Fyz. čas. SAV 21 (1971), No. 4

A CLASS OF DISTRIBUTIONS RELATED TO EXTRAPOLATION OF ANALYTIC FUNCTIONS

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Let D be a set of functions analytic in a region H bounded by a curve γ. We introduce a set D' of distributions defined on boundary values of functions from D. If Γ is an arc on the boundary γ, $T \in D'$ and $f \in D$, we can express $T[f]$ as a functional defined only on $f(z)$ reduced to $\Gamma : T[f] = \lim_{n\to\infty} \int_\Gamma t_n(z) f(z) dz$, where $\{t_n(z)\}$ are successive approximations to a known function or to a sequence of known functions. This is related to questions of uniqueness of analytic functions. Analytic extrapolation is thus reduced to approximation of functions. Since $T_a[f] = f(a)$, $(a \in H)$ belongs to D' we can construct in this way effective methods of analytic extrapolation of e. g. from factors and scattering amplitudes in particle physics.

Possible applications of distributions D' to theories of singular integral equations and of biorthogonal series are also mentioned. In a general plane distributions D' link together theorems on uniqueness of analytic functions with theorems on approximations by analytic functions.

I. INTRODUCTION

From factors and scattering amplitudes in particle physics are analytic functions of relevant variables (energy, cosine of the scattering angle, momentum transfer etc.). Two typical situations are displayed in Figs. 1a and 1b,

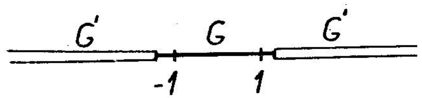

Fig. 1a. A typical analytic structure of the real (or imaginary) part of a scattering amplitude as a function of the scattering angle. Experimentally accessible region G is denoted by crosses.

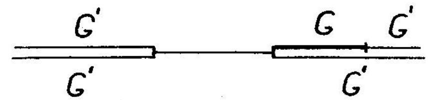

Fig. 1b. A typical analytic structure of a scattering amplitude as a function of energy squared at a fixed value of momentum transfer. Experimental region is denoted by crosses.

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where regions with experimental data available are denoted by crosses. To obtain a scattering amplitude outside the experimental region one is faced with the problem of analytic extrapolation. A similar situation can also be encountered in other problems where quantities in question are analytic functions. In cases shown in Figs. 1a and 1b the problem can be transformed [1—5] by a conformal mapping to an extrapolation off a part of the boundary of simply or doubly connected regions shown in Figs. 2a and 2b, respectively. A method for analytic extrapolation in situations in Figs. 2a and 2b was recently proposed in [1—3][1].

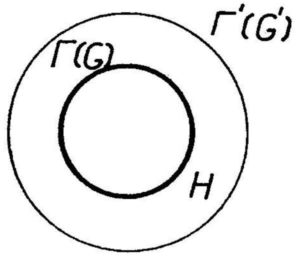

Fig. 2a. A conformal mapping of the analyticity region in Fig. 1a, suitable for analytic extrapolations. Before the mapping an additional cut along G was made.

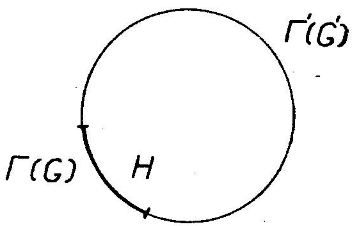

Fig. 2b. A conformal mapping of the analyticity region in Fig. 1b, suit able for analytic extrapolations.

The purpose of the present paper is to provide a general mathematical basis of extrapolations like those in [1—3].

To indicate the main idea let us consider the simple situation shown in Fig. 2b.

Let $f(z)$ and $g(z)$ be analytic in H: $z < 1$ and continuous in H. The boundary of H is denoted as C and consists of the „unknown" part Γ' and the „experimental part" Γ on which $f(z)$ is given. Under these conditions we can write:

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma+\Gamma'} f(z')\left[\frac{1}{z'-a} - g(z')\right] dz', \qquad a \in H. \tag{1}$$

According to [6] there exists a series of polynomials $\{g_n(z)\}$ converging uniformly to $(z-a)^{-1}$ on Γ'. Hence we obtain from (1):

$$f(a) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\Gamma} f(z')[(z'-a)^{-1} - g_n(z')] dz', \tag{2}$$

[1] The method used in [1—3] is based in fact on E-distributions described below. An alternative method is based on polynomial expansions. It has recently been improved in important papers by Cutkosky and Deo [4] and by Ciulli [5].

expressing thus $f(a)$ in terms of $f(z)$ on the „experimental" part of the boundary. If we were interested in a linear combinatios of derivatives of $f(z)$ in a point $a \in H$, we could proceed in the same way to obtain

$$\sum_{0}^{N} b_k f^{(k)}(a) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\Gamma} f(z') \left\{ \sum_{0}^{N} \frac{b_k k!}{(z'-a)^{k+1}} - g_n(z') \right\} dz'. \tag{3}$$

Here the sequence $\{g_n(z)\}$ uniformly approximates the first term in braces in eq. (3).

