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IMPOSITION OF CROSSING CONSTRAINTS IN THE UNITARIZED $\pi\pi$ VENEZIANO MODEL[1 2]

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The present paper reports on some work which was done in collaboration with R. Baier and F. Widder [1] and which aims at imposing unitarity on the partial wave amplitudes that follow from the Veneziano model for $\pi\pi$ scattering. Crossing symmetry which is an essential feature of the Veneziano model is violated, but by the use of exact conditions this violation is mimized. Problems of general interest are also reviewed.

I.

a. Information on $\pi\pi$ Scattering from Recent Experiments

The extrapolation of the forward-backward asymmetry to the pion pole in $\pi^- p \to \pi^+ \pi^- n$ [2] is the most model-independent analysis available for δ_0^0. This gives a unique solution which passes through $\pi/2$ near 720 MeV with a width of about 200 MeV. A compliation of all available data [2] shows two sets of phase shifts which pass through $\pi/2$ at about 720 MeV and 900 MeV, respectively. (Further discussion of these data is given by Morgan and Shaw [3].) The second piece of new information is the ration a_0^0/a_0^2. Already in 1968 Olsson and Turner [4] showed in the framework of effective Lagrange models that low energy pion production is consistent with the Weinberg value $a_0^0/a_0^2 = -3{\cdot}5$. Recently, Gutay et al. [5] — assuming a linear form in s, t, u for the real part of the amplitude at low energies and imposing the Adler consistency condition — werw able to determine $a_0^0/a_0^2 = -3{\cdot}2 \pm 0{\cdot}1$, which might be weakended to $-3{\cdot}2 \pm 1{\cdot}1$ with possible quadratic terms included. Also Cline et al. [6] have deduced the ratio $\sin \delta_0^0/\sin \delta_0^2 = -3{\cdot}1 \pm \pm 1{\cdot}1$ in the range 300—400 MeV from the ratios $\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^+)$ and $\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^-)$. On the other hand it has been argued by Clegg [7] that at

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least in $\pi^- p \to \pi^0\pi^0 n$ the final state $\pi^0\Delta^0$ contributes more than usually accepted and determination of phase shifts is obscured by this fact.

b. Low Energy Properties of the Veneziano Model for $\pi\pi$ Scattering

The one term Lovelace-Veneziano model [8] and the Weinberg current algebra model have two essential properties in common: The existence of the Adler zero (if $\alpha_\varrho(m_\pi^2) = 1/2$) and the suppression of the $I = 2$ channel. Both models are crossing symmetric, but lack unitarity.

The different isospin amplitudes may be written as

$$V^0(s, t, u) = \frac{3}{2}\left[V(s, t) + V(s, u)\right] - \frac{1}{2} V(t, u)$$

$$V^1(s, t, u) = V(s, t) - V(s, u)$$

$$V^2(s, t, u) = V(t, u)$$

$$V(s, t) = f^2(\alpha(s) + \alpha(t) - 1) \times B(1 - \alpha(s), 1 - \alpha(t)),$$

where α is the real linear exchange degenerate $\varrho - f$ trajectory. The scale f^2 may be determined from the ϱ width. Despite the fact that the Pomeron contribution is neglected in this model the evaluation of low energy parameters makes sense. This point has recently been greatly clarified by Kugler [9], who has shown that is it completely consistent to build up low energy $\pi\pi$ scattering by resonances alone, (as it is done in the Veneziano model). With $\alpha_\varrho(0) = 0{\cdot}483$ one gets for the scattering lengths $a_0^0/a_0^2 = -3{\cdot}9$ and (using $f^2/4\pi = 2{\cdot}1$) $a_0^0 = 0{\cdot}2065\, m_\pi^{-1}$ and $a_0^2 = -0{\cdot}0535\, m_\pi^{-1}$.

The Veneziano model being a pole approximation (and a solution of the finite energy sum rules), the question of how to incorporate the unitarity arises. An approach to this problem will be described in Section III. We turn now to a discussion of exact conditions for partial wave amplitudes.

II. CROSSING CONSTRAINTS

Whereas crossing symmetry of the scattering amplitudes, say $A(s, t, u)$, may be checked quite easily, the subject becomes quite involved when one uses partial wave amplitudes. The reason why one uses partial wave amplitudes is the much simpler unitarity relation, which is of great advantage in dispersion theory. Martin and others have derived a great number of conditions on partial wave amplitudes for $\pi\pi$ scattering in the subthreshold region $0 < s < 4$, which restrict the possible forms of amplitudes quite stringently. Instead of giving a complete list of conditions derived up to now I will rather give examples of the various kinds and refer for details to the literature. (From now on $m_\pi = 1$.)

