acta phys. slov. 23 (1973), No. 2

OPERATOR PRODUCT EXPANSIONS AND HIGH MASS PHENOMENOLOGY[1]

WOLFGANG KUMMER,* Wien

In this lecture a description of the product expansion of operators at light-like distances is given. It is shown that the field of light-cone physics includes not only the „traditional" applications of deep inelastic electron--proton scattering and similar situations, where a large fourmomentum is transferred from an *external* lepton line. The light cone is tested as well in certain cases where an *internal* integration is dominated by cut-offs or propagators with large masses.

I. INTRODUCTION

One common feature of the general concepts which have been abstracted from field theoretic models and applied to hadron physics was always their basic simplicity. This simplicity is shared by the recently studied product expansions of operators [1]. It can be shown in the perturbation theory that the product of two field operators $A(x)B(y)$ at different space-time points can be expressed as a sum of operators $O^{(n)}(y)$ at one of these points multiplied by c-number functions depending on the distance $z = x - y$

$$A(x)B(y) \underset{z\sim 0}{=} \sum_{n=0}^{N} C_n(z)O^{(n)}(y), \tag{1}$$

where the number N of the field operators $O^{(n)}$ is finite. The $C_n(z)$ are singular with singularities determined simply from the dimensions of the fields A, B, and $O^{(n)}$ [1] and multiplied possibly with powers of logarithms. Although there exist certain applications of expansions at small distances ([1] and see below) the step towards a more general applicability has been performed by

[1] Talk given at Elementary Particle Physics Seminar at Pezinská Baba, September 22—25, 1971.

* Institut für Hochenergiephysik der Österreichischen Akademie der Wissenschaften, 1050 WIEN, Nikolsdorfergasse 18, Austria.

Brandt and Preparata [2] who suggested an analogous expansion near the light cone $z^2 \sim 0$

$$A(x)B(y) \underset{z^2\sim 0}{=} \sum_n O_n(z;\, y)(-z^2 + i\varepsilon z_0)^{-p_n}, \tag{2}$$

where we have now omitted logarithmic factors. The bilocal operators O_n can be thought of as expansions

$$O_n(z;\, y) = \sum_{i,m=0} C^{(n)}_{m,i} O^{(n)}_{i,\alpha_1\ldots\alpha_m}(y)\, z^{\alpha_1}\ldots z^{\alpha_m}. \tag{3}$$

The additional indices $\alpha_1 \ldots \alpha_m$ after the comma in $O^{(n)}_i$ describe derivatives of the local operator $O^{(n)}_i$. Eqs. (2) and (3) amount to just taking *all* (infinitely many) terms in (1) constructed from one density alone, and selecting those with the same denominator $(-z^2)^{p_n}$. The $i\varepsilon z_0$ is introduced to damp high energy intermediate states [1].

For the applications to be discussed below, (2) is assumed to hold for electromagnetic and weak *current densities*. If this is true, the vacuum expectation value of an expansion like (2) for current densities is relevant for the high energy limit of the total $e^+ + e^-$ — cross-section, the proton-proton matrix element becomes important for the Bjorken scaling limit [3] of deep inelastic electron-proton scattering [2], [4] and also for an analogous limit in exclusive reactions like $e^+ + e^- \to$ hadrons $+ \gamma$, photoproduction or electroproduction of lepton pairs of high invariant mass [5] etc. In the latter examples the T-product of (2) is to be used which entails the simple replacement $i\varepsilon z_0 \to i\varepsilon$. Another example are the nowadays very popular photon-photon reactions which may be observed in positron-electron or even better in electron-electron storage rings. In Section II we present our ansatz for current densities, Section III is devoted to the applications. For further details we refer to ref. [6].

II. CURRENT DENSITIES ON THE LIGHT-CONE

In order to obtain the general expansion which — hopefully — is valid for commutators as well as T-products etc. for light-like relative distances $z^2 \sim 0$, one may use [2] — to guide us — the phenomenology at the tip of the light-cone, $z \sim 0$, first.