Proceeding to a further generalization, we can consider a class of functionals (hereafter referred to as E-distributions) defined on boundary values of functions analytic in H. In section 2 we shall show that (under suitable restrinctions on $f(z)$ a deneral functional T of this type is always representable by a sequence of continuous functions $\{\tau_n(z)\}$ defined on C and such that

$$T[f] = \lim_{n\to\infty} \int_C f(z)\tau_n(z) dz. \tag{4}$$

According to the theorem by Walsh [6], there exists a sequence of polynomials $\{g_n(z)\}$ (n does not denote the order of the polynomial) such that

$$|\tau_n(z) - g_n(z)| < \frac{1}{n} \qquad \text{for any } n \text{ and } z \in \Gamma'. \tag{5}$$

Hence

$$T[f] = T_{(\Gamma)}[f] = \lim_{n\to\infty} \int_{\Gamma} f(z)[\tau_n(z) - g_n(z)] dz. \tag{6}$$

Equation (6) shows that any functional T whose support is the whole boundary C is equal to a functional $T(\Gamma)$ whose support is only Γ. This is not quite surprising since a function analytic in H is uniquely determined by its values on Γ. Note that eq. (6) and its derivation shows also a close connection between approximations by analytic functions and theorems on uniqueness of analytic functions.

The derivation of eq. (6) is based on two points:

i) a functional defined on boundary values of analytic functions (an E-distribution) can be expressed in the form of eq. (4). Conditions under which this theorem is valid are given together with the proof in section 2.

ii) any function continuous on an arc Γ' can be approximated uniformly to any desired accuracy by polynomials. The relevant theorem by Walsh [6] and its generalization by Mergelyan [7] are quoted in detail in section 2.

A particular of case eq. (2), which is perhaps most important for practical applications, depends on item ii) and on

iii) a function $f(z)$ can be expressed in terms of its Cauchy integral over the boundary of the analyticity region. Most general results in this respect are due to Privaloff [8] and are described together with simple generalizations in sec. 3.

The rest of the paper is organized as follows: section 2 is devoted to E-distributions in the case when analyticity region of basic functions is the unit circle. In sec. 3 we discuss analytic extrapolations (like that in eq. (2)) in multiply connected regions and with less restrinctions on boundary values of extrapolated functions. In sections 4 and 5 a possibility to use E-distributions in theories of singular integral equations and of biorthogonal series is briefly mentioned. Conclusions and comments are given in section 6.

II. E-DISTRIBUTIONS IN A CASE OF THE UNIT CIRCLE

In the present section we shall define E-distributions as linear continuous functionals on boundary values of analytic functions. We consider here explicitely only the case of the unit circle. The reasoning is also valid (with unimportant modifications) for simply connected domains which can be conformally mapped into the unit circle. A part of the procedure is almost identical to the one used in the theory of distributions[2] on functions of the real variable. In such places we omit any comments or details. We define first the space D of basic functions as follows:

Definition 1: *A function $f(z) \in D$ if $f(z)$ is analytic in the interior of the unit circle K: $|z| < 1$, continuous in K: $|z| \leqslant 1$, and all the derivatives of the function $f(z)$ exist on the unit circle C: $|z| = 1$ and are finite.*

If a linear combination of functions is defined in the usual way, the space D becomes a vector space.

Definition 2: *A sequence $\{f_j\}$, $f_j \in D$ is said to converge to $f \in D$ if any sequence of derivatives $\{f_j^{(p)}\}$ converges uniformly to $f^{(p)}$ for $z \in C$: $|z| = 1$.* The convergence is denoted as $f_j \Rightarrow f$.

Definition 3: *An E-distribution is a linear continuous* (in the sense of Def. 2) *functional on D.* The set of E-distributions is denoted as D'.

