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THE GROUP THEORETICAL STRUCTURE OF HADRON MASS SPECTRA IN SU(2) ⊗ SU(2) DYNAMICS

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The Weinberg algebraic realization of the chiral SU(2) ⊗ (SU 2)symmetry is considered for the case when only s-wave and p-waves pions are taken into account. Making use of the generally accepted assumption of the absence of exotic states it is proved that the mass spectrum operator of hadrons behaves as a sum of scalar and a component of a 35-dimensional totally antisymmetric third rank tensor of the group SO(7).

I. INTRODUCTION

Recently Weinberg [1] has derived extremely powerful and elegant algebraic relations involving the pion-hadron decay amplitudes and hadron masses. These have following form:

$$[X^\alpha, X^\beta] = i\varepsilon^{\alpha\beta\gamma} I^\gamma \tag{1}$$

and

$$[X^\alpha, [m^2, X^\beta]] = -m_4^2 \delta^{\alpha\beta}, \tag{2}$$

where $\alpha, \beta = 1, 2, 3$ are isospin indices of the pion. The meaning of the various symbols in the previous two equations is as follows. $(X^\alpha)_{ba}$ is a matrix element in the space of the quantum numbers b and a such as isospin, spin, hypercharge, parity, etc. It is related to the invariant Feynman amplitude $M^\alpha_{ba}(p', q: p)$ for any helicity conserving transition process

$$a(p, \lambda) \to b(p', \lambda') + \pi^\alpha(q), \tag{3}$$

of the massless pion π^α by

$$M^\alpha_{ba}(p', q: p) = 2F_\pi^{-1}(m_a^2 - m_b^2)(X^\alpha)_{ba}, \tag{4}$$

where $a(p, \lambda)$ and $b(p', \lambda')$ denote hardons with momenta p and p', helicities

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λ and λ', and masses m_a and m_b, respectively. I^α is the isospin generator matrix, m^2 is the diagonal mass-squared matrix, m_4^2 is an isoscalar, and F_π is the pion decay amplitude approximately equal to 190 MeV. The matrices X^α are diagonal in helicities, i. e.

$$(X^\alpha)_{b\lambda' a\lambda} = \delta_{\lambda\lambda'}(X^\alpha)_{ba}. \tag{5}$$

The essential assumptions used by Weinberg in his derivation of the aforementioned relations were:

a. Three-graph contributions to the forward scattering amplitude of massless pions by hadrons calculated within the framework of the chirally invariant Lagrangians must not violate the asymptotic behaviour of the actual amplitude given by the Regge pole theory.

b. There should be no so-called exotic states having an isospin $I = 2$.

The algebraic relations (1) along with the standard relations involving the isospin generator matrices I^α of the isospin group $SU(2)_I$ define the Lie algebra of the chiral group SU(2) ⊗ SU(2), and this implies that hadron states must, for each helicity and various spins and isospins, form a basis for the unitary (reducible or irreducible) representation of the chiral group. The commutator (1) then determines the transition amplitudes among hadrons accomodated in the single unitary representation of the group in question. Once the matrices X^α are known they can be inserted in the second commutator in order to find the form of the mass spectrum of hadrons under consideration.

One sees that the method demonstrated by Weinberg posseses a great amount of physical appeal since it gives a scheme for calculating the pion hadron transition aplitudes and hadron mass spectra, which is, in some sense, the part of the aim of strong interaction physics.

This treatment has been extended to the SU(3) group by Ogievetsky [2], to multipion production processes by Mc Donald [3], and also to the higher chiral SU(3) ⊗ SU(3) group by Ram Mohan [4].

Unfortunately, relations (1) and (2) do not provide any information on how hadrons with different helicities are related to each other. As pointed out by Weinberg [1], the helicity and therefore spin dependence of the matrices X^α can be determined if one assumes that only a few partial waves predominate in the pion hadron transition processes (3).