It is undeniable from the success of the current algebra that local operators, transforming like the current and axial current densities of the free Dirac field, play a fundamental role even in hadron physics alone. Thus one is inclined to express the product of two densities of this type by operators which we identify again as current densities (j_μ) and axial current-densities $(j_{(5)\mu})$. For completeness we include moreover the other densities of the Dirac field as well (scalar S, pseudoscalar P, tensor T).

Hence the most general expansion near $z \sim 0$ reads for a product of conserved currents

$$j_\mu(x) j_\nu(y) \underset{z\sim 0}{=} P_{\mu\nu}[a_0 \delta^4(z) + a_1 z_\alpha j^\alpha(y)/z^2 + a_2 \delta(y)/\sqrt{z^2}] +$$

$$+ a_3 \varepsilon_{\mu\nu\alpha\beta} j^\beta_{(5)}(y) \partial^{(z)\alpha} z^{-2} + a_4 P_{\mu\nu\alpha\beta} T^{\alpha\beta}/\sqrt{z^2} + O(z^{-2}), \tag{4}$$

where

$$P_{\mu\nu} = g_{\mu\nu} \Box^{(z)} - \partial^{(z)}_\mu \partial^{(z)}_\nu$$

$$P_{\mu\nu\alpha\beta} = g_{\mu\nu} \partial^{(z)}_\alpha \partial^{(z)}_\beta - g_{\nu\alpha} \partial^{(z)}_\mu \partial^{(z)}_\beta - g_{\mu\beta} \partial^{(z)}_\nu \partial^{(z)}_\alpha + g_{\nu\alpha} g_{\mu\beta} \Box^{(z)} \tag{5}$$

and the shorthand z^2 for $-z^2 + i\varepsilon z_0$ is used. One may argue that especially the success of the current algebra — which forces the equal time commutators as derived from (4) to be finite — excludes anomalous dimensions and possible logarithmic factors in the a_i. From hermiticity, causality and C-parity it is easy to determine the symmetry coupling, if, e.g., all densities behave like octets (or nonets) in SU(3).

The transition analogous to the one from (1) to (2) is of course an assumption. Possibly it is the only manifestation of scale invariance in nature [7]. Generalizing also (5) to obtain correct current conservation on the light cone [6]

$$S_{\mu\nu} = g_{\mu\nu}(\partial^{(x)} \partial^{(y)}) - \partial^{(y)}_\mu \partial^{(x)}_\nu \tag{6}$$

$$S_{\mu\nu\alpha\beta} = g_{\mu\nu} \partial^{(y)}_\alpha \partial^{(x)}_\beta - g_{\nu\alpha} \partial^{(y)}_\mu \partial^{(x)}_\beta - g_{\mu\beta} \partial^{(y)}_\alpha \partial^{(x)}_\nu + g_{\mu\beta} g_{\nu\alpha} (\partial^{(x)} \partial^{(y)})$$

$$S^{(\pm)}_{\mu\nu\gamma} = (\varepsilon_{\nu\varrho\alpha\gamma} \partial^{\varrho(y)} S^\alpha_\mu \pm \varepsilon_{\mu\varrho\alpha\gamma} \partial^{\varrho(x)} S^\alpha_\nu)$$

one arrives at

$$j_\mu(x) j_\nu(y) \underset{z^2\sim 0}{=} S_{\mu\nu} R_1 + S_{\mu\nu\alpha\beta} R^{\alpha\beta}_2 + S^{(+)}_{\mu\nu\gamma} R^{\gamma(+)}_3 + S^{(-)}_{\mu\nu\gamma} R^{\gamma(-)}_3 \tag{7}$$

with a huge freedom in bilocal operators to be obtained in the R_i. We list a selection based on the restriction to operators without derivatives in the basic term (cf. (4))