Definition 4: *The p-th derivate of $T \in D'$ is defined by the relation*

$$T^{(p)}[f] = (-1)^p T[f^{(p)}]. \qquad (7)$$

Our further aim is to show that any $T \in D'$ can be represented in the form (4). To start with we introduce some notation. Let Q: $|z| > 1$ denote the complement

[2] A detailed exposition of the subject may be found in [9–11].

of K, and $K_a(z) \equiv (2\pi i)^{-1}(a - z)^{-1}$ is the Cauchy kernel. If $a \in Q$, then $K_a(z) \in D$. The function

$$\hat{T}(a) = T[K_a],$$

is said to be the Cauchy representation of a distribution $T \in D'$. It is easy to show that $T(a)$ is analytic in Q, and that

$$\hat{T}^{(n)}(a) = \frac{d^n}{da^n} \hat{T}(a)\, T\left[\frac{n!(-1)^n}{2\pi i(a - z)^{n+1}}\right], \qquad a \in Q.$$

The desired theorem is now obtained in the following form:

Theorem 1: *Let $T \in D'$, $f \in D$ then*

$$T[f] = \lim_{r \to 1} \int_C T(x/r) f(x) dx.$$

The proof of the „central" Theorem 1 is based on two lemmas. In their formulation we shall use auxiliary functions $f_r(z)$ defined as follows: Let $f \in D$, then $f_r(z) = rf(rz)$, where $0 < r < 1$. Noting that $f_r(z)$ is defined and analytic for $|z| < \frac{1}{r}$ we can write

$$fr(z) = \frac{1}{2\pi i} \int_{C(1/r)} dx f(rx) \frac{r}{x - z} = \int_C f(x) K_{(x/r)}(z) dx, \qquad (8)$$

where $C_{(1/r)}$ is the circle with radius $(1/r)$ and $K_{(x/r)}(z)$ is given by the previous notation for the Cauchy kernel. Let $I_N(z)$ be the integral sum corresponding to the last expression in eq. (8)

$$I_N(z) = \sum_1^N f(x_K) K_{(x_K/r)}(z) \Delta x_K. \qquad (9)$$

The function $I_N(z) \in D$ and it is easy to prove that $I_N(z) \Rightarrow f_r(z)$. The proof which is omitted for its simplicity rests on the fact that the Cauchy kernel $K_{(x_K/r)}(z)$, $0 < r < 1$, $|x_K| = 1$, has no singularities in K. Thus we have:

Lemma 1: *Let $f \in D$ and $I_N(z)$ be given by eq.* (9). *Then $I_N(z) \Rightarrow f_r(z)$, $z \in K$, $f_r(z) = rf(rz)$ and $0 < r < 1$.*

The following Lemma 2 shows that the functions $f_r(z)$ converge (in the sense of definition 2) to $f(z)$.

Lemma 2: *Let $f \in D$ and $f_r(z) = rf(rz)$, $0 < r < 1$. Then $f_r(z) \Rightarrow f(z)$ for $r \to 1-$.*

Proof: By the definition $f_r(z) \in D$. We have therefore only to prove that the

p-th derivate of $f_r(z)$ converges uniformly to $f^{(p)}(z)$ for $z \in C$ and for any natural p.

This follows from the inequality:

$$|f_r^{(p)} - f^{(p)}(z)| = |r^{(p+1)}f^{(p)}(rz) - f^{(p)}(z) + p^{p+1}f^{(p)}(z) - r^{p+1}f^{(p)}(z)| \leq$$
$$\leq p^{p+1}|f^{(p)}(rz) - f^{(p)}(z)| + |f^{(p)}(z)|(1 - r^{p+1}).$$

Recalling now that $\max_{z\in K}|f^{(p)}(z)| = M_p < \infty$ and tha $f^{(p)}(z)$ is uniformly continuous on K we can see that $f_r^{(p)}$ converges uniformly to $f^{(p)}$ on C. This completes the proof.