In particular, transition between states of nearly the same mass and the same parity involve only p-wave pions, as, for example, in the decays $\Delta \to N\pi$, $Y_1^* \to \Lambda\pi$, $Y_1^* \to \Sigma\pi$, $\Xi^* \to \Xi\pi$ etc. Just as a starting point assume that all pion decay processes involve only p-wave pions. This implies that the matrix X^α transforms as a third component of a three-vector D_3^α, so that

$$X^\alpha \equiv D_3^\alpha, \tag{6}$$

since X^α must be diagonal in the helicities. Assuming that all commutators entering the theory do not contain any terms carrying the isospin $I = 2$, one can show that the matrices of the isospin I^α, the angular momentum J_i, and thepion-hadron coupling matrices D_i^α may form the closed Lie algebra of the group SU(4) [1], namely,

$$[I^\alpha, I^\beta] = i\varepsilon^{\alpha\beta\gamma} I^\gamma, \tag{7a}$$

$$[J_i, J_j] = i\varepsilon_{ijk} J_k, \tag{7b}$$

$$[J_i, I^\alpha] = 0, \tag{7c}$$

$$[I^\alpha, D_i^\beta] = i\varepsilon^{\alpha\beta\gamma} D^\gamma, \tag{7d}$$

$$[J_i, D_j^\alpha] = i\varepsilon_{ijk} D_k^\alpha, \tag{7e}$$

and

$$[D_i^\alpha, D_j^\beta] = i\delta_{ij}\varepsilon^{\alpha\beta\gamma} I^\gamma + i\delta^{\alpha\beta}\varepsilon_{ijk} J_k, \tag{7f}$$

where $i, j, k = 1, 2, 3$ are angular momentum indices. These commutation relations imply that the hadron states must for various isospins and spins be accomodated in an unitary (reducible or irreducible) representation of the SU(4) group.

In this approximation the second Weinberg algebraic relation (2) has the form

$$[D_3^\alpha, [m^2, D_3^\beta]] = -m_4^2 \delta^{\alpha\beta}. \tag{8}$$

Once the group structure of the pion hadron coupling matrices D_i^α is known, one is able to show that the mass matrix m^2 is given as the sum

$$m^2 = m_0^2 + m_4^2, \tag{9}$$

where m_0^2 behaves under commutation with D_i^α, J_i and I^α as an SU(4) scalar and m_4^2 transforms as a member of a 20-dimensional representation of the SU(4) group. This representation is characterized by two rows and two columns in the Young diagram. The applications of Weinberg's algebraic relations when only p-waves pions were taken into account have been intensively discussed in the works of ref. [5].

To proceed further suppose one wishes to have a more realistic situation allowing an s-wave and a p-wave pion transition, one writes then

$$X^\alpha = \sin\Theta S^\alpha + \cos\Theta D_3^\alpha, \tag{10}$$

where S^α is an isovector three-scalar matrix representating the S-wave pion interaction and Θ denotes a mixing angle between an s-wave and a p-wave pion in decay products. Weinberg has shown [1] also that the matrices S^α

along with the matrices D_i^α, I^α, J_i and with three additional matrices K_i from the closed Lie algebra of the group SO(7), namely,

$$[S^\alpha, S^\beta] = i\varepsilon^{\alpha\beta\gamma} I^\gamma, \tag{11a}$$

$$[I^\alpha, S^\beta] = i\varepsilon^{\alpha\beta\gamma} S^\gamma, \tag{11b}$$

$$[S^\alpha, D_i^\beta] = i\delta^{\alpha\beta} K_i, \tag{11c}$$

$$[S^\alpha, K_i] = -iD_i^\alpha, \tag{11d}$$

$$[D_i^\alpha, K_j] = i\delta_{ij} S^\alpha, \tag{11e}$$

$$[J_i, K_j] = i\varepsilon_{ijk} K_k, \tag{11f}$$

$$[K_i, K_j] = i\varepsilon_{ijk} K_k. \tag{11g}$$

The last set of commutation relations along with the commutators of the Lie algebra of the SU(4) group given by Eqs. (7) are in fact the Lie algebra of the group SO(7).

