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GROUPS AND DYNAMICS OF PARTICLES[1]

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This survey concentrates on a classification of various dynamical models which have given rise to dynamical algebraic relations identical with the Lie albebras of certain Lie groups. These groups are not necessarily symmetry groups of the Hamiltonian of the physical system and are referred to as dynamical groups, non-invariant groups, or spectrum generating groups.

I. STATISTIC BOOTSTRAP MODEL AND STRONG COUPLING GROUP

Among several dynamical models which have the properties mentioned above we start by mentioning the Chew static bootstrap model [1] that was quite popular several years ago. According to the Chew bootstrap philosophy any hadron is a bound state or a resonance consisting of all hadrons. Forces which are responsible for binding this compcsite system together are due to the exchange of all possible hadrons in the crossed channels. This idea can be very simply demonstrated on the static meson-baryon interaction described by the Chew—Low equation [2].

Consider the static meson-baryon scattering of the form

$$\pi + a \rightarrow \pi + b, \tag{1}$$

where a and b denote baryons with their quantum numbers. Let A be a baryon beeing a bound state or a resonance in the partial wave denoted by $f_A(\omega)$, where ω is the initial pion energy. It is clear that the mass of the baryon A is associated with the pole of the partial wave amplitude $f_A(\omega)$ and the residue in this pole is simply a product of two reduced pion-baryon coupling constants G_A^a and G_A^b. Using the N/D method one can write the following equations

$$G_A^a G_A^b = -\frac{N_A(\omega_A)}{D'_A(\omega_A)} \tag{2}$$

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and

$$D_A(\omega_A) = 0, \tag{3}$$

where ω_A is the position of the pole of the partial wave amplitude and $D'_A(\omega_A)$ is the first derivative of the D function at this pole.

The application (3) of Eq. (2) under the assumption that forces binding the composite system A are well approximated by single particle exchanges in the u channel leads to simple algebraic relations involving pion-baryon coupling constants

$$G_A^a G_A^b = \sum_B C_{AB} G_B^a G_B^b, \tag{4}$$

where C_{AB} are crossing matrix elements from the u to s channel. When the solution to Eq. (4) is known then the analysis of Eq. (3) gives the mass spectrum of baryons as the function of their spins J and isospins I of the form [4]

$$m(I, J) = m_0 + \alpha I(I+1) + \beta J(J+1), \tag{5}$$

where m_0, α, and β are constants.

It was recognized by Cook, Goebel and Sakita [5] that the Chew static bootstrap model can be completely reworded in the group theoretic language. Use was made of the Chew-Low equation in the so-called strong coupling limit. The strong coupling theory requires that in the limit when all pion-baryon coupling constants tend to infinity, the scattering amplitude given by the Chew-Low equation must be finite.

Let us define the pion-baryon $\pi^\alpha + a \to b$ coupling constant as a matrix element of an operator X_α

$$\langle b\,|X_\alpha|\,a\rangle \tag{6}$$

taken between baryon states $|a\rangle$ and $|b\rangle$. Here α represents quantum numbers of the pion π^α. The mass of the baryon a, m_a, is also defined as the matrix element of a diagonal mass matrix M as

$$m_a = \langle a|M|a\rangle. \tag{7}$$

The invariance of the interaction under a symmetry group of the Hamiltonian requires that the operators X_α transform as components of the proper tensor under symmetry group transformations. This yields

$$[K_\alpha, K_\beta] = \mathrm{i} f_{\alpha\beta\mu}\, K_\mu \tag{8}$$

and

$$[K_\alpha, X_\beta] = \mathrm{i} d_{\alpha\beta\mu}\, X_\mu, \tag{9}$$

where K_α denote the generators of the symmetry group K, $f_{\alpha\beta\mu}$ are the structure constants of this group and $d_{\alpha\beta\mu}$ specify the tensorial character of the operators X_α. The strong coupling theory gives rise to two additional dynamical relations, namely,

$$[X_\alpha, X_\beta] = 0 \tag{10}$$

and

$$\Lambda_{\alpha\beta} = \sum_\mu \Lambda_{\alpha\mu}\Lambda_{\mu\beta} \tag{11a}$$