Using Lemma 1 and 2 we are now in a position to give the proof of Theorem 1: Using the definition of $T(z)$ we have:

$$\lim_{r\to 1} \int_C \hat{T}(x/r)f(x)\mathrm{d}x = \lim_{r\to 1} \int_C T[K_{(x/r)}(z)]f(x)\mathrm{d}x. \tag{10}$$

In eq. (10) the distribution T „acts" only on the variable z. The second integral in eq. (10) may be rewritten by using integral sums (9) such as

$$\int_C T[K_{(r/x)}(z)]f(x)\mathrm{d}x = \lim_{N\to\infty} \sum_1^N T[K_{(x_K/r)}(z)]f(x_K)\Delta x_K =$$
$$= \lim_{N\to\infty} T[\sum_1 K_{(x_K/r)}(z)f(x_K)\Delta x_K] = \lim_{N\to\infty} T[I_N(z)]. \tag{11}$$

According to Lemma 2 $I_N(z) \Rightarrow f_r(z)$ and since T is a continuous functional we have

$$\lim_{N\to\infty} T[I_N(z)] = T[\lim_{N\to\infty} I_N(z)] = T[f_r(z)].$$

In Lemma 2 we have shown that $f_r(z) \Rightarrow f(z)$ for $r \to 1-$ and therefore $\lim_{r\to 1} T[f_r(z)] = T[f]$. This shows finally that

$$\lim_{r\to 1-} \int_C \hat{T}(x/r)f(x)\mathrm{d}x = T[f], \tag{12}$$

and completes the proof. The proof was a rather cumbersome one. Note, however, that simple proofs of Theroem 1 known from the theory of distributions on real functions [12, 13] cannot be used here, since the E-distribution T is defined only on boundary values of analytic functions. This excludes some steps used in [12, 13].

In the Introduction we have indicated how any E-distribution whose support is the whole boundary can be expressed in a form of an E-distribution whose support is an arc on the boundary. In fact this property of E-distributions may be formulated in a more general way. As a hint one can use the theorem on the determination of an analytic function by its boundary values:

Theorem 2 (Privaloff [8], chapt. I, § 6). *Let $f(z)$ be analytic and bounded in K: $|z| < 1$. Let further the radial boundary values of $f(z)$ be equal to zero on a set of non zero measure on the unit circle $|z| = 1$. Then $f(z) \equiv 0$ in K.*

Paraphrasing the Privaloff theorem one may say that a bounded analytic function is uniquely defined by its boundary values on a set of non zero measure.

A natural conjecture arising from the Privaloff theorem is that any E-distribution on the unit circle can be expressed as an E-distribution whose support is an arbitrary non zero measure set on the unit circle. This is, with an additional restriction, really true.

Let Γ be a non zero measure set on C: $|z| = 1$ such that the set $E = C - \Gamma$ is closed. Then the set E is closed, bounded, has no internal points and does not separate the plane. The function $T(x/r)$ is continuous for $x \in E$. Conditions for the applicability of the Mergelyan theorem are fulfilled.

Theorem 3 (Mergelyan [7])[3]. *If E is a closed, bounded set not separating the plane, and if $f(z)$ is continuous on E and analytic in the interior points of E, then $f(z)$ can be uniformly approximated on E as closely as desired by polynomials.* According to the Mergelyan theorem there exist polynomials $P_r(z)$ (r does not denote the degree) such that

$$|\hat{T}(x/r) - P_r(x)| \leq 1 - r, \qquad x \in E, \qquad 0 < r < 1. \tag{13}$$

If $f \in D$, then $\int_C f(x)P_r(x)\mathrm{d}x = 0$ and we obtain for any $T \in D'$:

$$T[f] = \lim_{r\to 1-} \int_C [\hat{T}(x/r) - P_r(x)]f(x)\mathrm{d}x = \lim_{r\to 1-} \int_\Gamma [\hat{T}(x/r) - P_r(x)]f(x)\mathrm{d}x. \tag{14}$$

In deriving eq. (14) we have used eqs. (12, 13) and the boundedness of $f(x)$ on C. Eq. (14) is the desired expression of an E-distribution T as a functional whose support is the set 5. E-distributions $T_a[f] = f(a)$, $a \in K$ (the point a may lie also on the boundary) can also be expressed in the form of eq. (12). If $a \in K$, we have $T_a(z) = (2\pi i)^{-1}(z - a)^{-1}$ and if $a \in C$, we can write:

$$f(a) = \frac{1}{2\pi i} \lim_{r\to 1-} \int_C \frac{1}{\frac{x}{r} - a} f(x)\mathrm{d}x.$$

[3] The Mergelyan theorem is the most general result on approximation of functions by polynomials in the complex plane. The previous and less general form of the theorem is due to Walsh [6]. The Walsh theorem says that a function continuous on a Jordan arc of the finite complex plane can be uniformly approximated to any desired accuracy by polynomials.