However, the tensorial character of the mass matrix m^2 within the SO(7) group has not been derived yet. The purpose of the present paper is to fill this gap and to show that the mass squared matrix has also simple group properties, namely, that it behaves as the sum of a scalar and a component of a 35 dimensional totally antisymmetric tensor under the SO(7) group transformations.

II. SO(7) PROPERTIES OF THE MASS MATRIX

As mentioned above it is possible to prove that the mass matrix behaves as the sum of two parts which transforms as a scalar and as a component of the 35 dimensional totally antisymmetric irreducible tensor under the SO(7) group trnaformations. The known tensorial character of the mass matrix provides the straightforward method for writing down the mass spectrum of hadrons as a sum of the Clebsch-Gordan coefficients of the group SO(7). To do this use is made of the two following definitions

$$[S^\alpha, m^2] \equiv im^\alpha \tag{12}$$

and

$$[D_3^\alpha, m^2] \equiv im_3^\alpha, \tag{13}$$

which, when the relation (10) is taken into account, are inserted in Eq. (2). This yields

$$i\sin^2\Theta[S^\alpha, m^\beta] + i\sin\Theta\cos\Theta\{[S^\alpha, m^\beta] + [D_3^\alpha, m^\beta]\} + i\cos^2\Theta[D_3^\alpha, m_3^\beta] = m_4^2\delta^{\alpha\beta}. \tag{14}$$

The 3-scalar and 3-vector parts of this equation must be separately valid, thus

$$[S^\alpha, m^\beta] = -im_4^2\delta^{\alpha\beta}, \tag{15}$$

$$[S^\alpha, m_i^\beta] = -[D_i^\alpha, m^\beta]. \tag{16}$$

In the derivation of Eqs. (15) and (16) we have used Eq. (8).
Next we consider the Jacobi identity for S^α, D_i^β, and m^2, which gives

$$[S^\alpha, m_i^\beta] + \delta^{\alpha\beta}[m_i^2, K_i] - [D_i^\beta, m^\alpha] = 0. \tag{17}$$

One can prove using the Jacobi identity for S^α, D_i^β and

$$\mu_k^\varkappa \equiv i\varepsilon^{\varkappa\delta\omega}\varepsilon_{ilk}[D_j^\delta, m_l^\omega],$$

that the commutator

$$[m^2, K_i] = 0, \tag{18a}$$

which gives rise to the following equation

$$[S^\alpha, m_i^\beta] = [D_i^\beta, m^\alpha]. \tag{18b}$$

In order to proceed further we have found it very convenient to define an antisymmetric tensor $J_{\mu\nu}$ and an isovector and four vectors X_μ^α in a four dimensional space $\mu, \nu = 1, 2, 3, 4$ as

$$J_{ik} \equiv -\varepsilon_{ijk}J_k, \quad i, j = 1, 2, 3 \tag{19a}$$

$$J_{i4} \equiv K_i, \tag{19b}$$

$$X_i^\alpha \equiv D_i^\alpha \tag{19c}$$

and

$$X_4^\alpha \equiv S^\alpha, \tag{19d}$$

where the convention has been used that the superscripts like α, β, γ label isospin indices, while subsripts like μ, ν, ϱ, σ are connected with an abstract four dimensional space. These definitions allow us to rewrite the sets of commutators (7) and (11) in a compact form as

$$[I^\alpha, I^\beta] = i\varepsilon^{\alpha\beta\gamma}I^\gamma, \tag{20a}$$

$$[J_{\mu,\nu}, J_{\varrho\sigma}] = i(\delta_{\nu\varrho}J_{\mu\varrho} - \delta_{\nu\sigma}J_{\mu\varrho} - \delta_{\mu\varrho}J_{\nu\sigma} + \delta_{\mu\sigma}J_{\nu\varrho}), \tag{20b}$$