where

$$\Lambda_{\alpha\beta} = [X_\alpha, [M, X_\beta]]. \tag{11b}$$

It is evident that the commutation relations (8—10) define the Lie algebra of the Lie group beeing the semidirect product of the invariance group K and the Abelian group T generated by the mutually commuting operators X_α. Unitary irreducible representations of this algebra determine the pion-baryon coupling constants which are exactly the same as those following from the Chew static bootstrap model. This is so because the bootstrap equation (4) is nothing else but the matrix element of the commutator (10). Once the matrix elements of the operators X_α are known they can be inserted into Eq. (11) and a dynamical equation determining the mass spectrum of baryons is obtained [5, 6]. The solution to the mass equation is again exactly the same as the form of the mass spectrum (5) following from the bootstrap theory. It should be also noted that a strong coupling group makes the Chew-Low equation solvable.

These results show how analyticity and unitarity of the scattering amplitude completed by the bootstrap ideas can be expressed in an elegant group theoretic approach.

II. THE CAPPS BOOTSTRAP MODEL

The Capps bootstrap model [7] leads also to group theoretical considerations and is based on superconvergence relations of the scattering amplitude for the fixed momentum transfer. The bootstrap ideas in this model are represented by the two following assumptions:

(i) Superconvergence relations for the forward scattering amplitude can be saturated by single particle states that result from composites in all possible channels. Such saturation must not spoil the proper Regge behaviour.

(ii) The set of composites must be the same as the set of external particles.

These two assumptions are strong enough to prove that hadron states must form the basis for the representations of some unitary semisimple Lie groups.

Before proceeding with the demonstration of the Capps bootstrap model one is tempted to explain the first assumption of this model in greater detail. It is obvious that the contribution from the three graphs for the forward scattering amplitude cannot have the proper asymptotic behaviour of the actual scattering amplitude unless some cancellation among the three graphs is require. The first assumption is telling us that the rapidly growing terms contributed by the three graphs should cancel among themselves and not with a continuum part of the scattering amplitude.

To demonstrate the Capps bootstrap model we consider a forward scattering process of massless pions by hadrons of the form

$$\pi^\alpha(q) + a(p) \;\rightarrow\; \pi^\beta(q') + b(p') \tag{12}$$

realized in the storage rings. Here α, β are isospin indices of pions, a, b denote hadrons and p, q, q' and p' are the respective four momenta given by

$$\begin{aligned} q_\mu &- \omega n_\mu \\ q'_\mu &= \omega' n_\mu \\ n_0 &= |\boldsymbol{n}| = 1 \\ \boldsymbol{p} &= -\boldsymbol{n}|\boldsymbol{p}|,\ p_0 = (\boldsymbol{p}^2 + m_a)^{1/2} \\ \boldsymbol{p}' &= -\boldsymbol{n}|\boldsymbol{p}'|,\ p'_0 = (\boldsymbol{p}'^2 + m_b)^{1/2}. \end{aligned} \tag{13}$$

Energy and momentum conservation laws give the relations

$$\begin{aligned} |\boldsymbol{p}| + p_0 &= |\boldsymbol{p}'| + p'_0 \equiv E, \\ s &= (p+q)^2 = m_a^2 + 2E\omega = m_b^2 + 2E\omega', \\ u &= (p'-q)^2 = m_a^2 - 2E\omega' = m_b^2 - 2E\omega, \\ \omega' &= \omega + (m_a^2 - m_b^2)/2E, \end{aligned} \tag{14}$$

and angular momentum conservation yields the conservation of hadron helicities.

This process is described by the invariant Feynman amplitude denoted by $M_{ba}^{\beta\alpha}(\omega)$. The crossing symmetry between s and u channels imposes on M the restriction

$$M_{ba}^{\beta\alpha}(\omega) = M_{ba}^{\alpha\beta}(-\omega'). \tag{15}$$

It will be seen to be convenient to devide M into parts symmetric and antisymmetric in the pion isovector indices α and β given by

$$M^{\beta\alpha(+)}_{ba}(\omega) = \tfrac{1}{2}\,[M^{\beta\alpha}_{ba}(\omega) + M^{\alpha\beta}_{ba}(\omega)] \tag{16}$$

and

$$M^{\beta\alpha(-)}_{ba}(\omega) = (\omega + \omega')^{-1}[M^{\beta\alpha}_{ba}(\omega) - M^{\alpha\beta}_{ba}(\omega)]. \tag{17}$$

