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CONSTRAINTS FOR PARTIAL WAVES FROM REGGE ASYMPTOTICS AND PION-NUCLEON SCATTERING IN P_{11} STATE[1,2]

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The constraints for partial wave scattering amplitudes from Regge asymptotics are studied and imposed on an appropriately chosen N/D ansatz. It turns out that generally subtractions in the N and D functions (i. e. CDD poles) have to be included, whose parameters are fixed by the constraints. We discuss an ansatz where the lower energy region can be treated by means of a cutoff theory. The cutoff indicates the region where the Regge behaviour of partial waves is essentially given by an expansion with respect to inverse powers of the logarithm of energy and of the energy itself. Application to pion-nucleon scattering in the P_{11} state yields a phase shift in agreement with the experimental one (including a second resonance at about 1750 MeV). Here at least one CDD pole is necessary in order to satisfy already the leading asymptotics. All parameters are determined by the asymptotic constraints, except the cutoff. As required the latter comes out sufficiently high ($\ln \Lambda \gg 1$) and we find that the amplitude depends only weakly on it.

I. INTRODUCTION

We study the constraints for partial wave scattering amplitudes as implied by Regge behaviour of the total amplitude. Under obvious assumptions we find [1] generalizing well-known results [2—5] that the asymptotic behaviour of the partial waves is given by an expansion with respect to the inverse of the logarithm of energy and of the energy itself. The absence of terms of pure power behaviour $1/s^n$ ($n = 0, 1, \ldots$) leads to the constraints. We incorporate the constraints in an appropriately chosen N/D ansatz where generally subtractions in the N and D functions (i. e. CDD poles) have to be included, whose parameters are fixed by the constraints [6]. We understand

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[1] Talk given at Elementary Particle Physics Seminar at Smolenice, June 1—2, 1970

[2] Part of this work was done while one of the authors (F. K.) was staying at JINR Dubna and at CERN Geneva.

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the whole mechanism in such a way that in the lower energy region only a small number of subtractions is necessary, which has to be enlarged if one tends to higher energies (thus depending on the accuracy to which one fits a partial wave to a given asymptotics). Our special N/D ansatz is chosen such that the lower energy region can be treated by a cutoff theory. Here the cutoff indicates the region where the asymptotic expansion is valid. For the sake a simplicity we approximate the nearby singularities by pole terms.

In section 2 we state the results concerning the asymptotic behaviour for a scalar model, i.e. for a two-particle scattering of scalar identical particles. The detailed derivation and discussion can be found in ref. [1]. In section 3 we apply the theory to pion-nucleon scattering in the P_{11} state. Here at least one CDD pole is necessary (as a consequence of the special spin structure) in order to satisfy already the leading asymptotics. All parameters including the pion-nucleon constant and the nucleon mass are in principle determined by the constraints, except the cutoff. The calculated phase shift imposing in the N/D equations as input N^*, ϱ and N exchange is in agreement with the experimental one, i. e. in the lower energy region we find the zero, the Roper resonance and a second resonance at about 1750 MeV. As required the needed cutoff is sufficiently high ($\ln\Lambda \gg 1$).

II. THE SCALAR MODEL

We consider a two-particle scattering of scalar identical particles with the mass m and write the partial wave scattering amplitude in the form

$$A_l(s) = \frac{e^{i\delta_l(s)} \sin \delta_l(s)}{\varrho(s)\, R_l(s)} . \tag{1}$$

Here $\delta_l(s)$ is the phase shift, $\varrho(s)$ denotes the phase space function

$$\varrho(s) = \sqrt{\frac{s - 4m^2}{s}} \tag{2}$$

and

$$R_l(s) = \frac{\mathrm{Im} A_l(s)}{\varrho(s)\, |A_l(s)|^2} \geqslant 1 \tag{3}$$

is the ratio of the total to the elastic cross section in the l^{th} partial wave. The asymptotic behaviour of the total amplitude as it is relevant for the partial wave projection will be assumed in the form

$$A(s,z) \underset{s\to\infty}{\longrightarrow} \frac{1}{\pi}\, \beta_P(t)\Gamma(1-\alpha_P(t))\,(1+e^{-i\alpha_P(t)})\, s^{\alpha_P(t)} + (t \to u) + \dots \tag{4}$$