1. Inequalities relating partial waves $f_l^I(s)$ for different values of s [10]. Examples are $\left(f_{00} = \frac{1}{3}(f_0^0 + 2f_0^2) \text{ is the } \pi^0\pi^0 \to \pi^0\pi^0 \ S \text{ wave amplitude} \right)$

$$f_{00}(4) > f_{00}(0) > f_{00}(3{\cdot}189)$$

$$\frac{1}{3} f_0^0(0{\cdot}2937) + \frac{2}{3} f_0^2(0{\cdot}2937) - 1{\cdot}229\, f_1^1(0{\cdot}2937) > \frac{7}{9} f_0^0(2{\cdot}4226) - $$
$$- \frac{4}{9} f_0^2(0{\cdot}4226) + 1{\cdot}2552\, f_0^1(2{\cdot}4226)$$

2. Relations between integrals and amplitude at a single point [10]:

$$\frac{1}{2}\int_2^4 f_{00}(s)\,\mathrm{d}s \leqslant f_{00}(0)$$

3. Derivatives of partial wave amplitudes [11]

$$f'_{00}(0) < -\frac{1}{4}[2f_{00}(4) - f_{00}(2) - f_{00}(0)]$$

$$f'_{00}(s) < 0 \qquad 0 < s < 1{\cdot}127$$

$$f'_{00}(s) > 0 \qquad 1{\cdot}697 < s < 4$$

$$f''_{00}(s) > 0 \qquad 1{\cdot}127 < s < 1{\cdot}697$$

The last three conditions imply the existence of a unique minimum of f_{00}.

4. Inequalities concerning integrals only [12], e.g.

$$\int_0^4 \mathrm{d}s(4-s)\,(10s^2 - 32s + 16)\, f_{00}(s) \geqslant 0.$$

Relations of type 1.—4. follow from the sole use of crossing symmetry and the positivity of the absorptive part ($\mathrm{Im}\, f_l^I(s) \geqslant 0$ for $s \geqslant 4$).

5. Using also the nonlinear properties of unitarity Bonnier and Vinh Mau [13] deduced $f_{00}(4) > -4$.

6. Equalities relating integrals over partial wave amplitudes [14] follow from crossing symmetry alone. Restricting ourselves to S and P waves there are five relations, for instance

$$\int_0^4 \mathrm{d}s(4-s)\,(2f_0^0(s) - 5f_0^2(s)) = 0.$$

7. Only recently Balachandran and Blackmon [15] could derive several infinite sets of integral inequalities for S- and P-waves only, using crossing symmetry and the positivity of the $l = 2, 4, 6, \ldots$ partial wave amplitudes in the region $0 \leqslant s \leqslant 4$. One of these sets of relations reads

$$\int_0^4 \mathrm{d}s(4-s)\,[s^n(4-s) - 2s(4-s)^n/(n+1)]\, f_{00}(s) \geqslant 0,\ n = 0, 1, 2, \ldots$$

Relations 1.—6. for S and P waves constitute a set of about 50 constraints, some of them have already been applied to model calculations of $\pi\pi$ scattering [16]. It has also been shown that the nonunitarized Veneziano model fulfills them, difficulties arise only when D-waves are included. It should also be stressed that both „unitarizations" proposed by Lovelace, K-matrix unitarization as well as giving the trajectory an imaginary part above threshold, fail in this respect [17].

III. UNITARIZATION

In introduction we have pointed out that scattering lengths may be predicted from the Veneziano model and that they might be good predictions despite the neglection of Pomeranchuk contributions. Lovelace [18] proposed to identify the Veneziano amplitudes with K-matrix elements. These may have poles on the real axis. We therefore have

$$S_l^I(s) = \frac{1 + \mathrm{i}K_l^I(s)}{1 - \mathrm{i}K_l^I(s)} = 1 + 2\mathrm{i}\sqrt{(s-4)/s}\, f_l^I(s)$$

$$f_l^I(s) = \sqrt{s/(s-4)}\,\frac{K_l^I(s)}{1 - \mathrm{i}K_l^I(s)} = \frac{V_l^I(s)}{1 - \mathrm{i}\sqrt{(s-4)/s}\; V_l^I(s)}$$

with

$$K_l^I(s) = \sqrt{(s-4)/s}\; V_l^I(s)$$

and

$$f_l^I(s) = \sqrt{s/(s-4)}\; \mathrm{e}^{\mathrm{i}\delta}\sin\delta, \qquad \mathrm{Im}(f_l^I)^{-1} = -\sqrt{(s-4)/s}\,.$$