Proceeding as above we can construct a serie of polynomials $P_r(z)$ such that $\left|\left(\frac{x}{r} - a\right)^{-1} - P_r(x)\right| < 1 - r$ for any $z \in C - \Gamma$ and get

$$f(a) = \frac{1}{2\pi i} \lim_{r\to 1-} \int_{\Gamma} \left[\frac{1}{\frac{x}{r} - a} - P_r(x) \right] f(x)\mathrm{d}x. \qquad (15)$$

To extrapolate to a point on the boundary we can proceed also in a different way. Denoting $a = \exp(i\vartheta_0)$ and constructing a series of functions

$$t_{a,\sigma}(z) = \frac{1}{iz\sigma\sqrt{2\pi}} \exp\left\{\frac{1 - \cos(\arg z - \vartheta_0}{\sigma^2}\right\}$$

we easily obtain

$$f(a) = \lim_{\sigma\to 0} \int_{C} t_{a,\sigma}(z) f(z)\mathrm{d}z, \qquad a \in C. \qquad (16)$$

Since $t_{a,\sigma}$ are continuous on C one may again use the preceding procedure to reexpress $f(a)$ in terms of boundary values of $f(z)$ over Γ. A similar approach was used in [2].

III. EXTRAPOLATION OF AN ANALYTIC FUNCTION OFF A PART OF THE BOUNDARY OF ANALYTICITY REGION IN A MORE GENERAL CASE

In the present section we shall extend some of the considerations of section 2 to more general cases. Arguments of sec. 2 were essentially based on the central Theorem 1 and on the Mergelyan theorem. The Mergelyan theorem is easily applicable also to multiply connected regions, while the extension of Theorem 1 to such situations remains only a conjecture (we were unable to prove such a generalization). We shall therefore concentrate here only on the practically important case of an extrapolation of a function off a part of the boundary. We shall first generalize eq. (15) to the most general class of functions analytic in a simply connected region and later on we shall generalize eq. (15) to multiply connected regions.

A basis point in the derivation of eq. (15) was item iii) mentioned in the Introduction.

The most general class of functions analytic in a simply connected region G and representable in terms of their Cauchy integral over the boundary γ was found by Privaloff [8]. The class is denoted as E_1 and defined as follows:

Definition 5 (Privaloff [8]: *A function $f \in E_1$ if $f(z)$ is analytic in a simply connected region G bounded by a rectifiable curve γ and there exists a constant C and a sequence of curves $\{\gamma_K\}$, $\gamma_K \subset G$, topologically converging to γ and such that $\int_{\gamma_K} |f(z)||\mathrm{d}z| \leqslant C$ for any γ_K.*

According to [8] for any $f \in E_1$ there exist non tangential boundary values of $f(z)$ on γ (except perhaps for a zero measure set) and $f(z)$ is representable by a Cauchy integral over the boundary γ.

For any $f \in E_1$ we can therefore apply the procedure of section 2 and express $f(a)$ for any $a \in G$ in terms of boundary values on a part of γ. The result is then given by eq. (2).

Next we shall discuss the generalization of eq. (15) to multiply connected regions.

Let $G_1, G_2, \ldots G_n$ be simply connected disjoint regions bounded by rectifiable curves $\gamma_1, \gamma_2, \ldots, \gamma_n$. Let $G_1, G_2, \ldots, G_n$ lie in the interior of a region G_0 bounded by a rectifiable curve γ_0. A multiply connected region $\boldsymbol{g}$ is then defined as $\boldsymbol{g} = G_0 - nl \bigcup_1^n G_j$ and the boundary of $\boldsymbol{g}$ is equal to $\bigcup_0^n \gamma_j$ (see Fig. 3). An analogon $[E_1]$ of the space E_1 is then defined by.