$$[I^\alpha, J_{\mu\nu}] = 0, \tag{20c}$$

$$[I^\alpha, X_\nu^\beta] = i\varepsilon^{\alpha\beta\gamma}X_\nu^\gamma, \tag{20d}$$

$$[J_{\mu\nu}, X_\varrho^\alpha] = i(\delta_{\nu\varrho}X_\mu^\alpha - \delta_{\mu\varrho}X_\nu^\alpha), \tag{20e}$$

and

$$[X_\mu^\alpha, X_\nu^\beta] = i\delta_{\mu\nu}\varepsilon^{\alpha\beta\gamma}I^\gamma - i\delta^{\alpha\beta}J_{\mu\nu}. \tag{20f}$$

Now consider Eq. (18a) along with the fact that the mass matrix m^2 conserves spins and the isospin, hence

$$[m^2, I^\alpha] = [m^2, J_{\mu\nu}] = 0. \tag{21}$$

This property of the mass matrix is used in the Jacobi identity for X_μ^α, X_ν^β and m^2 to get the relation

$$[X_\mu^\alpha, [X_\nu^\beta, m^2]] = [X_\nu^\beta [X_\mu^\alpha, m^2]]. \tag{22}$$

This implies that the 144 matrices

$$B_{\mu\nu}^{\alpha\beta} \equiv [X_\mu^\alpha [X_\nu^\beta, m^2]], \tag{23}$$

are symmetric with respect to the interchange of the pairs of indices (α, μ) and (β, ν) and therefore their number is reduced to 78 independent matrices. Hence, the most general decomposition of the double commutator (23) takes the form

$$[X_\mu^\alpha, [X_\nu^\beta, m^2]] = \varepsilon^{\alpha\beta\gamma\gamma}{}_{\mu\nu}A + U_{\mu\nu}^{\alpha\beta}, \tag{24}$$

where $A_{\mu\nu}^\gamma$ behaves as an 18 dimensional isovector and an antisymmetric four tensor, while $U_{\mu\nu}^{\alpha\beta}$ is a 60 dimensional symmetric isotensor and a symmetric four tensor. The restriction of Eq. (24) to $\mu = \nu = 3$ and to $\mu = \nu = 4$ must give Eq. (8) and (15). This implies

$$[X_3^\alpha, [X_3^\beta, m^2]] = U_{33}^{\alpha\beta} = \delta^{\alpha\beta}m_4^2 \tag{25a}$$

and

$$[X_4^\alpha, [X_4^\beta, m^2]] = U_{44}^{\alpha\beta} = \delta^{\alpha\beta}m_4^2. \tag{25b}$$

The last relations tell that $U_{\mu\nu}^{\alpha\beta}$ behaves like an isoscalar and an isotropic symmetric four tensor. There is only one such tensor, namely.

$$U_{\nu\mu}^{\alpha\beta} = \delta^{\alpha\beta}\delta_{\mu\nu}m_4^2, \tag{26}$$

where m_4^2 behaves as an isoscalar and a four scalar.
In order to put Eq. (24) into group theoretic terms, we define an isovector and a four vector Z_μ^α by

$$[X_\mu^\alpha, m^2] \equiv -iZ_\mu^\alpha \tag{27a}$$

and write Eq. (24) as

$$[X^\alpha_\mu, Z^\beta_\nu] = \mathrm{i}\delta^{\alpha\beta}\delta_{\mu\nu}m_4^2 + \mathrm{i}\varepsilon^{\alpha\beta\gamma}A^\gamma_{\mu\nu}. \tag{27b}$$

