i^{n_0} t_i\rangle, \tag{3}$$

with real α_i^j, $i = 1, 2, 3, 4$; $j = 1, \ldots, n_0$.

This manifold is a manifold of bases of linear subspaces of the Poincaré algebra. But these subspaces are not yet Lie algebras in general. By an algebraic completion we get subalgebras of P

$$\langle A^1 \mid \beta_i^1 t_i\rangle, \ldots, \langle A^{n_0} \mid \beta_i^{n_0} t_i\rangle, t^{n_0+1}, \ldots, t^n, \tag{4}$$

where the β's come from the α's by restriction of their ranges caused by the occurrence of the elements $t^{n_0+1}, \ldots, t^n$.

Now we form the conjugacy classes relative to such elements of $\mathscr{P}_+^{\uparrow}$ which leave the homogeneous part H_0 invariant. For each of the disjoint conjugacy classes we write down one representative subalgebra

$$\langle A^1 \mid \gamma_i^1 t_i\rangle, \ldots, \langle A^{n_0} \mid \gamma_i^{n_0} t_i\rangle, t^{n_0+1}, \ldots, t^n, \tag{5}$$

where the ranges of the γ's are restrictions of the ranges of the β's.

After this we add step by step further translation elements to (5) and form the conjugacy classes relative to $\mathscr{P}_+^{\uparrow}$. Each conjugacy class is then represented by one subalgebra from it.

IV. RESULTS

Starting from the subalgebras of the Lorentz algebra we get by such a constructive method the Table 2 of subalgebras of the Poincaré algebra with one and only one subalgebra from each conjugacy class with respect to $\mathscr{P}_+^{\uparrow}$. (The one-dimensional subalgebras of P are written down in addition to the complete table at the beginning of Table 2). A_i denotes an infinitesimal rotation around the x_i-axis. B_i denotes an infinitesimal pure Lorentz transformation in the (x_i, x_4)-plane. The infinitesimal null rotation or so-called singular Lorentz transformations are denoted by A, B, C, D without indices. $\langle A_1 \mid \lambda\ t_1\rangle$ gives an infinitesimal screw that means an infinitesimal rotation coupled with an infinitesimal translation. Figures 1–3 show lattices of subalgebras with elements of this type. $A_1 \to A_2$ means that the algebra A_2 or a conjugated algebra is a subalgebra of A_1.

By the one-to-one correspondence of subalgebras and connected subgroups,

Table 2

Subalgebras of P

Different parameters and different signs denote different conjugacy classes. Notation: $\langle A|t\rangle = A + t$, T_4 = Lie algebra of the translation group

One-dimensional subalgebras of P:

A_1		$A_1 + B_2$
$\langle A_1 \mid \lambda t_1\rangle$	$0 < \lvert\lambda\rvert < \infty$	$\langle A_1 + B_2 \mid \pm t_1\rangle$
$\langle A_1 \mid \lambda_1 t_4\rangle$	$0 < \lambda_1 < \infty$	$\langle A_1 + B_2 \mid t_4 - t_3\rangle$
$\langle A_1 \mid \pm(t_4 + t_1)\rangle$		
B_1		t_1
$\langle B_1 \mid \lambda_1 t_2\rangle$	$0 < \lambda_1 < \infty$	t_4
C_α	$0 < \alpha < \pi,\ \alpha \neq \frac{\pi}{2}$	$t_4 + t_1$

Subalgebras of P (arranged relatively to their homogeneous part):