As all these relations are valid only for $s > 4$ one has to generalize them if one wants an expression for all s. One can write

$$f_l^I(s) = V_l^I(s)/(1 - \varrho_l^I(s)\; V_l^I(s)) \qquad (1)$$

where the imaginary part of ϱ_l^I is given by unitarity for $s > 4$. (For simplicity we use elastic unitarity only.)

$$\mathrm{Im}\varrho_l^I(s) = \sqrt{s/(s-4)}, \qquad s > 4.$$

For the left hand cut discontinuity Lovelace proposed

$$\mathrm{Im}\,\varrho_l^I(s + i\varepsilon) = \sqrt{s/(s-4)}, \qquad s < 0.$$

With this choice one can use an unsubtracted dispersion relation for $\varrho_l^I(s)$ and one gets a unique solution. This solution, unfortunately, violates most of the crossing constraints given in the previous section. Also the scattering length a_0^0 is changed by about 40 %, which seems to be very much for unitarity corrections.

In our attempt [1] to impose unitarity we wanted to save crossing symmetry at low energy as much as possible. For this we had to evaluate the left hand cut discontinuity near the physical region. In the region $-32 < s \leqslant 0$, where the partial wave expansion converges, we can use

$$\mathrm{Im}f_l^I(s + i\varepsilon) = \tag{2}$$

$$= -\frac{2}{4-s}\int_4^{4-s} dt\, P_l\left(1 + \frac{2t}{s-4}\right)\sum_{I',l'}(2l'+1)\,\beta_{II'}\,\mathrm{Im}\,f_{l'}^{I'}(t)\,P_{l'}\left(1+\frac{2s}{t-4}\right)$$

and from the definition of $\varrho_l^I(s)$

$$\mathrm{Im}\,\varrho_l^I(s) = \frac{1}{2\mathrm{Im}\,f_l^I(s)}(1 - \{1 - [2\mathrm{Im}\,f_l^I(s)\,(\mathrm{Re}\,\varrho_l^I(s) - 1/V_l^I(s))]^2\}^{1/2}).$$

For convergence we write a subtracted dispersion relation

$$\varrho_l^I(s) = \varrho_l^I(s_0) + \frac{s-s_0}{\pi}\int_4^\infty \sqrt{(s'-4)/s}\,\frac{ds'}{(s'-s)(s'-s_0)} + \tag{3}$$

$$+ \frac{s-s_0}{\pi}\int_{-32}^{0}\mathrm{Im}\varrho_l^I(s)'\frac{ds'}{(s'-s)(s'-s_0)} + \frac{(s-s_0)}{\pi}\sum_i \frac{R_{li}^I}{s-s_i},$$

where the far left hand cut has been approximated by several pole terms. The use of a subtracted dispersion relation is consistent with asymptotic Regge behaviour of the partial waves, i.e. the partial wave amplitude $f_l^I(s)$ become asymptotically $V_l^I(s) \sim s^{\alpha(0)-1} \sim s^{-1/2}$. Furthermore, the Adler condition remains valid if we choose $s_0 = 1$ and $\varrho_l^I(s_0) = 0$.

We have therefore an expression for the unitarity corrections to the Veneziano model which depend on the unknown constants R_{li}^I and s_i. (The latter, however, will be assumed to be fixed, because any variation in s_i can be compensated by a corresponding change in the R_{li}^I). Eqs. (1) and (2) are two relations for the unknown functions $\varrho_l^I(s)$ and $f_l^I(s)$ in terms of the Veneziano partial waves $V_l^I(s)$ and the parameters R_{li}^I. We have tried to solve these equations iteratively by using as a starting point an approximate ϱ_l^I obtained by neglecting all contributions from the left hand region and by demanding that the resulting $f_l^I(s)$ should fulfil the inequalities 1.—5. and should minimize the crossing integrals 6. It turned out that one needs two poles for each S-wave and three for the P-wave to meet these requirements, but the range of these seven parameters is not very restricted. The following additional requirements have therefore been made:

1. As the experimental mass and width of the ϱ meson have been used for the trajectory and the scale of the V model we require that the output m_ϱ and Γ_ϱ agree with the input. This fixes two parameters.