$z = \cos \Theta$

Here the index P refers to the Pomeron and lower lying trajectories are involved in the rest[3]. Actually the asymptotic behaviour (4) follows from the Regge theory only for $t = \text{const}$ or $u = \text{const}$, respectively, whereas for a fixed $z \neq \pm 1$ all three variables s, t, u tend simultaneously to infinity. Thus in general expression (4) will describe the asymptotic behaviour for a fixed $z \neq \pm 1$ only if for $s \to +\infty$ $A(s,z)$ is strongly restricted to forward and backward scattering, respectively. This is the case, for instance, for all amplitudes of the Veneziano type which for a $z \neq \pm 1$ vanish exponentially for $s \to +\infty$. The same is true for the Pomeron if one adopts, for instance, the extra-Pomeranchuk term discussed recently in ref. [7].

The partial wave projection of expression (4) has the form

$$A_l(s) \underset{s\to+\infty}{\longrightarrow} \int\limits_{-(s-4m^2)}^{0} dx B_P(x) \frac{s^{\alpha_P(x)}}{s-4m^2} P_l\left(1 + \frac{2x}{s-4m^2}\right) + \dots \tag{5}$$

Since the contributions for $s \to +\infty$ are assumed to be restricted to forward and backward scattering, respectively, we may replace the lower limit by some $-T < 0$ (T arbitrarily large, if necessary). Then by expansion of $B_P(x)$ at some $x_0 < 0$, repeating the argumentation for the remaining trajectories which are all to be assumed linearly rising

$$\alpha_k(x) = a_k + b_k x \tag{6}$$

we may derive the asymptotic structure [1]

$$\begin{aligned} A_l(s) \underset{s\to+\infty}{\longrightarrow} \frac{s}{s-4m^2} a_l \Bigg\{ & \frac{i}{\ln s} - \frac{\pi}{2\ln^2 s} + \frac{ic_p}{\ln^2 s} + O(\ln^{-3}s) + \\ & + \frac{ic_{Pl}^{(1)}}{s\ln^2 s} + O(s^{-1}\ln^{-3}s) + \dots + \Omega(s^{-Tb}) + \dots \Bigg\} + \\ & + \frac{s}{s-4m^2} \sum_k \frac{a_l^{(k)}}{s^{1-a_k}} \left\{ \frac{e_l^{(k)} + ib_l^{(k)}}{\ln s} + \dots + \Omega(s^{-Tb}) + \dots \right\} . \end{aligned} \tag{7}$$

This result is an extension of expressions derived in ref. [2—5]. The contribution

[3] For simplicity we assume the absence of the Regge cuts. However, there are good reasons to believe that the general results we derive do not depend on this.

given by the first curly bracket is from the Pomeron; $\Omega(s^\gamma)$ means $O(s^\gamma)$ up to logarithmic factors and for the parameters we find

$$a_l = \frac{[1+(-1)^l]}{16\pi b_P}\,\sigma_{tot}(\infty); \qquad c_P = \frac{16\pi\beta'(0)}{b_P\sigma_{tot}(\infty)}; \qquad c_l^{(k)} \ldots \text{real.} \tag{8}$$

Thus we see that the asymptotic Regge behaviour of partial waves is given by a power expansion with respect to $1/s$ coupled with an expansion with respect to inverse powers of $\ln s$. The important point is that no pure power terms in $1/s$ appear, i. e.

$$\text{no terms in } 1/s^n \quad (n = 0, 1, 2, \ldots..) \tag{9}$$

which will lead to constraints discussed below[4]. Of course, expression (7) says generally nothing about the behaviour for $s \to -\infty$, where in particular for the linearly rising trajectories (6) we expect an exponential growth as it is known, for instance, for the Veneziano model. Thus, if we write down a dispersion relation for $A_l(s)$, we must incorporate this behaviour explicitly in the discontinuity in the asymptotic region $s \to -\infty$. Concluding this discussion we mention the following asymptotic behaviour of the inelasticity function $R_l(s)$ (uneven l excluded)

$$R_l(s) = -\frac{\operatorname{Im} A_l^{-1}(s)}{\varrho(s)} \xrightarrow[s\to+\infty]{} \frac{\ln s}{a_l}\left[1 - \frac{c_p}{\ln s} + \ldots\right]. \tag{10}$$