Ia) A_1	
A_1, t_1	$\langle A_1 \mid \pm(t_4 + t_1)\rangle$
A_1, t_4	$\langle A_1 \mid \lambda t_1\rangle, t_4$
$A_1, t_4 + t_1$	$\langle A_1 \mid \lambda_1 t_4\rangle, t_1$
A_1, t_2, t_3	$\langle A_1 \mid \pm(t_4 + t_1)\rangle, t_4 - t_1$
A_1, t_1, t_4	$\langle A_1 \mid \lambda t_1\rangle, t_2, t_3$
A_1, t_1, t_2, t_3	$\langle A_1 \mid \lambda_1 t_4\rangle, t_2, t_3$
A_1, t_4, t_2, t_3	$\langle A_1 \mid \pm(t_4 + t_1)\rangle, t_2, t_3$
$A_1, t_4 + t_1, t_2, t_3$	$\langle A_1 \mid \lambda t_1\rangle, t_4, t_2, t_3$
A_1, T_4	$\langle A_1 \mid \lambda_1 t_4\rangle, t_1, t_2, t_3$
$\langle A_1 \mid \lambda t_1\rangle$	$\langle A_1 \mid \pm(t_4 + t_1)\rangle, t_4 - t_1, t_2, t_3$
$\langle A_1 \mid \lambda_1 t_4\rangle$	**With:** $0 < \lvert\lambda\rvert < \infty$, $0 < \lambda_1 < \infty$
Ib) B_1	
B_1, t_2	B_1, T_4
$B_1, t_4 + t_1$	$\langle B_1 \mid \lambda t_2\rangle$
B_1, t_1, t_4	$\langle B_1 \mid t_2\rangle, t_3$
B_1, t_2, t_3	$\langle B_1 \mid \lambda t_2\rangle, t_4 + t_1$
$B_1, t_4 + t_1, t_2$	$\langle B_1 \mid \lambda t_2\rangle, t_1, t_4$
$B_1, t_4 + t_1, t_2, t_3$	$\langle B_1 \mid \lambda t_2\rangle, t_4 + t_1, t_3$
B_1, t_4, t_1, t_2	$\langle B_1 \mid \lambda t_2\rangle, t_1, t_4, t_3$
	With: $0 < \lambda < \infty$
Ic) $C_\alpha = A_1 \sin\alpha + B_1 \cos\alpha$	
C_α, t_1, t_4	$C_\alpha, t_4 + t_1, t_2, t_3$
C_α, t_2, t_3	C_α, T_4
$C_\alpha, t_4 + t_1$	**With:** $0 < \alpha < \pi,\ \alpha \neq \frac{\pi}{2}$

Id) $A = A_1 + B_2$	A, T_4
A, t_1	$\langle A \mid t_4 - t_3\rangle$
$A, t_4 + t_3$	$\langle A \mid t_4 - t_3\rangle, t_1$
$A, t_4 + t_3, t_2$	$\langle A \mid t_4 - t_3\rangle, t_4 - t_3^2$
$A, t_4 + t_3, t_1$	$\langle A \mid t_4 - t_3\rangle, t_1 \sin\beta + t_2 \cos\beta, t_4 + t_3$
$A, t_4 + t_3, t_1 \sin\alpha + t_2 \cos\alpha$	$\langle A \mid t_4 - t_3\rangle, t_4 + t_3^2$
With: $0 < \alpha < \pi,\ \alpha \neq \frac{\pi}{2}$	$\langle A \mid t_4 - t_3\rangle, t_1, t_2, t_4 + t_3$
$A, t_4 + t_3, t_1, t_2$	$\langle A \mid \pm t_1\rangle$
$A, t_4 + t_3, t_4 - t_3, t_2$	$\langle A \mid \pm t_1\rangle, t_4 + t_3$
	$\langle A \mid \pm t_1\rangle, t_2, t_4 + t_3$
	$\langle A \mid \pm t_1\rangle, t_2, t_4 + t_3, t_4 - t_3$
IIa) A_1, B_1	
$A_1, B_1, t_4 + t_1$	A_1, B_1, t_2, t_3
A_1, B_1, t_4, t_1	$A_1, B_1, t_2, t_3, t_4 + t_1$
	A_1, B_1, T_4
IIb) $A = A_1 + B_2,\ B = A_2 - B_1$	
$A, B, t_4 + t_3$	**With:** $\frac{\pi}{4} \leqslant \alpha \leqslant \frac{5\pi}{4},\ \alpha \neq \frac{3\pi}{4}$
$A, B, t_1, t_4 + t_3$	$\langle A \mid t_1\rangle, \langle B \mid -t_2\rangle$
$A, B, t_1, t_2, t_4 + t_3$	$\langle A \mid \lambda t_1\rangle, \langle B \mid (t_4 - t_3)\rangle, t_2, t_4 + t_3$
A, B, T_4	**With:** $-\infty < \lambda < +\infty$
$\langle A \mid \lambda t_1\rangle, \langle B \mid t_1 + \beta t_2\rangle, t_4 + t_3$	$\langle A \mid \lambda t_1\rangle, \langle B \mid t_1 - \lambda t_2\rangle, t_4 + t_3$
With: $-\infty < \lambda, \beta < \infty,\ \lambda \neq -\beta$	**With:** $-\infty < \lambda < +\infty$
$\langle A \mid \lambda t_1\rangle, \langle B \mid t_1 - \lambda t_2\rangle$	$\langle A \mid t_1\rangle, \langle B \mid -t_2\rangle, t_4 + t_3$
With: $-\infty < \lambda < \infty$	$A, \langle B \mid t_4 - t_3\rangle, t_1, t_2, t_4 + t_3$
$\langle A \mid \cos\alpha . t_1\rangle, \langle B \mid \sin\alpha . t_2\rangle, t_4 + t_3$	$A, \langle B \mid \pm t_2\rangle, t_1, t_4 + t_3$
IIc) $A = A_1 + B_2, -B_3$	
$A, -B_3, t_1$	$A, \langle -B_3 \mid \beta_1 t_1\rangle$
$A, -B_3, t_4 + t_3$	**With:** $0 < \beta_1 < \infty$
$A, -B_3, t_2, t_4 + t_3$	$A, \langle -B_3 \mid \beta_1 t_1 + \beta_2 t_2\rangle, t_4 + t_3$
$A, -B_3, t_1, t_4 + t_3$	**With:** $0 < \beta_1 < \infty,\ 0 < \lvert\beta_2\rvert < \infty$ and $\beta_1 = 0,\ 0 < \beta_2 < \infty$
$A, -B_3, t_1 \sin\beta + t_2 \cos\beta, t_4 + t_3$	$A, \langle -B_3 \mid \lambda(\sin\alpha . t_1 - \cos\alpha . t_2)\rangle, \cos\alpha . t_1 + \sin\alpha . t_2, t_4 + t_3$
With: $0 < \beta < \pi,\ \beta \neq \frac{\pi}{2}$	**With:** $0 \leqslant \alpha < \pi,\ 0 < \lambda < \infty$
$A, -B_3, t_1, t_2, t_4 + t_3$	$A, \langle -B_3 \mid \beta_1 t_1\rangle, t_4 + t_3$
$A, -B_3, t_2, t_4 + t_3, t_4 - t_3$	**With:** $0 < \beta_1 < \infty$
$A, -B_3, T_4$	$A, \langle -B_3 \mid \beta_1 t_1\rangle, t_2, t_4 + t_3, t_4 - t_3$
	With: $0 < \beta_1 < \infty$
IIIa) A_1, A_2, A_3	
A_1, A_2, A_3, t_4	$A_1, A_2, A_3, t_1, t_2, t_3$
	A_1, A_2, A_3, T_4
IIIb) B_1, B_2, A_3	
B_1, B_2, A_3, t_3	$B_1, B_2, A_3, t_1, t_2, t_4$
	B_1, B_2, A_3, T_4