The matrices m_4^2 and $A^\gamma_{\mu\nu}$ can now be expressed in terms of X^α_μ and Z^β_ν as

$$m_4^2 = -\frac{\mathrm{i}}{12}[X^\alpha_\mu, Z^\alpha_\mu] \tag{28a}$$

and

$$A^\gamma_{\mu\nu} = -\frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta\gamma}[X^\alpha_\mu, Z^\beta_\nu]. \tag{28b}$$

Making use of the Jacobi identity the commutators $[X^\alpha_\mu, m_4^2]$ and $[X^\alpha_\mu, A^\gamma_{\varrho\sigma}]$ are given as follows (see the Appendix)

$$[X^\alpha_\mu, m_4^2] = -\mathrm{i}Z^\alpha_\mu \tag{29a}$$

and

$$[X^\alpha, A^\beta_{\mu\nu}] = \mathrm{i}\varepsilon^{\alpha\beta\gamma}(\delta_{\varrho\mu}Z^\gamma_\nu - \delta_{\varrho\nu}Z^\gamma_\mu) + \mathrm{i}\delta^{\alpha\beta}R_{\varrho\mu\nu}. \tag{29b}$$

Here $R_{\varrho\mu\nu}$ is defined as

$$R_{\varrho\mu\nu} \equiv -\frac{\mathrm{i}}{3}[X^\alpha_\varrho, A^\alpha_{\mu\nu}]$$

and it is an isoscalar and totaly antisymmetric four tensor of the third rank obeying the commutation relations:

$$[X^\alpha_\sigma, R_{\varrho\mu\nu}] = -\mathrm{i}(\delta_{\varrho\sigma}A^\alpha_{\mu\nu} - \delta_{\sigma\mu}A^\alpha_{\varrho\nu} + \delta_{\sigma\nu}A^\alpha_{\varrho\mu}). \tag{29c}$$

It should be also noted that matrices m_4^2, Z^α_μ, $A^\gamma_{\mu\nu}$ and $R_{\varrho\mu\nu}$ transform as the irreducible tensor under rotation in the isospin space and in the four dimensional space and therefore fulfil the standard commutation relations

$$[I^\alpha, m_4^2] = [J_{\mu\nu}, m_4^2] = [I^\alpha, R_{\mu\nu\varrho}] = 0, \tag{30a}$$

$$[I^\alpha, A^\beta_{\mu\nu}] = \mathrm{i}\varepsilon^{\alpha\beta\gamma}A^{\mu\nu}_\gamma, \tag{30b}$$

$$[I^\alpha, Z^\beta_\mu] = \mathrm{i}\varepsilon^{\alpha\beta\gamma}Z^\gamma_\mu, \tag{30c}$$

$$[J_{\mu\nu}, Z^\alpha_\varrho] = \mathrm{i}(\delta_{\nu\varrho}Z^\alpha_\mu - \delta_{\mu\varrho}Z^\alpha_\nu), \tag{30d}$$

$$[J_{\mu\nu}, R_{\varrho\sigma\omega}] = \mathrm{i}(\delta_{\nu\varrho}R_{\mu\sigma\omega} - \delta_{\nu\sigma}R_{\mu\varrho\omega} + \delta_{\nu\omega}R_{\mu\varrho\sigma} - \delta_{\mu\varrho}R_{\nu\sigma\omega} + \delta_{\mu\sigma}R_{\nu\varrho\omega} - \delta_{\mu\omega}R_{\nu\varrho\sigma}), \tag{30e}$$

$$[J_{\mu\nu}, A^{\varrho\sigma}_\alpha] = \mathrm{i}(\delta_{\nu\varrho}A^\alpha_{\mu\sigma} - \delta_{\nu\sigma}A^\alpha_{\mu\varrho} - \delta_{\mu\varrho}A^\alpha_{\nu\sigma} + \delta_{\mu\sigma}A^\alpha_{\nu\varrho}). \tag{30f}$$

The sets of relations (29) and (30) show that the matrices m_4^2, Z^α_μ, $A^\gamma_{\mu\nu}$ and $R_{\mu\nu\varrho}$ may form a $1 + 12 + 18 + 4 = 35$, a 35-dimensional tensor of the group SO(7).

