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GROUPS AND DYNAMICS[1]

Finite dynamical symmetry transformations for the Kepler motion

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In recent years there has been considerable interest in the dynamical symmetry groups of simple systems. Particular attention has been devoted to the classical one-particle problem in a $-1/r$ potential, i.e. the Kepler motion. For this problem, characterized by the Hamiltonian function

$$H = \frac{p^2}{2m} - \frac{g}{q}, \tag{1}$$

two well-known vectorial constants of motion exist (see e.g. [1]): the angular momentum

$$\mathbf{L} = \mathbf{q} \times \mathbf{p}, \tag{2}$$

and the Laplace-Runge-Lenz vector

$$\mathbf{K} = \pm \frac{mg}{p_0}\left(\frac{\mathbf{q}}{q} - \frac{\mathbf{p} \times \mathbf{L}}{mg}\right), \tag{3}$$

where

$$p_0 = (\mp 2mH)^{1/2} \tag{4}$$

(upper and lower sings refer to negative and positive energies, respectively). The constants of motion $\mathbf{L}$ and $\mathbf{K}$ satisfy the Poisson bracket relations of the Lie algebra of the SO(4) or SO(3,1) groups:

$$(L_i, L_j)_{qp} = \varepsilon_{ijs}L_s, \ (L_i, K_j)_{qp} = \varepsilon_{ijs}K_s, \ (K_i, K_j)_{qp} = \pm\, \varepsilon_{ijs}\, L_s; \tag{5}$$

the notation

$$(A, B)_{qp} = \frac{\partial A}{\partial q_s}\frac{\partial B}{\partial p_s} - \frac{\partial A}{\partial p_s}\frac{\partial B}{\partial q_s} \tag{6}$$

has been used here (summation over repeated indices is understood). The finite canonical transformations of the basic dynamical variables $\mathbf{q}$ and $\mathbf{p}$, generated by the angular momentum $\mathbf{L}$, can easily be derived from the differential equations of the group, since these are linear and can easily be integrated. In the case, however, of the full group generated by $\mathbf{L}$ and $\mathbf{K}$ the integration of the differential equation is no longer possible since these equations are nonlinear [2].

In the present paper we propose to consider the following transformation of the basic variables [3]:

$$\mathbf{x} = p_0\mathbf{q}, \qquad \mathbf{y} = p_0^{-1}\mathbf{p}. \tag{7}$$

(An analogous change of variables is usually performed in order to find solutions of the Schrödinger eigenvalue problem of the H atom.) The transformation (7) is not canonical in the usual, restricted sense;

$$\mathbf{q} \to \mathbf{x},\ \mathbf{p} \to \mathbf{y},\ t \to \tau = t - \frac{\mathbf{pq}}{2H},\ H \to H, \tag{8}$$

however, defines an extended canonical transformation [4]. This new scheme allows us to introduce a new Poisson bracket expression defined by

$$(F,G)_{xy} = \frac{\partial F}{\partial x_s}\frac{\partial G}{\partial y_s} - \frac{\partial F}{\partial y_s}\frac{\partial G}{\partial x_s}. \tag{9}$$

The constants of motion (2) and (3) can be written as follows:

$$\mathbf{L} = \mathbf{x} \times \mathbf{y}, \quad \mathbf{K} = \tfrac{1}{2}(1 \mp y^2)\mathbf{x} \pm (\mathbf{xy})\mathbf{y}. \tag{10}$$

These obey, together with the quantities

$$\mathbf{X} = \tfrac{1}{2}(1 \pm y^2)\mathbf{x} \mp (\mathbf{xy})\mathbf{y}, \qquad \mathbf{Y} = x\mathbf{y}, \tag{11}$$
$$T = \mp \mathbf{xy}, \qquad U = \tfrac{1}{2}x(1 \mp y^2), \qquad N = \tfrac{1}{2}x(1 \pm y^2),$$

simple Poisson bracket relations. Define the following 6×6 skew-symmetric scheme:

$$(G_{IJ}) = \left[\begin{array}{ccc|c|c|c} 0 & L_3 & -L_2 & & & \\ -L_3 & 0 & L_1 & (\pm)^{-\frac{1}{2}}\mathbf{K} & -\mathrm{i}\mathbf{Y} & (\pm)^{-\frac{1}{2}}\mathbf{X} \\ L_2 & -L_1 & 0 & & & \\ \hline & -(\pm)^{-\frac{1}{2}}\mathbf{K} & & 0 & -(\mp)^{\frac{1}{2}}U & -\mathrm{i}\,T \\ \hline & \mathrm{i}\mathbf{Y} & & (\mp)^{\frac{1}{2}}U & 0 & (\mp)^{\frac{1}{2}}N \\ \hline & -(\mp)^{-\frac{1}{2}}\mathbf{X} & & \mathrm{i}\,T & -(\pm)^{\frac{1}{2}}N & 0 \end{array}\right]. \tag{12}$$

This notation makes it possible to condense all Poisson bracket relations between the quantities (10), (11), into a single formula:

$$(G_{IJ}, G_{KL})_{xy} = \delta_{IL}G_{KJ} + \delta_{JK}G_{LI} + \delta_{IK}G_{JL} + \delta_{JL}G_{IK}. \tag{13}$$

One has further

$$G_{IS}G_{SJ} = 0, \tag{14}$$
$$\varepsilon_{IJPQRS}G_{PQ}G_{RS} = 0.$$

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The basic variables $\mathbf{x}$, $\mathbf{y}$ can be expressed through the G_{IJ} as

$$\mathbf{x} = \mathbf{X} + \mathbf{K},$$

$$\mathbf{y} = (U + N)^{-1} \mathbf{Y}.$$

The linear Poisson bracket relations (13) allow us to determine, e. g., finite transform of the quantities $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{K}$, U, N, and, by virtue of (15), those of the basic variables $\mathbf{x}$ under the full invariance symmetry group generated by $\mathbf{L}$ and $\mathbf{K}$. It should be st that this group is not identical with that based on the Poisson bracket relations (5 investigated in [2]. These two kinds of groups are, however, isomorphic; both act t tively on the manifold of isoenergetic orbits. Moreover, finite transformations of th as well as of the basic variables $\mathbf{x}$ and $\mathbf{y}$, can be obtained for any element of the S full dynamical group of canonical transformations, generated by the G_{IJ}.

An alternative discussion of the integration problem has been given in [5].

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