[2]In diference to my preprint TUL 33 (1970) this conjugacy class has been corrected by a hint from H. Bacry, P. Combe and P. Sorba (C. N. R. S., Marseille Oct. 72).

IIIc) $A = A_1 + B_2,\ B = A_2 - B_1,$ $C_\alpha = \cos\alpha \,.\, A_3 + \sin\alpha \,.\, B_3$ **With:** $0 \leqslant \alpha < \pi$ $A, B, C_\alpha, t_4 + t_3$ $A, B, B_3, t_1, t_4 + t_3$ $A, B, C_\alpha, t_1, t_2, t_4 + t_3$ A, B, C_α, T_4 $\langle A \mid t_1\rangle, \langle B \mid t_2\rangle, A_3, t_4 + t_3$	$\langle A \mid -t_1\rangle, \langle B \mid -t_2\rangle, A_3, t_4 + t_3$ $A, B, \langle A_3 \mid \pm (t_4 + t_3)\rangle$ $A, B, \langle B_3 \mid \lambda t_1\rangle, t_4 + t_3$ **With:** $0 < \lambda < \infty$ $A, B, \langle B_3 \mid \lambda t_1\rangle, t_2, t_4 + t_3$ **With:** $0 < \lambda < \infty$ $A, B\ \langle A_3 \mid \pm)t_4 - t_3)\rangle, t_1, t_2, t_4 + t_3$
IV) $A_1, B_1, C = A_3 + B_2, D = A_2 - B_3$ $A_1, B_1, C, D, t_4 - t_1$	$A_1, B_1, C, D, t_4 - t_1, t_2, t_3$ A_1, B_1, C, D, T_4

Translations

t_1 t_4 $t_4 + t_1$	t_1, t_2, t_3 t_4, t_1, t_2 $t_4 + t_3, t_1, t_2$
t_1, t_2 (space-like plane) t_4, t_1 (time-like plane) $t_4 + t_3, t_2$ (null-plane)	T_4

Table 2 is also a table of all conjugacy classes of connected subgroups of the Poincaré group.