Let us now turn to the derivation of the dispersion relation for $A_l(s)$ incorporating the asymptotic behaviour (7). We assume first that expression (5) with the lower integral limit $-T$ is valid also for $s \to -\infty$. Then, restricting ourselves to the first leading term of relation (7) and observing from expression (5) (with lower limit $-T$) that in this order $A_l(s)$ is uneven in s, we may write down the following decomposition in asymptotic and non-asymptotic contributions

$$A_l(s) = \frac{s}{\pi}\int_{s_c}^{\infty} ds' \frac{a_l}{s'\ln s'(s'+s)} + \frac{1}{\pi}\int_{-s_c}^{-s_L} ds' \frac{\omega_l(s')}{s'-s} - \frac{g^2}{s-m^2}\delta_{l,0} + $$
$$+ \frac{1}{\pi}\int_{s_R}^{s_c} ds' \frac{\varrho R_l|A_l|^2}{s'-s} + \frac{s}{\pi}\int_{s_c}^{\infty} ds' \frac{a_l}{s'\ln s'(s'-s)}. \tag{11}$$

[4] We may expect that this general property is also right in the case of the Regge cuts and for trajectories which are not strictly linear. Moreover, if even pure power terms appear (with definite coefficients) they lead to constraints.

Here $(-\infty, -s_L)$ and $(s_R, +\infty)$ is the left-hand and right-hand cut, respectively, and $\omega_l(s)$ is the discontinuity across the left-hand cut. We denote the asymptotic region by $|s| > s_c$, where the expansion (7) is applicable (which means obviously $\ln|s| \gg 1$). We took into account the symmetry property for the asymptotic parts which certainly need one subtraction. However, the subtraction constant is determined and equal to zero since from the formula [5]

$$\frac{s}{\pi} P\int_{s_c}^{\infty} ds' \frac{1}{s'\ln s'(s' \mp s)} \xrightarrow[s\to+\infty]{} \mp \frac{1}{\pi}\ln\left[\frac{\ln s}{\ln s_c}\right] -$$
$$-\binom{1}{\frac{1}{2}}\frac{\pi}{3\ln^2 s} + O(\ln^{-4} s) + \frac{1}{s}\frac{s_c}{\pi\ln s_c} + \ldots \tag{12}$$

one verifies easily the desired asymptotic behaviour of the dispersion representation (11) (analogously one proceeds with the non-leading terms of expression (7)). We note that the compensation of the dominant (real) terms of relation (12) is actually necessary to get the required asymptotics: vanishing and purely imaginary (which is automatically guaranteed by the symmetry properties in s). We mention the following with respect to a possible blow-up of $A_l(s)$ for $s \to -\infty$: we describe this by replacing a_l by $a_l(s)$ in the asymptotic left-hand cut in relation (11), i. e. in the first term on the right-hand side, where $a_l(s)$ has an exponential growth for $s \to -\infty$ and tends to 1 stronger than any power for $s \to +\infty$. What we assume is that $a_l(s)$ is also of order 1 in the lower energy region ($|s| \ll s_c$), where the first integral is than of order $\Omega(s_c^{-1})$.

For the whole expansion (7) incorporated in a dispersion relation like eq. (11) one can show [8] that all ln-terms of the asymptotic expansion of the dispersion relation are in agreement with the required asymptotics (7), and this is true in any order $\Omega(s^{-n})$. The reason is that this dependence is due to the upper limit $+\infty$ of the asymptotic integrals in the dispersion relation. However, for the contribution from the finite integration region in a relation like (11) we get constraints according to condition (9), thus, e. g. for $n = 1$ ($n = 0$: subtraction constant equal to zero in eq. (11)[5]

$$\frac{2}{\pi}\frac{a_l s_c}{\ln s_c} + O(\ln^{-2} s_c) - \frac{1}{\pi}\int_{-s_c}^{-s_L} ds'\omega_l(s') - g^2\delta_{l,0} - \frac{1}{\pi}\int_{s_R}^{s_c} ds'\varrho R_l|A_l|^2 = 0. \tag{13}$$

[5] We assume uniqueness for the expansion (7) by means of the existence of a Mellin transform.