$$R_1 = a_0 \delta^4(z) + z^\alpha j^{(1)}_\alpha / z^2 + \delta^{(1)}/\sqrt{z^2} + \ldots \tag{8}$$

$$R^{\alpha\beta}_2 = (z^\alpha j^{(2)\beta}_{(\pm)} \pm (\beta \leftrightarrow \alpha))/z^2 + z^\alpha z^\beta z^\delta j^{(3)}_\delta / z^4 + \varepsilon^{\alpha\beta\gamma\delta} z_\gamma j^{(1)}_{(5)\delta}/z^2 +$$

$$+ z^\alpha z^\beta S^{(2)}/z^{3/2} + T^{(1)\alpha\beta}/\sqrt{z^2} + (z^\alpha T^{(2)\gamma\delta}_{(\pm)} \pm (\beta \leftrightarrow \alpha)) z_\delta / z^{3/2} + \ldots$$

$$R^{(\pm)\gamma}_3 = j^{(2)\gamma}_{(5)(\pm)} \ln z^2 + z^\gamma P^{(1)}_{(\pm)}/\sqrt{z^2} + \ldots$$

The bilocal operators $j^{(1)}_\alpha(z; y)$ etc. are hermitian and fulfil relations like

$$j_\alpha(z; y) = \pm j_\alpha(-z; x). \tag{9}$$

This follows from the causality

$$j_\alpha(x) j_\beta(y) = (j_\alpha(x) j_\beta(y))^+, \; z^2 < 0 \tag{10}$$

and the hermiticity (A, B are indices of, say, SU(3))

$$(j^A_\alpha(x), j^B_\beta(y))^+ = (j^A_\beta(x), j^B_\gamma(y))_{\substack{A\leftrightarrow B \\ \alpha\leftrightarrow\beta \\ x\leftrightarrow y}}. \tag{11}$$

Among the hermitian operators one may include also those with an "abnormal" C- parity consisting of

$$\tilde{j}_\alpha = i(\overline{\Psi}(x)\gamma_\alpha \Psi(y) - \overline{\Psi}(y)\gamma_\alpha \Psi(x)) \tag{12}$$

etc.

This term contains, e.g., the energy momentum operator as well [6].

III. TESTS OF LIGHT-CONE EXPANSIONS

We consider first the Fourier transform of the matrix element of two current densities

$$M_{\mu\nu} = (2\pi)^{-4} \int e^{-ikx} d^4x \langle \beta | j_\mu(x) j_\nu(0) | \alpha \rangle , \tag{13}$$

where the commutator or the T-product can be inserted equally well and where $|\alpha>$ and $|\beta>$ are general hadronic states with momenta $p^{(1)}, \ldots p^{(n)}$. For any timelike linear combination of some or all of these momenta

$$P = \sum_i c_i p_i \tag{14}$$

we calculate now

$$T_{\mu\nu} = \int d^4k \delta(q^2 - k^2) \delta[\nu - (Pk)] M_{\mu\nu} = \tag{15}$$

$$= \int d^4x \Phi(x, P, q^2, \nu) < \beta | j_\mu(x) j_\nu(0) | \alpha >.$$

In the system with $\vec{P} = 0$ one finds in the limit $\nu, q^2 \to \infty$, $P_0 \ll \nu$, $\omega = -q^2/2\nu$ = fixed (A-limit) for Φ the result [2]

$$\Phi \underset{A}{\rightarrow} -2\pi\tau^{-1} \sin\left(\frac{\nu x^2}{2\tau} + \omega\tau\right) \tag{16}$$

$$\tau^2 = (Px)^2 - P^2 x^2$$

from which one can conclude that the regions

$$\left|\frac{\nu x^2}{2\tau}\right| \lesssim 1, \; |\omega\tau| \lesssim 1$$

give the essential contributions to the integration in (15). This is equivalent to the region around the light cone

$$x^2 \lesssim 4/q^2.$$

This consideration has been applied originally [2] to the special case of deep inelastic lepton-proton scattering $|\alpha> = |\beta> = |p>$ and to the problem of formfactors (T-product, $\beta = 0$, $|\alpha> = |p>$), where only one timelike fourmomentum p is present. Performing the calculation in the same system $\vec{p} = \vec{P} = 0$ without change of integrations in (15) gives ($p^2 = m^2$)

$$T_{\mu\nu} \underset{A}{\rightarrow} \frac{2\pi|\nu|}{m^2} M_{\mu\nu}. \tag{17}$$

