Fyzikálny časopis 20 (1970), No. 3

THE CONSTRUCTION OF GENERATORS OF THE DYNAMICAL 2 + 1 LORENTZ GROUP FOR QUANTUM-MECHANICAL SYSTEMS

JOZEF LÁNIK, Bratislava

The connection between the Lie algebra of the postulated dynamical (non-invariance) 2 + 1 Lorentz group and the various forms of one-dimensional Hamiltonians characterizing quantum-mechanical systems is discussed under the assumption that the *compact* t_3 generator of the group is connected with the Hamiltonian H in such a way that $t_3 = \alpha H$, α being a real constant.

1. INTRODUCTION

The application of the group theory for the description of dynamical systems is one of the recent attempts in physics. The idea that physical systems might be characterized by dynamical (usually non-compact) groups and their unitary irreducible representations has been verified on a lot of quantum-mechanical examples [1—6]. This hypothesis has consequently been used in strong interactions, too [7—10].

Nevertheless, there still are some generally unsolved problems in this new methodology, even in quantum mechanics. For a given system they are connected with the choice of the right dynamical group and its representations and they also concern the identification of observables and transition operators with generators and elements of the dynamical group.

For solving these problems in quantum-mechanics, besides the standard method (see e.g. refs. [2—5]) based on determining the right dynamical groups for a given system with the known Hamiltonian H another method can be used as well [11—12]. In the latter approach a dynamical (non-invariance) group and an identification of observables with generators of the group are postulated and one tries to find the corresponding Hamiltonians (or potentials). It has been shown in our previous paper [11] that in the case of the simplest one-dimensional system and under the assumptions I, II and III listed below, the potential $V(x)$ can be found explicitely and has the form

$$V(x) = \frac{m}{8\hbar^2\alpha^2} x^2 + \frac{2\hbar^2}{m}\left(G + \frac{3}{16}\right) x^{-2}, \tag{1}$$

where G is a Casimir operator of the postulated dynamical algebra. The three assumptions mentioned above can be formulated as follows:

I. a dynamical group is the $2+1$ Lorentz group with the Lie algebra spanned by three basis elements t_1, t_2, t_3 with the commutation relations

$$[t_1, t_2] = -\mathrm{i}t_3, \quad [t_2, t_3] = \mathrm{i}t_1, \quad [t_3, t_1] = \mathrm{i}t_2, \tag{2}$$

II. an identification of the operator t_3 with the Hamiltonian H is of the form

$$t_3 = \alpha H, \tag{3}$$

where α is a real constant and the Hamiltonian of the system is

$$H = \frac{p_x^2}{2m} + V(x), \tag{4}$$

where $V(x)$ is the unknown potential and the operator $p_x = -\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}x}$,

III. the generators t_1, t_2 are differential operators of the second order.

We notice here that if the postulates (I)—(III) are changed one can also try to find the corresponding potentials. However, it is evident in that case that only certain combinations of those assumptions are consistent and allow us to obtain the potentials.

The purpose of the present paper is to investigate the problem of the consistency for the one-dimensional case when the only postulate (III) is changed. We shall suppose the generators t_1 and t_2 to be differential operators of the N-th order, where N is a finite number and $N \geqslant 3$. It will be shown in section 2 that the postulates are consistent only when $N = 3$ and the corresponding potentials will be found in an explicit form which is different from (1). In section 3 some conclusions are drawn.

2. DIFFERENTIAL OPERATORS OF THE N-th ORDER AS GENERATORS OF THE 2 + 1 LORENTZ GROUP

2.1. The case of the finite N and $N \geqslant 3$

With the aid of the substitution (see [11])

$$\xi = x\left[\frac{m}{2\hbar^2\alpha}\right]^{1/2}$$

we can rewrite the generator (3) in a more suitable form

$$t_3 = -\frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + c_0(\xi), \tag{5}$$

where $c_0(\xi)$ is connected with the unknown potential by the relation

$$c_0(\xi) = \alpha V\left(\xi\hbar\sqrt{\frac{2\alpha}{m}}\right). \tag{6}$$

For our aims we shall assume that a representation of the generators t_1 and t_2 is of the form

$$t_i = \sum_{j=0}^{N} A_{(i)j}(\xi)\frac{\mathrm{d}^j}{\mathrm{d}\xi^j}, \qquad i = 1, 2, \tag{7}$$

where at least one of the functions $A_{(1)N}(\xi)$, $A_{(2)N}(\xi)$ is not identically equal to zero for expressions (7) to represent differential operators of the N-th order.