Thus we see quite generally that any solution of the generalized eq. (11) needs arbitrary parameters (besides the particle pole parameters g^2 and m^2), in order to make it consistent with the Regge asymptotic. Otherwise the solution would be overdetermined. The number of parameters depends, of course, on the number of constraints (9) we wish to fulfil, i. e. on the accuracy we intend to satisfy the asymptotics with. In the N/D approach discussed below this means obviously that we have to include CDD poles whose parameters are fixed by the constraints. We expect that practically for the treatment of the lower energy region only a few CDD poles are sufficient (in some cases possibly none) and that the inclusion of further poles is only necessary if one wants to reach also higher energy regions.

We proceed now to the discussion of an appropriate N/D ansatz and the formulation of the corresponding constraints. To simplify the relation we approximate the left-hand cut by near-by poles at s_j and the asymptotic part $(-\infty, -s_c)$ as it is indicated on Fig. 1. The quantity s_c appears then as

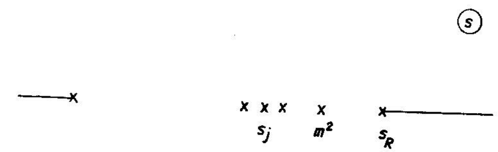

Fig. 1. Singularities in the s-plane

an adjustable parameter and we understand this approximation as reasonable if the required s_c is fitted as sufficiently high ($\ln s_c \gg 1$) and the solution depends only weakly on it. For convenience we put the particle pole in the N function and the asymptotic integrals in the D function. For the s-wave we write (dropping the partial wave index)

$$A(s) = \frac{N(s)}{D(s)} =$$

$$= \frac{f_0 + \sum\limits_j \frac{G_j}{s - s_j} - \frac{g^2 D(m^2)}{s - m^2}}{1 - \frac{s}{\pi}\int\limits_{s_R}^{s_c} ds' \varrho R \frac{N(s')}{s'(s'-s)} - \frac{s}{\pi}\int\limits_{s_c}^{\infty} ds' \varrho R \frac{N(s')}{s'(s-s')} - \frac{s}{\pi}\int\limits_{s_c}^{\infty} ds' (\varrho R)_L \frac{N(-s')}{s'(s'+s)}} . \quad (14)$$

Here $(\varrho R)_L$ is $-\operatorname{Im} A^{-1}(s)$ on the asymptotic left-hand cut which is known as in eq. (11) (as well as ϱR in the physical asymptotic region). We included one CDD pole in the sense that we use a once-subtracted D function, although

the particle pole is involved in $N(s)$. The reason is that we want to incorporate the Regge behaviour beyond the leading term. We mention that the parameter f_0 in $N(s)$ is necessary to get the Regge behaviour in the leading term with a once-subtracted D function (the purely imaginary behaviour is due to the compensation of the dominant real parts of the asymptotic integrals in $D(s)$). We mention that the case of no CDD pole is simply retained by dropping the particle pole in $N(s)$, whose parameters are then given by standard relations

$$D(m^2) = 0; \quad g^2 = -\frac{N(m^2)}{D'(m^2)} . \quad (15)$$

Here we have one free parameter f_0 to be determined by one asymptotic constraint (corresponding to $n = 0$ in eq. (5)). The ansatz (14) involves three free parameters f_0, g^2, m^2 to be fixed by three constraints. The explicit calculation [1] shows that they are given by (we write $N(s) \to f_0 + f_{-1}/s + \ldots$ for $s \to +\infty$)

$$\text{order } s^0: \quad \frac{1}{\pi}\int\limits_{s_R}^{s_c} ds \frac{\varrho R N}{s} + 1 + \Omega(s_c^{-1}) = 0 \quad (16)$$

$$\text{order } s^{-1}: \quad \frac{1}{\pi}\int\limits_{s_R}^{s_c} ds \varrho R N - \frac{2}{\pi}\frac{f_0}{a} s_c \ln s_c + O(s_c^{-1}) = 0 \quad (17)$$

$$\text{order } s^{-2}: \quad \frac{1}{\pi}\int\limits_{s_R}^{s_c} ds s \varrho R N - \frac{2}{\pi}\frac{f_{-1}}{a} s_c \ln s_c + O(s_c^{-1}) = 0. \quad (18)$$