To prove this we first rewrite the commutation relations (20) in a single compact form introducing the standard notation for the generators J_{ab} of the SO(7) group by

$$J_{ab} = J_{\mu\nu}, \qquad \text{if } a, b = 1, 2, 3, 4$$

$$J_{ab} = J_{\alpha\beta} \equiv -\varepsilon^{\alpha\beta\gamma}I^\gamma, \qquad \text{if } a, b = 5, 6, 7$$

$$J_{ab} = J_{\alpha\mu} \equiv X^\alpha_\mu, \qquad \text{if } a = 5, 6, 7, \quad b = 1, 2, 3, 4$$

The commutators (20) are then rewritten as

$$[J_{ab}, J_{cd}] = \mathrm{i}(\delta_{bc}J_{ad} - \delta_{bd}J_{ac} - \delta_{ac}J_{bd} + \delta_{ad}J_{bc}). \tag{31}$$

Next we define a totally antisymmetric third rank tensor of the group SO(7) given by

$$T_{abc} \equiv T_{\alpha\beta\gamma} \equiv \varepsilon_{\alpha\beta\gamma}m_4^2, \qquad \text{if } a, b, c = 5, 6, 7$$

$$T_{abc} \equiv T_{\mu\nu\varrho} \equiv R_{\mu\nu\varrho}, \qquad \text{if } a, b, c = 1, 2, 3, 4$$

$$T_{abc} \equiv T_{\mu\alpha\nu} \equiv A^\alpha_{\mu\nu}, \qquad \text{if } a, c = 1, 2, 3, 4, \quad b = 5, 6. 7$$

and

$$T_{abc} \equiv T_{\alpha\beta\mu} \equiv \varepsilon_{\alpha\beta\gamma}Z^\gamma_\mu, \qquad \text{if } a, b = 5, 6, 7, c = 1, 2, 3, 4.$$

This definition of the tensor T_{abc} is used to rewrite the sets of commutation relations (29) and (30) in a single compact form as

$$[J_{ab}, T_{cde}] = \mathrm{i}(\delta_{bc}T_{ade} - \delta_{bd}T_{ace} + \delta_{be}T_{acd} - \delta_{ac}T_{bde} + \delta_{ad}T_{bce} - \delta_{ae}T_{bcd}). \tag{32}$$

The last commutation relation proves that the matrices m_4^2, $A^\alpha_{\mu\nu}$, Z^α_μ and $R_{\mu\nu\varrho}$ are components of the same 35-dimensional totally antisymmetric third rank tensor of the group SO(7).

We now complete our proof by deducing from Eqs. (21), (27a) and (30a) that the difference $m^2 - m_4^2$ commutes with J_{ab}, i. e.,

$$[J_{ab}, m^2 - m_4^2] = 0. \tag{33}$$

This relation implies that $m^2 - m_4^2$ must behave as a scalar m_0^2 under the SO(7) group transformations and hence the mass squared matrix m^2 is given as the sum

$$m^2 = m_0^2 + m_4^2, \tag{34}$$

of the SO(7) scalar m_0^2 and the component m_4^2 of the 35-dimensional totally antisymmetric third rank tensor, as was to be proved.