Next we assume that the interaction Lagrangian is chirally invariant of the form

$$L_{int} = - F_\pi^{-1} A^\alpha_\mu D_\mu \varphi^\alpha + 2F_\pi^{-2} \varepsilon^{\alpha\beta\gamma} V^\alpha_\mu \varphi^\beta \partial_\mu \varphi^\gamma \tag{18}$$

where φ^α is the pion field, A^α_μ is the phenomenological axial vector current, F_π is the pion decay amplitude, $F_\pi = 190$ MeV, V^α_μ is the conserved phenomenological vector current, normalized so that

$$\int d^3\mathbf{x}\, V^\alpha_0(x) = 2I^\alpha, \tag{19}$$

and I^α is the generator of the isospin group. The chirally invariant Lagrangian for the process (12) yields low energy theorems [8]

$$M^{\beta\alpha(-)}_{ba}(0) = 8\mathrm{i}F_\pi^{-2} E\, \{ \varepsilon^{\alpha\beta\mu} (I^\mu)_{ba} + \sum_{\substack{n \\ (m_n = m_a)}}{}' (X^\beta)_{bn} (X^\alpha)_{na} - \sum_{\substack{n \\ (m_n = m_b)}}{}' (X^\alpha)_{bn} (X^\beta)_{na} \} \tag{20}$$

and

$$M^{\beta\alpha(+)}_{ba}(0) = 2F_\pi^{-2}\, \{ \sum_{\substack{n \\ (m_n = m_a)}}{}' (2m_n^2 - m_a^2 - m_b^2)(X^\beta)_{bn}(X^\alpha)_{na} + \sum_{\substack{n \\ (m_n = m_b)}}{}' (2m_n^2 - m_a^2 - m_b^2)(X^\alpha)_{bn}(X^\beta)_{na} \}, \tag{21}$$

where $(X^\beta)_{ba}$ is associated with the invariant Feynman amplitude M^β_{ba} for the process $a \to \pi^\beta + b$ as

$$M^\beta_{ba} = 2F_\pi^{-1} (m_a^2 - m_b^2)(X^\beta)_{ba}. \tag{22}$$

Low energy theorems allow us to make one subtraction in the dispersion relations for the antisymmetric and symmetric parts of the scattering amplitude

These dispersion relations are saturated by single patricle states according to the first Capps bootstrap assumption. After tedious but rather simple algebra one finds asymptonic behaviour of the tree graphs contributions for the amplitudes $M_{ba}{}^{(-)}(\omega)$ and $M^{\beta\alpha(+)}_{ba}(\omega)$ to be of the form

$$M^{\beta\alpha}_{ba}(\omega)_{\omega\to\infty} = 8F_\pi^{-2} E\, \{ \mathrm{i}\varepsilon^{\alpha\beta\mu} (I^\mu)_{ba} - \sum_n [(X^\alpha)_{bn}(X^\beta)_{na} - (X^\beta)_{bn} - (X^\alpha)_{na}]\} + 0\left(\frac{1}{\omega^2}\right) \tag{23}$$

and

$$M^{\beta\alpha(+)}_{ba}(\omega)_{\omega\to\infty} = 2F_\pi^{-2} \sum_n (2m_n^2 - m_a^2 - m_b^2)[(X^\beta)_{bn}(X^\alpha)_{na} + (X^\alpha)_{bn}(X^\beta)_{na}] + 0\left(\frac{1}{\omega^2}\right). \tag{24}$$

We now apply the second part of the first Capps bootstrap assumption telling us that the asymptotic behaviour of $M^{(-)}$ and $M^{(+)}$ saturated by single particle states should not spoil the expected Regge behaviour. The amplitude $M^{(-)}$ has pure isospin $I = 1$ exchanged in the t channel and has the asymptotic behaviour

$$M^{\beta\alpha}_{ba}(\omega)_{\omega\to\infty} \approx \omega^{\alpha_1(0)-1}, \tag{25}$$

where $\alpha_1(0)$ is the intercept of the dominant $I = 1$ trajectory. Presumably $\alpha_1(0) = \alpha_\varrho(0) \approx 0.5$. This shows that $M^{(-)}$ vanishes as $\omega \to \infty$. Hence the first Capps bootstrap assumption demands that the term in Eq. (23) which behaves as ω^0 must vanish itself and we get

$$\sum_n [(X^\alpha)_{bn}(X^\beta)_{na} - (X^\beta)_{bn}(X^\alpha)_{na}] = \mathrm{i}\,\varepsilon^{\alpha\beta\mu}(I^\mu)_{bn}. \tag{26}$$