Definition 6 *A function $f(z) \in [E_1]$ if $f(z)$ is analytic in $\boldsymbol{g}$ and there exist a constant C and $(n + 1)$ sequences of simple curves $\{\gamma Kj\}_{K=1}^{\infty}$, $j = 0, 1 \ldots, n$ such that $\{\gamma Kj\}_{K=1}^{\infty}$ converges topologically to γ_j for $j = 0, 1, \ldots, n$, and that $\int_{\gamma Kj} |f(z)||\mathrm{d}z| < C$.*

Then there exist simply connected regions H_1 and H_2 bonded by rectifiable curves L_1 and L_2 such that $H_1 \cup H_2 \subset \boldsymbol{g}$, $H_1 \cup H_2 = g$ (see Fig. 3). If $f(z) \in$

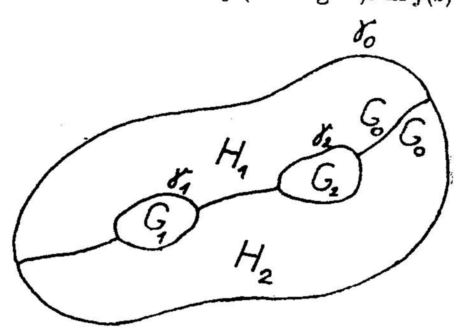

Fig. 3.

$\in [E_1]$ in g, then $f(z) \in E_1$ in the regions H_1 and H_2. According to [8] there exist non tangential boundary values of $f(z)$ on $\gamma \subset L_1 \cup L_2$ (except perhaps for a zero measure set). After a simple reasoning one also proves that $f(z)$ can by represented by its Cauchy integral over its non tangential boundary values on γ. Thus for any $a \in \boldsymbol{g}$ the following holds

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{x-a} dx. \qquad (17)$$

Let L be a non zero measure set on γ such that the set $\gamma - L$ is closed. Let a set E be defined as: $E = n \bigcup_0^n E_j$, where $E_j = L \cap \gamma_j$ if L and γ_j are not disjoint. The situation is shown in Fig. 4. We hawe assumed there that γ_0 and L are not disjoint. If this is not the case, we can always perform (prior to extrapolation) a conformal mapping so that the new γ_0 is not disjoint with L. The set E is bounded, closed and does not separate the complex plane. The function $(z - a)^{-1}$ is continuous in E and analytic in the interior of E. Sinec $\gamma - L \in E$, the Mergelyan theorem shows that there exists a set of polynomials $P_n(z)$ such that

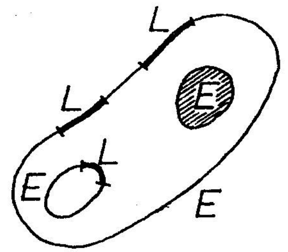

Fig. 4.

$$|(x-a)^{-1} - P_n(x)| < \frac{1}{n} \text{ for } a \in \mathbf{g},\ x \in \gamma - L. \qquad (18)$$

Using eq. (18) and proceeding as above we obtain

$$f(a) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_L f(x)\,[(x-a)^{-1} - P_n(x)]dx. \qquad (19)$$

The eq. (19) is again a constructive (at least in principle) paraphrase of the fact that an analytic function is uniquely given by its values on a part of the boundary of the analyticity region.

Finally let us note that we have so far used the Mergelyan theorem to show that there exists a sequence of polynomials approximating uniformly a given function or a given sequence of functions. This is actually more than one needs. For instance, in eqs. (18, 19) the functions $P_n(x)$ need only to be analytic and bounded in $\mathbf{g}$ (so that $f\ P_n$ belongs to E_1). Sometimes [3] it appears suitable to start with constructing the functions corresponding to $[(x-a)^{-1} - P_n]$ in eqs. (18, 19) directly, instead of constructing polynomial approximations to $(x-a)^{-1}$.

IV. POSSIBLE APPLICATION OF E-DISTRIBUTIONS TO SINGULAR INTEGRAL EQUATIONS

Here we shall show that the problem of analytic extrapolation, solved above in terms of E-distributions is equivalent to a particular class of singular integral equations.

Let $\Phi(z)$ be a function analytic in a region S^+ bounded by a simple smooth curve γ consisting of two arcs L and K (see Fig. 5a). Let $\Phi(z)$ be continuous in S^+ and let its bundary values on L and K be denoted as $\varphi(t)$ and $\psi(t)$, respectively. The complement of S^+ with respect to the whole complex plane is denoted as S^-. According to ref. [14] § 29 we have

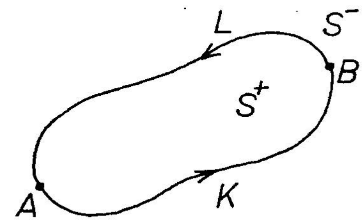

Fig. 5a.

$$\Omega(\partial) \equiv \int_L \frac{\varphi(t)dt}{t-\partial} + \int_K \frac{\psi(t)dt}{t-\partial} \equiv 0 \text{ for any } \partial \in S^-. \qquad (20)$$

If $z \in L$, we can perform the limit $\partial \to z$ to obtain

$$\varphi(z) + \frac{1}{i\pi} P \int_L \frac{\varphi(t)dt}{t-z} = -\int_K \frac{\psi(t)dt}{t-z} \quad \text{for } z \in L, z \neq \text{A, B}. \qquad (21)$$

where P denotes the principal value.