This consideration is valid also for the more general case and yields "S-wave sum rules" [6].

The most prominent example with the largest amount of experimental information is the case of the deep inelastic electron-proton scattering, where (q is the momentum transfer from the electron, $q^2 < 0$, $q_0 > 0$)

$$\begin{aligned} d\sigma/dq^2 d\nu \propto W_{\mu\nu}(p, q, W) &= \\ = \int e^{iqx}\, d^4x < W, p|[j_\mu(x), j_\nu(0)]|p, W > &= \\ = (q_\mu q_\nu - q^2 g_{\mu\nu})V_1 + \{(p_\mu q_\nu + p_\nu q_\mu)\nu - p_\mu p_\nu q^2 - g_{\mu\nu}\nu^2\}V_2 &+ \\ + \varepsilon_{\mu\nu\alpha\beta} q^\alpha W^\beta V_3 + \varepsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta (Wq) V_4. \end{aligned} \tag{18}$$

In this expression p and W represent the fourmomentum and the polarization fourvector of the target proton, $V_i = V_i(q^2, \nu)$. In the commutator only the d-coupling contributes ($\bar{j}$, $\bar{T}$ have "abnormal" C-parity, S, P, $j_{(5)}$ the normal one) and

$$\begin{aligned} (z^2 - i\varepsilon z_0)^{-1} - c.c. &= i\pi\varepsilon(z_0)\delta(z^2) \\ (-z^2 + i\varepsilon z_0)^{-1/2} - c.c. &= -i\varepsilon(z_0)\,\Theta(z^2)(z^2)^{-1/2} \\ \ln(-z^2 + i\varepsilon z_0) - c.c. &= i\pi\varepsilon(z_0)\,\Theta(z^2). \end{aligned} \tag{19}$$

With the Fourier transforms of (19)

$$\begin{aligned} \int d^4x e^{iqx} i\pi\varepsilon(x_0)\delta(x^2) &\propto \varepsilon(q_0)\delta(q^2) \\ \int d^4x e^{iqx} i\varepsilon(x_0)\,\Theta(x^2)(x^2)^{-1/2} &\propto \Box_q \varepsilon(q_0)\,\Theta(q^2)(q^2)^{-1/2} \\ \int d^4x e^{iqx} i\pi\varepsilon(x_0)\,\Theta(x^2) &\propto \varepsilon(q_0)\delta'(q^2) \end{aligned} \tag{20}$$

one obtains in a straightforward way the contribusion of all the matrix elements of (8).

The quickest way to see the effect of a specific term in (8) on the A-limit of deep inelastic electron-proton scattering is exhibited only for the first term in R_1. The contribution to $W_{\mu\nu}$ reads (time reversal allows only even powers of $\left(p \frac{\partial}{\partial q}\right)$, the "abnormal" $\bar{j}_\alpha$ states with $iz^\beta(\bar{\Psi}_{j\alpha}(y)\gamma_\beta \Psi(y) - \bar{\Psi}\gamma_\beta \Psi_{j\alpha}))$

$$P^{(q)}_{\mu\nu} \left[C_2 \left(p \frac{\partial}{\partial q}\right)^2 + C_4 \left(p \frac{\partial}{\partial q}\right)^4 + \ldots \right] \delta(q^2)\varepsilon(q_0) \tag{21}$$

so that the C_2-part gives

$$V_1 \underset{A}{\overset{\propto}{\rightarrow}} \left[\nu^2 \delta''(q^2) + \frac{p^2}{2}\delta'(q^2) \right] \propto \frac{\delta''(\omega)}{\nu} + O(\nu^{-2}).$$