Now we shall investigate the consistency of (5) and (7) with commutation relations (2). This has been done for $N = 2$ in ref. [11] and the potentials of the form (1) have been explicitely found. If $N = 0$ or 1, one can easily see that the relations (5) and (7) are not consistent with (2). To investigate the consistency of (5) and (7) with eqs. (2), if N is finite and $N \geqslant 3$, it is more convenient to introduce the new functions

$$a_j(\xi) = A_{(1)j}(\xi) + \mathrm{i}A_{(2)j}(\xi), \qquad j = 0, 1, 2, \ldots, N, \tag{8}$$
$$b_j(\xi) = A_{(1)j}(\xi) - \mathrm{i}A_{(2)j}(\xi), \qquad j = 0, 1, 2, \ldots, N,$$

where $\mathrm{i} = \sqrt{-1}$. The notations (8) correspond to the following ones

$$t_+ = t_1 + \mathrm{i}t_2 = \sum_{i=0}^{N} a_i(\xi)\frac{\mathrm{d}^i}{\mathrm{d}\xi^i}, \tag{9}$$

$$t_- = t_1 - \mathrm{i}t_2 = \sum_{i=0}^{N} b_i(\xi)\frac{\mathrm{d}^i}{\mathrm{d}\xi^i},$$

where $a_N(\xi)$ and $b_N(\xi)$ again cannot be simultaneously equal to zero. It is also useful to introduce the following notations

$$a_i = a_i(\xi), \quad b_i = b_i(\xi), \quad c_0 = c_0(\xi), \quad i = 0, 1, \ldots, N, \tag{10}$$

$$a_i^{(j)} = \frac{\mathrm{d}^j}{\mathrm{d}\xi^j} a_i(\xi), \text{ etc.,}$$

$$\binom{i}{j} = \frac{i!}{(i-j)!\, j!}, \qquad a_{-1} = b_{-1} = 0.$$

Substituting the expressions (5) and (7) into the operator equations (2) and using notations (8), (9) and (10), the following system of differential equations can be deduced after some simple but lengthy calculations

$$a_N^{(1)} = 0, \quad b_N^{(1)} = 0, \tag{11}$$

$$\sum_{i=j+1}^{N} \binom{i}{j} a_i c_0^{(i-j)} + \tfrac{1}{4} a_j^{(2)} + \tfrac{1}{2} a_{j-1}^{(1)} = -a_j, \quad j = 0, 1, \ldots, N, \tag{12}$$

$$\sum_{i=j+1}^{N} \binom{i}{j} b_i c_0^{(i-j)} + \tfrac{1}{4} b_j^{(2)} + \tfrac{1}{2} b_{j-1}^{(1)} = b_j, \quad j = 0, 1, 2, \ldots, N, \tag{13}$$

$$\sum_{i=0}^{N} \binom{i}{0} [a_i b_0^{(i)} - b_i a_0^{(i)}] = -2c_0, \tag{14}$$

$$\sum_{i=0}^{N} \binom{i}{0} [a_i b_2^{(i)} - b_i a_2^{(i)}] + \sum_{i=1}^{N} \binom{i}{1} [a_i b_1^{(i-1)} - b_i a_1^{(i-1)}] + \\ + \sum_{i=2}^{N} \binom{i}{2} [a_i b_0^{(i-2)} - b_i a_0^{(i-2)}] = \tfrac{1}{2}, \tag{15}$$

$$\sum_{i=0}^{N} \binom{i}{0} [a_i b_j^{(i)} - b_i a_j^{(i)}] + \sum_{i=1}^{N} \binom{i}{1} [a_i b_{j-1}^{(i-1)} - b_i a_{j-1}^{(i-1)}] + \\ + \ldots + \sum_{i=j}^{N} \binom{i}{j} [a_i b_0^{(i-j)} - b_i a_0^{(i-j)}] = 0, \quad j = 1;\ 3, 4, 5, \ldots, N, \tag{16}$$

$$\sum_{i=j}^{N} \binom{i}{j} [a_i b_N^{(i-j)} - b_i a_N^{(i-j)}] + \sum_{i=j+1}^{N} \binom{i}{j+1} [a_i b_{N-1}^{(i-j-1)} - b_i a_{N-1}^{(i-j-1)}] + \\ + \ldots + \sum_{i=N}^{N} \binom{i}{N} [a_i b_j^{(i-N)} - b_i a_j^{(i-N)}] = 0, \quad j = 1, 2, 3, \ldots, N. \tag{17}$$