On the other hand, eq. (21) implies that the function defined by eq. (20) (by construction $\Omega(\partial)$ is analytic in S^- has zero boundary values on L and is consequently identical equal to zero in S^-. This in turn implies that $\varphi(t)$ on L and $\psi(t)$ on K are boundary values of the function analytic in S^+.

To find a solution $\varphi(t)$ of eq. (21) is therefore equivalent to an analytic extrapolation (through S^+) of $\psi(t)$ on K to $\varphi(t)$ on L. The solution of this problem in terms of E-distributions was given in sections 2 and 3. Changing the role of S^+ and S^- (as shown in Fig. 5b) while keeping the same orientation of L and K, leads in the same way to the equation

$$-\varphi(z) + \frac{1}{i\pi} P \int_L \frac{\varphi(t)dt}{t-z} = -\frac{1}{i\pi} \int_K \frac{\psi(t)dt}{t-z} \quad z \in L. \qquad (22)$$

In this case the functions $\varphi(t)$ and $\psi(t)$ are boundary values of a function $\Phi(z)$ analytic in S^+ of Fig. 5b.

The eqs. (31 and 32) are very special cases of equations:

$$\pm\varphi(z)+\frac{1}{i\pi}P\int_L\frac{\varphi(t)dt}{t-z}=\alpha(t),\quad z\in L. \tag{23}$$

Note, however, that eqs. (23) are just those singular integral equations which cannot be solved by a reduction to the non homogenous Riemann-Hilbert mapping problem (ref. [14] § 96).

There arises naturally the conjecture that perhaps also the general type of irregular integral equations (23) has something to do with analytic extrapolations and E-distributions.

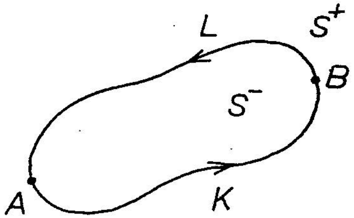

Fig. 5b.

V. E-DISTRIBUTIONS AND BIORTHOGONAL SERIES

A suitable set of E-distributions can represent a biorthogonal system to a set of functions which otherwise does not have its biorthogonal system.

Consider the set of functions $\{vz^n\}_1^\infty$ on an arc $\Gamma: |z|=1$ and $0<\alpha<\arg z<$ $<\beta<2\pi$. The system $\{vz^n\}$ is linearly independent on Γ. In fact if $\sum_0^N a_n z^n=0$ for any $z\in\Gamma$, then $a_n=0$ for $n=0,1,\ldots,N$. This is easily proved if one realizes that $\sum_0^N a_n z^n$ defines an entire function which is then equal on the arc Γ.

Within a set of functions continuous on Γ there does not exist a system $\{\varphi_n\}_1^\infty$ biorthogonal to $\{z^n\}$. If there were such a system, the following would hold (we use the simplest weight $1|dz|=dz/(iz)$):

$$\int_\Gamma |dz|\varphi_K(z)z^n=\delta_{n,K}\quad n=0,1,2,\ldots\ K=0,1,2,\ldots \tag{24}$$

This is however impossible. Consider the function φ_0^*/z. It is continuous on Γ and the theorem by Walsh [6] applies. Consequently to any positive ε there exists a polynomial $P_\varepsilon(z)$ such that

$$|\varphi_0^*(z)-zP_\varepsilon(z)|<\varepsilon\qquad\text{for } |z|=1,\quad z\in\Gamma. \tag{25}$$

Using now eqs. (22, 24) we obtain

$$\int_\Gamma|dz||\varphi_0(z)|)\leqslant\varepsilon\int_\Gamma|dz||\varphi_0(z)|. \tag{26}$$

Since eq. (26) holds true for any positive ε, it follows that $\varphi_0(z)=0$ on Γ. This contradicts eq. (21) taken for $n=k=0$ and completes the proof.

A system biorthogonal to $\{z^n\}_1^\infty$ is easily constructed in terms of E-distributions.

