Letters to the Editor

CHAIN FRACTION EXPANSION OF THE S-OPERATOR[1,2]

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This talk is a progress report on an investigation which is not yet finished. The basic idea is to replace the standard perturbation theory by some different expansion scheme. Naturally, Padé approximants come to one's mind when he tries to improve the perturbation theory. However, we do not want to relate matrix elements of the perturbation expansion to Padé coefficients in a Padé expansion. Rather, we would undertake to expand the S-operator before the matrix elements have been taken.

Formally, the S-operator can be written as

$$S = T\exp\left(ig \int d^4x\, H(x)\right). \tag{1}$$

The perturbation theory requires an expansion of the exponential into a power series with the formal time ordering operator T taken under the integrals in each term separately. Instead of a power series, we can easily use a formal Padé expansion or a chain fraction expansion which is just a special case of the Padé expansion (just as the power series!). In order to do so, we remember the diagonal Padé expansion

$$e^x = \frac{1+\frac{x}{2}}{1-\frac{x}{2}} + O(x^3) = \frac{1+\frac{x}{2}+\frac{x^2}{12}}{1-\frac{x}{2}+\frac{x^2}{12}} + O(x^5) \tag{2}$$

as well as the chain fraction expansion,

$$e^x = 1 + \frac{x}{1-\frac{x}{2}} + O(x^3) = 1 + \frac{x}{1-\frac{x}{2}:\left(1+\frac{x}{6}\right)} + O(x^4). \tag{3}$$

Details about these expansions are nicely collected, for example. in [1]. It should b recalled that a diagonal Padé approximant as well as an even order chain fraction approximant as well as an even order chain fraction approximation of a unitary matrix are both unitary. This is, of course, not the case in the ordinary perturbation theory.

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When using Eq. (3) to expand the exponential in the S-operator (1), it remains to define the time ordering operator T in

$$S = I + T\,\frac{ig \int d^4x H(x)}{1-\frac{ig}{2}\int d^4y H(y)} + O(g^3). \tag{4}$$

One possibility is, of course, the use of Hori's formula [2]. It turns out to be rather cumbersome, however. We therefore try a direct interpretation, which agrees with the perturbation theory up to $O(g^3)$ in an expansion of the denominator of Eq. (4). It is given by

$$S = I + \frac{ig}{2}\int d^4x\, \{H(x)[1 - ig \textstyle\int d^4y H(y)\, \Theta\,(x-y)]^{-1} +$$
$$+ [1 - ig \textstyle\int d^4y H(y) \Theta(y-x)]^{-1} H(x)\} + O(g^3). \tag{5}$$

Since Eq. (5) coincides with the perturbation S-operator up to $O(g^3)$, it should give for example lowest order conrtibutions to self energies. However, in a matrix element of Eq. (5), an infinite number of higher order contributions is implicitely summed up and the result can therefore be damped and finite. It should be pointed out that an extension to higher order chain fraction approximants can be written down. Since it soon becomes rather lengthy, we shall omit to do so here.

One difficulty which arises in a practical calculation is the appearance of operators in the denominator. Corresponding matrix elements can, for example, be computed in a bubble approximation. For the Φ^3 theory this would give a damping factor

$$\left[1 + \frac{g^2}{64\pi^2 p_0}\int_{4m^2}^{\infty} \frac{ds}{\omega(p_0-\omega)^2}\sqrt{\frac{s-4m^2}{s}}\right]^{-1} \tag{6}$$

with

$$\omega = {}_{+}\sqrt{s - m^2 + p_0^2}\,. \tag{7}$$

Although this is a nice damping factor, it is not covariant. The reason is that the separation of terms in Eq. (5) seizes to sum up to a covariant result in an approximation of both parts of the sum. It is to be expected that an improved technique to compute the matrix elements of the functions of operators will help to overcome the difficulties of the present calculations. In this way, we hope to arrive at a satisfying method to compute matrix elements which diverge in the standard perturbation theory.

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