It is easily seen that in order to investigate the consistency of expressions

(5) and (7) with the operator equations (2) it is necessary and sufficient to investigate the consistency of the system of non-linear differential equations (11)—(17), supposing $N \geqslant 3$. We notice here that the case of $N = 2$ has been solved in [11].

The relations

$$a_N = \alpha_N, \quad a_{N-1}^{(1)} = -2\alpha_N, \qquad b_N = \beta_N, \quad b_{N-1}^{(1)} = 2\beta_N, \tag{18}$$

follow immediately from eqs. (11), (12) and (13) (if $j = N$), where α_N and β_N are the constants of integration. Combining eqs. (18) and (17) for $j = N - 2$, we get the following condition for the constants α_N and β_N

$$\alpha_N \beta_N = 0, \tag{19}$$

from which we generally obtain three possibilities

$$\text{(i)} \quad \alpha_N = 0, \quad \beta_N \neq 0, \qquad \text{(ii)} \quad \alpha_N \neq 0, \quad \beta_N = 0, \qquad \text{(iii)} \quad \alpha_N = 0, \quad \beta_N = 0. \tag{20}$$

In the following we shall not consider the most trivial case (iii) because it is the case when $A_{(1)N}(\xi) = A_{(2)N}(\xi) = 0$ in relations (7). Thus, let us discuss the case (i), i.e., $\alpha_N = 0$, $\beta_N \neq 0$. The following holds:

Lemma 1. If $b_N = \beta_N \neq 0$, then for a given integer $N \geqslant 3$ and any integer i from the interval $N - 1 \geqslant i \geqslant 2$, the solutions of the system of equations (11), (12), (13) and (17) must have the form

$$a_N = a_{N-1} = \ldots = a_{N-i+3} = a_{N-i+2} = 0, \qquad a_{N-i+1} = \alpha_{N-i+1} = \text{const.} \tag{21}$$

Proof. We shall use the method of mathematical induction. If $i = 2$, the proof of the lemma follows immediately from relations (18) and (19).

Thus when $b_N = \beta_N \neq 0$, let us assume that eqs. (21) hold for all $i = 2, 3, \ldots, k - 1$, where k is a given integer in the interval $2 \leqslant k - 1 \leqslant N - 2$, i. e., let us assume that

$$a_N = a_{N-1} = \ldots = a_{N-k+3} = 0, \qquad a_{N-k+2} = \alpha_{N-k+2} = \text{const.} \tag{22}$$

We shall show that eqs (21) hold for $i = k$, too, i.e., it will be also true that

$$\alpha_{N-k+2} = 0,$$

$$a_{N-k+1} = \alpha_{N-k+1} = \text{const.}$$

To prove the lemma in this way we combine eqs. (17) for $j = N - k \geq 1$ and the induction assumption (22). After some rearrangements we get

$$\binom{N-k+2}{N-k+1} \alpha_{N-k+2}\, b^{(1)}_{N-1} = \binom{N}{N-1} \beta_N a^{(1)}_{N-k+1}. \tag{23}$$

On the other hand, substituting relations (22) into eq. (12) (for $j = N - k + 2$), one obtains

$$a^{(1)}_{N-k+1} = -\, 2\alpha_{N-k+2}. \tag{24}$$

Combining relations (18), (23) and (24) we have (since $\beta_N \neq 0$)

$$(2N - k + 2)\, \alpha_{N-k+2} = 0,$$

from which it follows that $\alpha_{N-k+2} = 0$, if $k \leqslant N - 1$. Thus, this result combined with eq. (24) proves Lemma 1.