Next we apply the second bootstrap hypothesis of Capps that the set of internal hadrons denoted by n is the same as the set of the external hadrons denoted by a or b. This implies that X^α are matrices and Eq. (16) can be rewritten in the matrix form

$$[X^\alpha, X^\beta] = \mathrm{i}\,\varepsilon^{\alpha\beta\mu} I^\mu. \tag{27}$$

The isospin conservation tells us that the following commutation relation must be fulfilled

$$[I^\alpha, X^\beta] = \mathrm{i}\,{}_\beta{}^{\alpha\beta\mu} X^\mu \tag{28}$$

and, of course, I^α obeys the standard commutation relations

$$[I^\alpha, I^\beta] = \mathrm{i}\,\varepsilon^{\alpha\beta\mu} I^\mu. \tag{29}$$

The algebraic relations (27—29) are exactly the Lie algebra of the SU(2) ⊗ ⊗ SU(2) group and tell us that hadron states of the same helicities but with various spins and isospins must form a basis for unitary (generally reducible) representations of the group in question.

The amplitude $M^{(+)}$ has both isospins $I = 0$ and $I = 2$ exchanged in the t channel. The part with $I = 2$ can be separated as

$$M_{ba}^{\beta\alpha(I=2)}(\omega) = M_{ba}^{\beta\alpha(+)}(\omega) - \tfrac{1}{3}\,\delta^{\alpha\beta}\, M_{ba}^{\alpha\beta(+)}(\omega)\,. \tag{30}$$

This part has the following Regge behaviour

$$M_{ba}^{\beta\alpha}(\omega) \underset{\omega\to\infty}{=} \omega^{\alpha_2(0)}\,. \tag{31}$$

There are reasons to believe thas $\alpha_2(0) < 0$. If this is true, the Capps bootstrap model requires that the term in Eq. (24) which behaves as ω^0 must be an isoscalar. This condition gives rise to the matrix relation

$$[X^\alpha, [m^2, X^\beta]] = -\, m_4^2\, \delta^{\alpha\beta}\,, \tag{32}$$

where m^2 is the diagonal mass squared matrix and m_4^2 is an isoscalar defined by

$$m_4^2 = -\,\tfrac{1}{3}\,[X^\alpha, [m^2, X^\alpha]]\,. \tag{33}$$

The implication of Eq. (32) is that the mass squared matrix m^2 behaves as the sum of a scalar and a component of a four vector under $SU(2) \otimes SU(2)$ group transformations.

One can recognize that the dynamical Eqs (27) and (32) following from the Capps bootstrap scheme are exactly the so-called Weinberg algebraic realizations of chiral symmetry [8]. It was shown that they are consequences of the Capps bootstrap model applied to this particular problem. So far the Capps bootstrap model was applied only to problems with degenerase masses of hadrons. Here we have demonstrated that this model is powerful enough to determine the mass spectrum of hadrons.

III. DUALITY AND N-POINT FUNCTIONS

Clavelli and Ramond [9] have shown that duality can be regarded as a purely group theoretical concept, implying SU(1,1) invariance of the N-point function. Their concept is based on three generators of she SU(1,1) group, namely L_0, L_+ and L_-, which fulfil the Lie algebra

$$[L_0, L_\pm] = \pm\, L_\pm \tag{34}$$

$$[L_-, L_+] = L_0\,. \tag{35}$$

They have presented the minimal set of group theoretical rules allowing to construct the dual amplitude. These rules are:

(i) Associate the absorption of a particle of momentum k_μ, spin j, j_3 and of a set of internal quantum numbers λ, with a vertex operator $V(k_\mu, j, j_3, \lambda; z)$, where $z = \exp(-i\tau)$ is a complex variable on the unit circle.