2. a_0^2 has been fixed at $-0{\cdot}053$ (the V model gives $-0{\cdot}0535$) because unitarity corrections tend to increase the S-wave scattering lengths and we assume a smaller change in the $I = 2$ case than for $I = 0$.

3. As one finds that all $I = 0$ phase shifts go through $\pi/2$ for $s > 650$ MeV, we have selected four possibilities corresponding to a resonance at $m_\sigma =$ $= 725$ MeV, 850 MeV, 900 MeV and 1150 MeV and determined the remaining three parameters according to the procedure stated above.

The solutions show the following features: The $I = 0$ S-wave resonates at the chosen energies with resonance widths 290, 440, 490 and 700 MeV, resp. a_0^0 is always 0·224 which means that unitarity corrections to the scattering length are of the order of 10 % and $a_0^0/a_0^2 = -4{\cdot}2$. a_1^1 has not been changed by this procedure: $a_1^1 = 0{\cdot}0414$. The $I = 2$ wave is always negative in the region under consideration and falls down to a minimum of $-12°$ to $-15°$ in the ϱ region and increases slowly for increasing energy.

It is clear that our solutions could be still improved by further iterations but to judge the quality I have to mention that the crossing integrals 6. are fulfilled with an accuracy of about 10^{-3} compared to about 10^{-2} in the Pade approximation to the Φ^4 theory [19] and 10^{-1} in the Lovelace unitarization [18]. With respect to the other constraints the requirement of a minimum of f_{00} for $1{\cdot}127 < s < 1{\cdot}697$ seems to be one of the most stringent ones. A further check that the narrow resonances are transformed into finite width resonances in a correct fashion without disturbing the low energy contribution to the finite energy sum rules consists in demanding that the moments

$$S_n^I = \int_4^{50} ds(s-4)^n\,\mathrm{Im}f_l^I(s)$$

and

$$\bar{S}_n^I = \int_4^{50} ds(s-4)^n \, \mathrm{Im}\, V_l^I(s) \qquad n = 0, 1, \quad I = 0, 1, 2,$$

should not differ very much. This is indeed the case for $I = 0, 1$ and implies for $I = 2$ that $\sin^2 \delta_0^2$ should be small.

REFERENCES

[1] Baier R., Kühnelt H., Widder F., Nuovo Cim. Lett. *3* (1970), 594 and to be published.

[2] Scharenguivel J. H., et al., Phys. Rev. *186* (1969), 1387.

[3] Morgan D., Shaw G., Columbia Univ. Preprint NYo 1932 (2)-164.

[4] Olsson M., Turner L., Phys. Rev. Lett. *20* (1967). 1127.

[5] Gutay L. J., Meiere F., Scharenguivel J., Phys. Rev. Lett. *23* (1969), 431.

[6] Cline D., Braun K., Scherer V., Nucl. Phys. *B 18* (1970), 77

[7] Clegg, Nucl. Phys. *B 19* (1970), 160.

[8] Lovelace C., Phys. Lett. *28 B* (1968), 264;
Shapiro J., Phys. Rev. *179* (1969), 1345.

[9] Kugler M., Phys. Lett. *31 B* (1970), 379.

[10] Martin A., Nuovo Cim. *58 A* (1968), 303;
Common A., Nuovo Cim. *56 A* (1968), 524;
Auberson G., Brander O., Mahoux G., Martin A., Nuovo Cim. *65 A* (1970), 743.

[11] Auberson G., Saclay preprint DPh-T/69-68 (1969).

[12] Piquet O., Wanders G., Phys. Lett. *30 B* (1969), 418.

[13] Bonnier B., Vinh Mau, Phys. Rev. *165* (1968), 1923.

[14] Roskies R., Phys. Letters *30 B* (1969). 42; Nuovo Cim. *64 A* (1969), 585.

[15] Balachandran A. P., Blackmon M., Phys. Lett. *31 B* (1970), 655.

[16] Auberson G., Piquet O., Wanders G., Phys. Lett. *28 B* (1968), 41;
Castoldi P., Nucl. Phys. *B 12* (1969), 567;
Bonnier B., Gauron P., preprint (Paris) 1967;
Bonnier B., Nucl. Phys. *B 17* (1970), 525.

[17] Levers R., Schwela D., Bonn preprint (1970).

[18] Lovelace C., Proc. Argonne Conference (1969);
Dobson P., Nuovo Cim. Lett. *2* (1969), 761;
Lipinski H., Univ. of Wisconsin preprint Coo-264 (1969).

[19] Basdevant J., Cohen-Tannoudji G., Morel A., CERN prepr. TH 1063 (1969).

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