Now, using Lemma 1, eqs. (18) and (16) for $j = N \geqslant 3$, we obtain after some calculations (and since $\beta_N \neq 0$)

$$4a_2 = N a^{(1)}_1, \tag{25}$$

where $a_2 = \alpha_2$, α_2 being a constant due to Lemma 1. On the other hand, combining Lemma 1 and eq. (12) for $j = 2$, we get

$$a^{(1)}_1 = -\, 2\alpha_2. \tag{26}$$

For $N \geqslant 3$, eqs. (25) and (26) can simultaneously be fulfilled if and only if $a_2 = \alpha_2 = 0$. Thus, we have proved:

Lemma 2. For a given integer $N \geqslant 3$, if $b_N \neq 0$, then the solutions of equations (11), (12), (13), (17) and (16) must be

$$a_N = a_{N-1} = \ldots = a_2 = 0,$$

$$a_1 = \alpha_1 = \text{const.}$$

Lemma 3. For a given integer $N \geqslant 4$, if $b_N = \beta_N \neq 0$, then the solutions of equations (11), (12), (13), (17) and (16) must have a form

$$a_N = a_{N-1} = \ldots = a_1 = 0,$$

$$a_0 = \alpha_0 = \text{const.}$$

The proof of this lemma can be easily given by combining Lemma 2 with eqs. (16) (for $j = N - 1 \geqslant 3$) and (12) (for $j = 1$).

Now, using Lemma 3 and the notation (10) we get

$$a_0 = \alpha_0 = 0$$

from eq. (12) for $j = 0$.

It can be easily seen that for $N \geqslant 4$ these results are not consistent with eq. (15), if we suppose $b_N = \beta_N \neq 0$, i.e., in the case (i) (see relation (20)).

In the case (ii) (see (20)) the same result can be obtained in a similar way as in case (i). Thus, summing-up, we have proved that the operators (5) and (7) do not represent any generators of the $2 + 1$ Lorentz group if at least one of the $A_{(1)N}(\xi)$ and $A_{(2)N}(\xi)$ is not equal to zero and if the finite integer $N \geqslant 4$. When $N = 3$, Lemma 3 is not fulfilled and this case must be investigated separately. It will be done in the following.

2.2. The case of $N = 3$

In this case the system of equations (11)—(17) is consistent, and, using Lemma 2 (supposing e.g. $b_3 = \beta_3 \neq 0$) it can be solved. After some calculations we obtain solutions of the following form

$$a_3(\xi) = a_2(\xi) = 0, \quad a_1(\xi) = \frac{1}{16\beta_3}, \quad a_0(\xi) = \frac{-1}{8\beta_3}\xi + \alpha_0, \tag{27}$$

$$b_3(\xi) = \beta_3, \quad b_2(\xi) = 2\beta_3\xi - 16\beta_3^2\alpha_0,$$

$$b_1(\xi) = -\,4\beta_3\xi^2 + 4.16\beta_3^2\alpha_0\xi + \beta_1,$$

$$b_0(\xi) = -\,8\beta_3\xi^3 + 3.4.16\beta_3^2\alpha_0\xi^2 - 4.16.16\beta_3^3\alpha_0^2\xi - 8\beta_3\xi + 2\beta_1\xi + {} + 4.16\beta_3^2\alpha_0 - 16\beta_3\alpha_0\beta_1,$$

$$c_0(\xi) = \xi^2 - 16\beta_3\alpha_0\xi + 32\beta_3^2\alpha_0^2 + \frac{1}{4} - \frac{\beta_1}{8\beta_3},$$

where $\beta_3 \neq 0, \alpha_0, \beta_1$ are constants of integration. If we introduce a new constant

$$C = 32\beta_3^2\alpha_0^2 - \frac{1}{4} + \frac{\beta_1}{8\beta_3}, \tag{28}$$

then the calculated potential has the form (using (6))

$$V(x) = \frac{m}{2\hbar^2\alpha^2}\left(x - 8\beta_3\alpha_0\hbar\sqrt{\frac{2\alpha}{m}}\right)^2 - \frac{C}{\alpha}, \tag{29}$$

which is the potential of the harmonic oscillator. It can be easily shown that the constant C is connected with the Casimir operator G by the relation