C_4 is accompagnied by $\delta''''(\omega)/\gamma$ etc. The (in general unknown) constants C_{2n} determine therefore the even moments of a function $V(\omega)$ in

$$V_1 \underset{A}{\rightarrow} \frac{V(\omega)}{\nu} + O(\nu^{-2}).$$

All other bilocal operators constructed from Dirac densities yield the same type of contribution to V_1 starting with $\delta''(\omega)/\nu$, if they contribute at all to this order ν^{-1}. The absence of the zeroth moment

$$\int V(\omega)\, d\omega = 0 \tag{22}$$

is a simple expression of the absence of operator Schwinger-terms [4] in any ansatz constructed from Dirac-densities alone. Together with the positivity property of V_1 we conclude

$$V(\omega) = 0.$$

This result seems to be in agreement with the vanishing of σ_L/σ_T found in SLAC-experiments [8]. This property of our ansatz is not astonishing, since it is recognized now as a general feature of a theory based on interacting fermion fields of spin $\frac{1}{2}$ (quark, parton).

It is straightforward, but lengthy, to do the same calculation for the other terms in (8). From dimensional arguments a result ($B_i = B_i(\omega)$, $A_i = A_i(\omega)$, real)

$$\begin{aligned} V_1 &\rightarrow B_0/\nu + A_1/\nu^{3/2} + B_1/\nu^2 + O(\nu^{-5/2}) \\ V_2 &\rightarrow B_2/\nu^2 + A_2/\nu^{5/2} + O(\nu^{-3}) \\ iV_3 &\rightarrow A_3/\sqrt{\nu} + B_3/\nu + O(\nu^{-3/2}) \\ iV_4 &\rightarrow A_4/\nu^{3/2} + B_4/\nu^2 + O(\nu^{-5/2}) \end{aligned} \tag{23}$$

can be expected. Actually from $V = B_0 = 0$, also $A_3 = A_4 = 0$ can be concluded because of the inequalities for V_i's [9]. Thus despite the large number of possible terms in the ansatz (8) the result for the "scaling functions" is simple. The same is true for examples involving *amplitudes* (and hence the T-product), as in e.g. $e^+ + e^- \to$ hadros $+ \gamma$ [6].

We turn now to a simple example for "internal" virtual masses. The electromagnetic correction to the hadronic matrix element can be described by the effective Lagrangian (Λ is a cut-off)

$$L_{eff} = (2\pi)^{-4} e^2 \int d^4x j_\mu(x) j^\mu(0) \int d^4k e^{ikx} \Lambda^2 \{k^2(\Lambda^2 - k^2)\}^{-1} . \qquad (24)$$

After combinig the denominator à la Feynman, the integral

$$\Phi(x^2, \Lambda^2) = (2\pi)^{-4} \int e^{ikx} d^4k \, \Lambda^2 \{k^2(\Lambda^2 - k^2)\}^{-1} \qquad (25)$$

can be easily reduced to a finite integral over the first derivative with respect to the (mass)^2 of the causal Wightman-function:

$$\frac{\partial \Delta_F(x, \mu)}{\partial \mu^2} \propto (2\pi)^{-4} \int d^4k e^{-ikx}(k^2 - \mu^2)^{-2} = (16i\pi)^{-1} [2\Theta(-x^2) K_0(\mu \sqrt{-x^2}) - $$

$$- i\pi\Theta(x^2) H_0^{(2)} (\mu \sqrt{x^2}) \qquad (26)$$

so that

$$\Phi = \Lambda/8[\Theta(x^2) H_1^{(1)}(\Lambda\sqrt{x^2})/\sqrt{x^2} + 2\Theta(-x^2) K_1(\Lambda\sqrt{-x^2})/(i\pi\sqrt{-x^2}) , \qquad (27)$$