If one would like to impose more constraints one generally has to include more subtractions in N and D (i. e. polynomials of a higher degree and this in N and D in the same order). Observing that the asymptotic integrals in $D(s)$ are in the lower energy region ($|s| \ll s_c$) of the order $\Omega(s_c^{-1})$ and that because of the constraint (16) the first two terms in $D(s)$ can be written unsubtracted, we arrive at the following N/D ansatz for $A(s)$ valid in the lower energy region

$$A(s) = \frac{N(s)}{D(s)} = \frac{f_0 + \sum\limits_j \frac{G_j}{s - s_j} - \frac{g^2 D(m^2)}{s - m^2}}{-\frac{1}{\pi}\int\limits_{s_R}^{s_c} ds' \varrho R \frac{N(s')}{s'(s'-s)}} . \quad (19)$$

This means that the behaviour in the lower energy region can be described by a definite cutoff theory, where, however, the constraints (16)—(18) refer to the full Regge asymptotics. Concluding this section we mention the following asymptotic behaviour of the phase shift

$$\delta(s) \to (n + \tfrac{1}{2})\pi + \frac{\pi}{2 \ln s} + \dots \qquad (20)$$

which violates the standard Levinson bound; n corresponds to the number of subtractions. We assume, of course, that further subtractions in the D function in eq. (14) lead to no zeroes (i. e. no poles of $A(s)$ excluding thereby common zeroes of N and D) and interprete the case where the corresponding subtractions in the N function do not impose zeroes on it as CDD poles at infinity (which obviously do not occur in the pole approximation).

III. PION-NUCLEON SCATTERING IN THE P_{11} STATE

We applied the above theory to πN scattering in the P_{11} state. It is well-known that here a one-channel N/D approach without CDD poles is not sufficient (at least, if one considers only nearby singularities as input. Analogously as in eq. (19) we use following ansatz for the amplitude

$$h_{11}(\omega) = \frac{N_{11}(\omega)}{D_{11}(\omega)} = \frac{\frac{16}{9}\frac{\gamma_{33} D_{11}(-\omega_{33})}{\omega + \omega_{33}} + \frac{c_{11}}{2 i \operatorname{Im}\omega}\left[\frac{D_{11}(\omega_\varrho)}{\omega - \omega_\varrho} - \frac{D_{11}(\omega_\varrho^*)}{\omega - \omega_\varrho^*}\right] + \frac{\lambda_{11}}{\omega}}{-\frac{1}{\pi}\int_1^{\Lambda} d\omega' \varrho_{11} R_{11} \frac{N_{11}(\omega')}{\omega' - \omega}}. \qquad (21)$$

The inputs are approximated by poles as indicated in Fig. 2 and we use the variable $\omega = W - M$, where $W = \sqrt{s}$ and M the nucleon mass.[6] The first term in $N_{11}(\omega)$ represents the N^* exchange, the second the ϱ exchange and the third the nucleon pole including the small contribution from the N exchange. The phase space function is used in the form [9] (the pion mass is put equal to 1)

[6] We drop here the question of the McDowell symmetry since we are only interested in the nearby-singularity region for the inputs. Of course, the treatment of the asymptotics requires a special discussion given elsewhere.

$$\varrho_{11}(\omega) = (\omega^2 - 1)^{3/2} \frac{1}{2} \frac{2 + \omega/M}{(1 + \omega/M)^3} \xrightarrow[\omega \to +\infty]{} \frac{M^2}{2}\omega \qquad (22)$$

where the threshold behaviour is handled in the standard way. The asymptotic behaviour is now of the form

$$h_{11}(\omega) \xrightarrow[\omega \to +\infty]{} i \frac{a_{11}}{\omega \ln \omega} + \dots, \qquad (23)$$

$$a_{11} = \frac{\sigma_{tot}^{(1)}(\infty)}{16\pi b M^2}$$

where $\sigma_{tot}^{(1)}(\infty)$ is the total pion-nucleon cross section in the isospin state $\frac{1}{2}$ and the inelasticity function $R_{11}(\omega)$ goes for large ω like

$$R_{11}(\omega) \xrightarrow[s \to +\infty]{} \frac{2}{M^2}\frac{\ln \omega}{a_{11}} + O(1). \qquad (24)$$

Because of relation (23) (with the extra factor $1/\omega$) there is now no constant (f_0) in $N_{11}(\omega)$ (compared to the ansatz (19) with the asymptotics $\sim i/\ln s$). This