$$G = C^2 - \tfrac{1}{4},$$

where G is defined as follows

$$G = t_3^2 - t_1^2 - t_2^2.$$

Finally we remark that the operators t_1, t_2 and t_3 can be rewritten in a more suitable form using the adjoint creation and annihilation operators a^+ and a, respectively, defined by the relations

$$a^+ = -\frac{1}{2}\frac{d}{d\xi} + \xi - 8\beta_3\alpha_0, \tag{30}$$

$$a = \frac{1}{2}\frac{d}{d\xi} + \xi - 8\beta_3\alpha_0.$$

They fulfil the following commutation relation

$$[a, a^+] = 1. \tag{31}$$

Combining expressions (5), (9), (27) and using relations (28) and (30) we get

$$t_3 = \tfrac{1}{2}(aa^+ + a^+a) - C, \tag{32}$$

$$t_+ = -\frac{1}{8\beta_3}a^+,$$

$$t_- = -8\beta_3(t_3 - C + \tfrac{1}{2})a.$$

We mention here that this form of the operators is consistent with the results of ref. [13]. It can be easily seen that the operators (32) do not fulfil the Hermiticity conditions $t_3 = t_3^+$, $t_-^+ = t_+$, thus the corresponding representations of the group cannot be unitary.

3. CONCLUSION

In the present paper we have investigated the connection between the Lie algebra of the postulated non-invariance $2 + 1$ Lorentz group and the various forms of one dimensional Hamiltonians (or more precisely, potentials) under the assumption that the generator t_3 of the group is connected with the Hamiltonian H in such a way that $t_3 = \alpha H$, where α is a real constant. It has been shown that this problem has no solutions if the generators t_1 and t_2 are supposed to be differential operators of the N-th order, where N is a finite integer and $N \geqslant 4$. The solution exists only when $N = 2$ or 3 and the corresponding potentials are given by relations (1) or (29), respectively.

Finally we would like to mention that the results of this paper suggest the construction of generators of dynamical groups as differential operators of

approximately equal orders if they are supposed to be of the finite orders and they are constructed in terms of the dynamical variables the number of which is lower than the number of generators. In this connection see also ref. [14].

The author wants to express his thanks to Dr. M. Blažek, Dr. J. Niederle, and Mr. J. Černuch for valuable discussions.

Note added in proof. Recently the solution of the problem formulated in the present paper has been done by Cordero and Ghirardi (Trieste preprint IC/70/26) for three-dimensional systems under the assumption that the only postulate (II) is changed.

REFERENCES

[1] Barut A. O., Phys. Rev. *135* (1964), B 839.
[2] Barut A. O., Böhm A., Phys Rev. *139* (1965), B 1107.
[3] Barut A. O., Budini P., Fronsdal C., Proc. Roy. Soc. *A 291* (1966), 106.
[4] Mukunda N., O'Raifeartaigh L., Sudarshan E. C. G., Phys. Rev. Lett. *15* (1965), 1041.
[5] Barut A. O., In: *High-energy Physics and Elementary Patricles*. International atomic energy agency, Vienna 1965, 679.
[6] Budini P., Nuovo Cimento *44* (1966), 363.
[7] Dothan Y., Gell-Mann M., Ne'eman Y., Phys. Lett. *17* (1965), 148.
[8] Fronsdal C., Phys. Rev. *171* (1968), 1811.
[9] Barut A. O., *Implications of the 0 (4, 2) model of strong interactions for Regge pole and high-energy phenomenology*. Trieste preprint IC/68/88.
[10] Kuriyan J. G., Sudarshan E. C. G., Phys. Rev. *162* (1967), 1650.
[11] Lánik J., Nuclear Physics *B 2* (1967), 263.
[12] Doebner H. D., Melsheimer O., J. Math. Phys. *9* (1968), 1638.
[13] Hwa R. C., Nuyts J., Phys. Rev. *145* (1966), 1188.
[14] Гельфанд И. М., Кириллов А. А., *О телах связанных с обертывающими алгебрами алгебр ЛИ*. Труды Мат. инс. им. В. А. Стеклова АН СССР, Москва 1965.

Received February 20th, 1970

Fyzikálny ústav SAV,
Bratislava