Let us introduce a set of functionals τ_n defined on any polynomial $P(z)$ by the relation

$$\tau_n(P)=\lim_{\varepsilon\to0}\frac{1}{2\pi i}\int_\Gamma\frac{dz}{z}\left[\frac{1}{z^n}-zQ_{n,\varepsilon}(z)\right]P(z), \tag{27}$$

where $Q_{n,\varepsilon}$ is a polynomial whic satisfies the inequality[4] $|z^{-(n+1)}-Q_{n,\varepsilon}(z)|<\varepsilon$ on Γ'. The existence of $Q_{n,\varepsilon}$ is again secured by the Walsh theorem [6].

It is now easy to prove that

$$\tau_n\,[z^K]=\delta_{n,K}\qquad\text{for } n,K=0,1,\ldots, \tag{28}$$

hence the system of functional $\{\tau_n\}$ is biorthogonal to $\{z^n\}_1^\infty$.

Note that τ_n are just the E-distributions which assign to any function $f(z)\in D$ (see section 2) its derivative in the origin multiplied by $(n!)^{-1}$:

$$\tau_n[f]=\frac{1}{n!}f^{(n)}(0).$$

VI. COMMENTS AND CONCLUSIONS

The basic difference between E-distributions and distributions acting on functions of a real variable lies in the concept of support. A function of a real variable has, loosely speaking, to be defined locally. An analytic function is determined uniquely by its values on any arc Γ on the boundary of the analyticity region. E-distributions share this property and any E-distribution can be expressed so that Γ is its support.

The concept of E-distribution shows also a close connection between theorems about the unique determination of analytic functions and theorems about approximations of functions by analytic functions. As a trivial example one can for instance prove — by using the Walsh theorem [6] and the procedure outlined above — that an analytic function is uniquely given by its

[4] The set Γ' is given as $\Gamma': |z|=1, z\notin\Gamma$.

values over an arc on the boundary of the analyticity region. As a non trivial example one could perhaps use the Mergelyan theorem [7] and the procedure of section 3 to prove general theorems about the unique determination of an analytic function by its boundary values.

In practical applications E-distributions can be useful when a constructive prescription for an extrapolation off a part of a boundary is needed. Attempts in this direction were done in [1–3]. Let us recall briefly the problems arising in such attempts on practical extrapolations[5]. Within the present approach an extrapolated quantity is in general given in the following form:

$$A = \lim_{n\to\infty} \int_{\Gamma} a_n(z) f(z) dz + \lim_{n\to\infty} \int_{\Gamma} a_n(z) f(z) dz = \lim_{n\to\infty} \int_{\Gamma} a_n(z) f(z) dz. \tag{29}$$

Here $\{a_n(z)\}$ is a sequence of functions (uniformly converging to zero on $\gamma - \Gamma$), Γ is the arc we extrapolate off, γ is the whole boundary and $f(z)$ on Γ represents the experimental data. Sequences $\{a_n(z)\}$ show increasing oscillation and increase in the max $|a_n(z)|$ on Γ with increasing n. This is quite natural since a function $f(z)$ should be „seized well" by $\{a_n(z)\}$ before being extrapolated well. Experimental data are always subject to experimental errors and then instabilities in the determination of A develop. They forbid to take the limit $n \to \infty$ in (29). The best that can then be done is to find an n_0 which minimalizes the error in the determination of A and truncate the limiting procedure in (29) on that n_0. Note that for $n_0 < \infty$ the arc $(\gamma - \Gamma)$ on which there are no experimental data contributing to A. The evaluation of n_0 then requires some modest a priori information about $f(z)$ on $(y - \Gamma)$.

The sequence $\{a_n(z)\}$ in eq. (29) is generally not given in a unique way (even if the definition of A and arc Γ are given). Then there arises a natural question of finding an optimal sequence $\{a_n(z)$s. This might practically amount to finding a sequence with minimal oscillations or with the slowest increase of the max $|a_n(z)|$. The understanding of these questions would improve the reliability of practical extrapolation methods.

The authors would like to thank Dr. J. Fischer (Prague) for stimulating discussions and correspondence. One of the authors (J. P.) is indebted to Professor R. E. Cutkosky (Pitsburg and Cambridge) for valuable conversations.

[5] The most complicated problems in analytic extrapolations are connected with the stability of extrapolated quantities and are common to all extrapolation procedures ([1–5]).

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Received Marc h 26th, 1970