which shows that only the region $|x^2| \lesssim \Lambda^{-2}$ is important. As long as (24) is considered between states with a low fourmomentum, the tip of the light-cone alone $x \sim 0$ contributes. This happens because in this case

$$x = \xi \Lambda^{-1}$$

$$|\xi^2| \lesssim 1$$

can be introduced and this makes all higher terms in (3) together smaller by at least a factor Λ^{-1} with respect to the first one — as long as the first term $m = 0$ alone yields by itself a finite contribution on the regularized integral. Otherwise a knowledge of the whole light-cone is required.

If our bilocal ansatz (8) contains the $S^{(i)}$-term, belonging to an SU(3)-octet, there emerge quite naturally (linearly diverging) tadpoles with $I = 1$, and $I = 0$, $I_3 = 0$ from the local operator S in (4) for electromagnetic mass-differences as required by the Coleman-Glashow-analysis [10]. Note, however, that the full light-cone contributes in other high energetic hadronic reactions with radiative corrections.

An example which does not require any regularization is the semileptonic box-graph in the renormalizable theory of weak interactions [11]. Here scalar densities occur. The vector term j_α in the expansion at $z \sim 0$

$$S(x)S(y) \underset{z \sim 0}{=} \ldots + a j_\alpha(y) z^\alpha/z^4$$

turns out to represent just the hadronic current in semileptonic weak interactions which may be assumed to be conserved (CVC). Here the tip of the light-cone alone is essential for low energetic (decay-) processes.

With respect to nonleptonic weak interactions, which can be treated just like the radiative corrections, but with an intermediate W-meson instead of the photon, we just note the amusing fact that the intermediate — vector — boson theory does not yield "tadpoles" of about equal strength for parity conserving and parity violating amplitudes of hyperon decays. The renormalizable theory with scalar bosons on the other hand gives a consistent picture [6].

Essentially similar considerations — only slightly modified — can be applied to radiative corrections of weak decays (in the V-A-theory) as well [6].

REFERENCES

[1] Wilson K., Phys. Rev. *179* (1969), 1499.
Brandt R. A., Ann. Phys. (N. Y.) *44* (1967), 221.
Zimmermann W., Communs. Math. Phys. *6* (1967), 161.

[2] Brandt R. A., Preparata G., Nucl. Phys. *B 27* (1971), 541.
Brandt R. A., *The light-cone and symmetry breaking*, CERN-preprint TH 1402 (1971).

[3] Bjorken J. D., Phys. Rev. *179* (1969), 1547.

[4] Brandt R. A., Phys. Rev. Letters *23* (1964), 1547.
Jackiw R., van Royen R., West G. B., Phys. Rev. *D 2* (1970), 77.
Leutwyler H., Stern J., Nucl. Phys. *B 20* (1970), 77.
Gálfi G., Kuti J., Patkos A., Phys. Letters *31 B* (1970), 465.

[5] Georgelin Y., Stern J., Jersák J., Nucl. Phys. *B 27* (1971), 493.

[6] Kummer W., Nucl. Phys. *B 42* (1972),141.

[7] Fritzsch H., Gell-Mann M., *Scale invariance and the light-cone*. In: Proc. of the Coral Gable Conf. on fundamental interactions at high energies, Miami 1971 (to be published).

[8] Taylor R. E., SLAC report No. SLAC-PUB-677 (1970).

[9] Gálfi L., Gnädig P., Kuti J., Niedermayer F., Patkos A., *Deep inelastic scattering of polarized electron beam from polarized nucleon target*. ITP-report 281, Budapest 1970.

[10] Coleman S., Glashow S. C., Phys. Rev. *134* (1964), B 671.

[11] Kummer W., Segrè G., Nucl. Phys. *64* (1965), 585.
Kummer W., Acta Phys. Austr. *32* (1970), 387.

Received September 22nd, 1971