(ii) Requiere $V(k_\mu, j, j_3, \lambda; z)$ to transform under SU(1,1) transformations as a spin J_s representation. This implies

$$[L_0, V] = -\, z\, \frac{dV}{dz} \tag{36}$$

$$[L_\pm, V] = -\,(2)^{-1/2} z^{\pm 1} \left(z\, \frac{d}{dz} \mp J_s \right) V\,. \tag{37}$$

Here J_s is in general a function of the Casimir operasors of the Poincare group and internal symmetry group,

$$J_s = J\,(m^2, j, \{\lambda\}\,.)_s \tag{38}$$

(iii) The transformation properties of the vertex operator under the Lorentz and internal symmetry group are the same as she transformation properties of the field of the absorbed particle.

(iv) Any number of external particles $1, 2, 3, \ldots, i, \ldots, k, \ldots, N$ can interact in a dual manner only if shey belong to the same representation of SU(1,1), i. e.,

$$J_s\,(m_i^2, j_i, j_{i3}, \{\lambda_i\}) = J_s\,(m_k^2, j_k, j_{k3}, \{\lambda_k\})\,. \tag{39}$$

This gives a correlation among all possible external and internal quantum numbers of different particles.

(v) The factorizable dual amplitude A_N describing the scattering of the particles $1, 2, 3, \ldots, i, \ldots, k, \ldots, N$ in that order is given by

$$A_N = \frac{1}{C} \oint <0 \left| \prod_{e=1}^{N} \left[\frac{dz_e}{z_e}\, |z_e - z_{e+1}|^{-1-J_s}\, \Theta\,(\arg z_{e+1} - \arg z_e)\; V(k_{n\mu}, j_e, j_{e3}, \lambda_e; z_e) \right] \right| 0 >\,, \tag{40}$$

where C is the integrated Haar measure and the integration is taken around the unit circle.

One can see that once the proper representaticn of SU(1,1) group is chosen to descirbe hadrons then the simple group theoretical considerations determine the scattering amplitude and the mass spectrum condition which is in some sense the goal of strong interaction physics.

IV. DYNAMICAL GROUPS IN QUANTUM MECHANICS

The hypothesis that the dynamics of a given quantum mechanical system can be completely described by some dynamical group as well as by the Schrödinger equation has been verified for almost all interesting and important quantum mechanical problems [10]. In quantum mechanics we postulate a Hamiltonian $\mathscr{H}$ which is usually a complicated differential operator and then a solution to the Schrödinger equation

$$\mathscr{H}\Psi_n = E_n\Psi_n$$

determines the energy levels E_n and the set of the quantum numbers n of a given quantum mechanical system, which is completely described by the wave functions Ψ_n .In the approach — using dynamical groups — one starts from a chosen dynamical group G and phenomenologically identifies its generators with operators of physical observables rather than postulating the Hamiltonian. In addition, the quantum mechanical wave funcsions Ψ_n are assumed to form a basis for a single unitary irreducible representation of the group in question and in such a way measurable physical quantities can be straightforwardly calculated. The same idea was consequently generalized and used in particle physics.

V. CONCLUSION

This survey of group theoretical methods in the dynamics of particle interacsions does not claim to be complete. Many other very interesting dynamical considerations leading in their final results to group theoretical conclusions have been omitted. The aim of this talk was not give a complete survey of this subject but rather to illustrate the division of theoreticians into two antagonistic groups, a division made by H. Lipkin [11] at the Lund International Conference on High Energy in 1969. According to Lipkin there are stu physicists and IBY physicists. The stu physicists study the behaviour of the scattering amplitude $T(s, t, u; IBY)$ as a function of the continuous kinematic variables s, t and u for fixed values of discrete internal quantum numbers IBY. The IBY physicists do the reverse. The stu physics is called dynamics and uses mainly functions of several complex variables. IBY physics is called symmetry and uses algebras, mainly group theory. It was said that the members of these two schools did not talk to one another, because some of the stu physicists were proud of knowing nothing about the group theory and vice versa some of the IBY members were proud of knowing nothing about complex planes. It was also said that these two groups were brought together by the

use of finite energy sum rules. This survey has perhaps been an attempt to indicate the real situation, i.e. that either the antagonism between the two aforementioned scholls of physics has never been as deep as one could imagine from Lipkin's talk or that the dialogue between these two antagonistic schools of thoughs proceeds very satisfactorily.

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