APPENDIX

The purpose of this appendix is to derive the commutation relations which were used in the proof of the tensorial character of the mass squared matrix. We start by considering the matrix relation (27b) which is of the form

$$[X_\mu^\alpha, Z_\nu^\beta] = \mathrm{i}\delta^{\alpha\beta}\delta_{\mu\nu}m_4^2 + \mathrm{i}\varepsilon^{\alpha\beta\gamma}A_\gamma^{\mu\nu}. \tag{A.1}$$

The matrices m_4^2 and $A_{\mu\nu}^\gamma$ are now rewritten in terms of X_μ^α and Z_ν^β as

$$m_4^2 = -\frac{\mathrm{i}}{12}[X_\mu^\alpha, Z_\mu^\alpha] \tag{A.2}$$

and

$$A_\gamma^{\mu\nu} = -\frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta\gamma}[X_\mu^\alpha, Z_\nu^\beta]. \tag{A.3}$$

The commutator $[X_\varrho^\beta, m_4^2]$ and $[X_\varrho^\beta, A_\mu^\gamma]$ can then be written by making use of the Jacobi identity as

$$[X_\varrho^\beta, m_4^2] = \frac{\mathrm{i}}{12}\{[Z_\nu^\alpha, [X_\varrho^\beta, X_\mu^\alpha]] + [X_\mu^\alpha, [Z_\nu^\alpha, X_\varrho^\beta]]\} \tag{A.4}$$

and

$$[X_\varrho^\beta, A_{\mu\nu}^\gamma] = \frac{\mathrm{i}}{2}\varepsilon^{\alpha\delta\gamma}\{[Z_\nu^\delta, [X_\varrho^\beta, X_\mu^\alpha]] + [X_\mu^\alpha, [Z_\nu^\delta, X_\varrho^\beta]]\}. \tag{A.5}$$

Carrying out the algebraic reduction by using the commutation relations (20), the following intermediate reults are obtained

$$11[X_\varrho^\beta, m_4^2] = -5\mathrm{i}Z_\varrho^\beta + \varepsilon^{\beta\alpha\gamma}[X_\mu^\alpha, A_{\varrho\mu}^\gamma] \tag{A.6}$$

and

$$2[X_\varrho^\beta, A_{\mu\nu}^\gamma] = \mathrm{i}\varepsilon^{\beta\gamma\delta}(\delta_{\varrho\mu}Z_\nu^\alpha - \delta_{\varrho\nu}Z_\mu^\alpha + \delta_{\mu\nu}Z_\varrho^\alpha) + [X_\mu^\beta, A_{\varrho\nu}^\gamma] - \delta^{\gamma\beta}[X_\mu^\alpha, A_{\varrho\nu}^\alpha]. \tag{A.7}$$

The last equation (A.7) is then used to determine the second term on the right-hand side of Eq. (A.6). This yields the result which when inserted into Eq. (A.6) gives the final form for the commutator in question, namely

$$[X_\varrho^\beta, m_4^2] = -\mathrm{i}Z_\varrho^\beta. \tag{A.9}$$

Summing over the β and γ indices in Eq. (A.7), the following relation is obtained

$$[X_\varrho^\alpha, A_{\mu\nu}^\alpha] = -[X_\mu^\alpha, A_{\varrho\nu}^\alpha], \tag{A.10}$$

which implies that a matrix $R_{\varrho\mu\nu}$ defined by

$$R_{\varrho\mu\nu} \equiv -\tfrac{1}{3}\mathrm{i}[X_\varrho^\alpha, A_{\mu\nu}^\alpha], \tag{A.11}$$

behaves as an isoscalar and totally antisymmetric third rank four tensor. Making use of the definition (A.11) along with the relation (A.7), the commutator $[X_\mu^\beta, A_{\varrho\nu}^\gamma]$ can be calculated which, when inserted into Eq. (A.7), yields the result

$$[X_\varrho^\alpha, A_{\mu\nu}^\beta] = \mathrm{i}\varepsilon^{\alpha\beta\gamma}(\delta_{\varrho\mu}Z_\nu^\gamma - \delta_{\varrho\nu}Z_\mu^\gamma) + \mathrm{i}\delta^{\alpha\beta}R_{\varrho\mu\nu}. \tag{A.12}$$

The commutator $[X_\sigma^\beta, R_{\varrho\mu\nu}]$ can be evaluated in a similar way to complete the results which were used in the second Section.

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