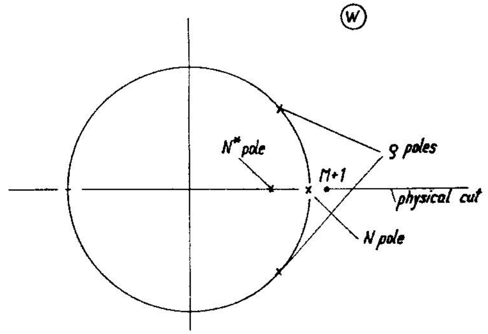

Fig. 2. Singularities in the W-plane. N pole $W = M$; N^* pole $W = M - 2{\cdot}2$; ϱ poles $W = M - 1{\cdot}8 \pm 4{\cdot}6\,i$.

means that here the introduction of one CDD pole is necessary, since otherwise the N/D system would be overdetermined, and this already for satisfying the leading asymptotics (compare the discussion following the formulae (14) and (15) and use there $f_0 = 0$). For Λ at about 20 000 ($\ln \Lambda \approx 10$) we find indeed sufficient agreement with the experimental data as shown in Fig. 3. For the inputs we used standard values. We represent the parameter dependences in Tabl. 1. The inelasticity $R_{11}(\omega)$ was interpolated smoothly from its known experimental values in the lower-energy region to its asymptotic value (24). In order to fix both λ_{11} and the nucleon pole position ($\omega = 0 \sim W = M$) we imposed a second constraint analogously to eq. (17). This relation depends only weakly on the nucleon pole position but determines the Pomeron

Table 1

Parameter dependences of the solution (the last both columns refer to the second constraint in the sense of eq. (17)).

Λ	g_{11}	$\frac{16}{9}\gamma_{33}$	c_{11}	zero of $N_{11}(\omega)$	zeroes of Re $D_{11}(\omega)$		$\frac{1}{\pi}\int \varrho R N d\omega$	$\frac{2}{\pi}\frac{f_1}{a}\Lambda \ln \Lambda$
50000	0·21	0·22	1·84	2·0	4·0	6·3	8·43.10³	8·52.10³
30000	0·22	0·22	1·93	1·0	4·0	6·3	7·83·10³	7·92.10³
10000	0·22	0·22	2·14	1·4	4·0	5·9	3·50.10³	3·42.10³
30000	0·20	0·23	1·64	1·2	2·8	6·4	0·90.10³	1·02.10³
70000	0·20	0·22	1·43	1·4	3·0	8·8	1·77.10³	2·00.10³

slope to be $b_P \approx 0.05$ GeV^{-2}.[7] It is very satisfying that the free cutoff parameter Λ must be chosen sufficiently high ($\ln \Lambda \gg 1$) to get agreement with the experimental data (in particular the zero, the Roper resonance and a second resonance at about 1750 MeV.

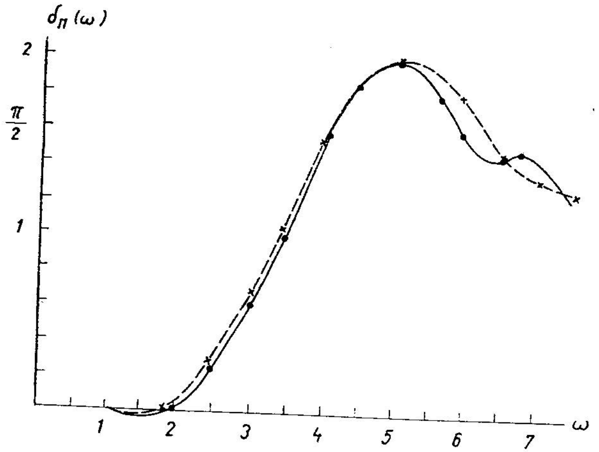

Fig. 3. Experimental — and theoretical —— phase shift $\delta_{11}(\omega)$ for the parameter values $\Lambda = 30000$, $\lambda_{11} = 0{\cdot}22$, $16/9\ \gamma_{33} = 0{\cdot}22$.

Now a program is in progress, where improved inputs were used and the McDowell symmetry is taken into account. It would be of great interest to impose also more constraints, i. e. to apply more subtractions, and to study the behaviour of the extended solutions in the lower energy region.

[7] A determination of the nucleon mass should probably need also the P_{33} amplitude in the sense of a reciprocal bootstrap.

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