Theorem: *Each connected Lie subgroup of the Poincaré group is closed.*

We shall not give the proof here, see [10]. But we shallillu strate the case of a non-closed subgroup by a known example. The 2-dimensional torus surface thought of as a group of translations on itself is given by the group SO(2)×SO(2).

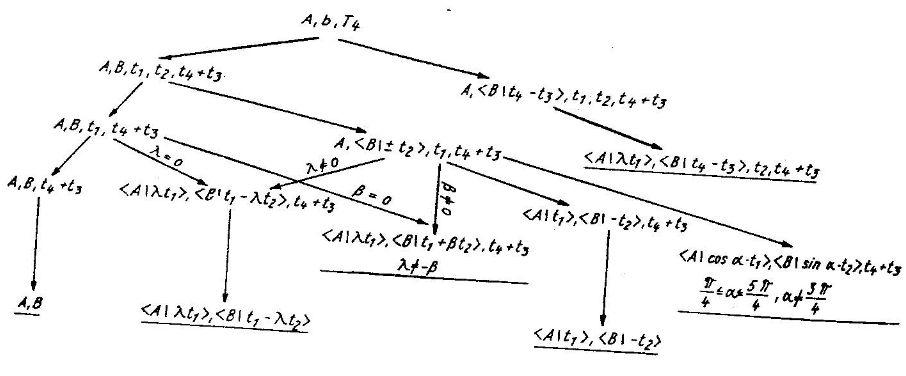

Fig. 1. Lattice of subalgebras.

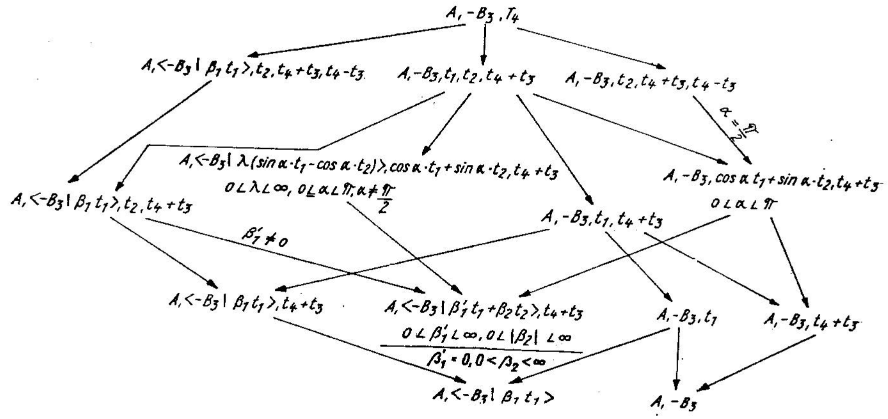

Fig. 2. Lattice of subalgebras.

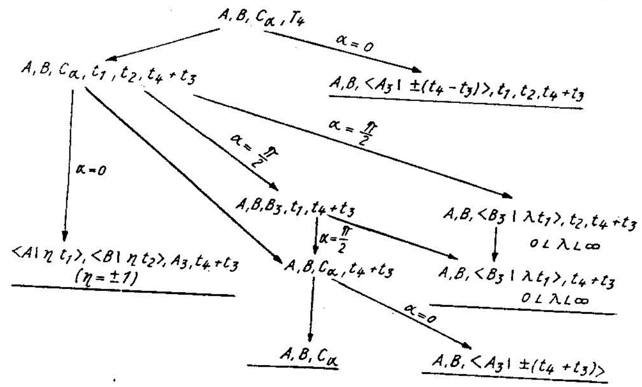

Fig. 3. Lattice of subalgebras.

The 1-dimensional subgroup of translations with an irrational slope is then an infinite screw line which is wound around the torus and which comes never back to the origin. This line is dense in the torus. The closure of this 1-dimensional subgroup is the 2-dimensional torus.

Our theorem asserts that such pathological cases do not occur among the subgroups of the Poincaré group. It is interesting that the conformal group possesses such non-closed subgroups. Its maximal compact subgroup U(2) × SU(2) has a 1-dimensional subgroup described above.

Since the coset space $\mathscr{G}/\mathscr{H}$ for a closed sugroup $\mathscr{H}$ is a Hausdorff space ([11] p. 243) using our theorem we can say: Table 2 is also a table of transitive realizations of the Poincaré group on Hausdorff spaces.

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