ON STRESS-STRAIN RELATIONS IN THE THEORY OF NON-LINEAR VISCO-ELASTICITY

RUDOLF PŘEVRÁTIL, Bratislava

It is supposed in the theory of linear visco-elasticity that the relation

$$\sigma(t) = E\int\limits_0^t \psi(t-\tau)\, \varepsilon'(\tau)\, \mathrm{d}\tau \tag{1}$$

holds between the stress $\sigma(t)$ and the strain $\varepsilon(t)$. The symbols E and $\psi(t)$ denote the modulus of elasticity and the relaxation function of material, respectively. In the theory of non-linear viscoelasticity, certain generalizations of relation (1) are often used. An important case may be obtained by replacing the strain by some of its non-linear measures $\eta[\varepsilon(t)]$, so that the relation

$$\sigma(t) = E\int\limits_0^t \psi(t-\tau)\, \eta'(\tau)\, \mathrm{d}\tau \tag{2}$$

is obtained instead of relation (1).

The specific work $W(\varepsilon; T)$ done by the stress on the strain in the time interval $\langle 0, T\rangle$ is given by the relation

$$W(\varepsilon; T) = \int\limits_0^T \sigma(t)\, \varepsilon'(t)\, \mathrm{d}t. \tag{3}$$

Some basical thermodynamical principles as well as other considerations lead to the condition that the specific work given by (3) should be non-negative for every $\varepsilon(t)$.

The purpose of this paper is to find the conditions restricting the shape of the functions $\psi(t)$ and $\eta(\varepsilon)$ which ensure the non-negativity of the specific work. However, the paper deals with a specific case: it is supposed that $\varepsilon(t)$ is a closed deformation cycle:

$$\varepsilon(0) = \varepsilon(T) = 0. \tag{4}$$

The conditions ensuring the non-negativity of the specific work — provided that (4) holds — are given in Theorem 2, proved at the end of the paper. The function $\psi(t)$ should be positive, decreasing and convex from below in the interval $\langle 0, \infty)$; the function $\eta(\varepsilon)$ should be non-decreasing in the interval $(-1, \infty)$ provided that $\eta(0) = 0$.

1. INTRODUCTION

We shall investigate the relation between the stress $\sigma(t)$ and the strain $\varepsilon(t)$ in a homogenous body subjected to pure tension or compression. Symbol t denotes time. We shall confine our considerations to the time interval $0 \leqslant t < \infty$ and assume that the body was undisturbed until the instant $t = 0$.

It is assumed in the theory of linear visco-elasticity (see, for instance, [3]) that there exists a linear relation

$$\sigma(t) = E\int_0^t \psi(t-\tau)\, \varepsilon'(\tau)\, d\tau \tag{1.1}$$

between the stress and the strain. The symbol E denotes the instantaneous modulus of elasticity of the material, symbol $\psi(t)$ denotes the relaxation function of the material. This constant and this function define fully the mechanical behaviour of the material and can be determined experimentally. On the other hand, there are certain conditions based on theoretical considerations which limit the shape of relaxation functions. These limitations are implied by the condition that the work done by the stress $\sigma(t)$ on the strain $\varepsilon(t)$ per unit volume of the body in the time interval $\langle 0, T\rangle$ should be positive (or non-negative) for every $\varepsilon(t)$ non-vanishing in the interval $\langle 0, T\rangle$. This condition is a consequence of the uniqueness of the solution of a certain class of boundary-value problems [1] (or a consequence of basic thermodynamical principles [4]).

The theory of linear visco-elasticity describes satisfactorily the behaviour of many visco-elastic bodies if stress and strain are sufficiently small. However, the results of some experiments (see, for instance, [5, 7]) indicate that the stress-strain relations are strongly non-linear in the case of a large stress and strain. Moreover, there exists a large number of materials of practical importance in which strong non-linearity occours even in the case of a small stress and strain. The dependence between stress and strain is described in these cases by various types of non-linear integral relations [2, 7], which generalize the relation (1.1).

An important special case can be obtained if we substitute for the strain $\varepsilon(t)$ in (1.1) some non-linear function $\eta[\varepsilon(t)]$ of the strain. In this case, the stress depends linearly not on the strain but on a certain non-linear measure of deformation. There are some materials in which the validity of such a dependence was proved experimentally under certain conditions [7]. On the other hand, such a generalization seems to be natural because the strain is a measure of deformation which is chosen — from the geometrical point of view — quite arbitrarily.

In this paper, we shall discuss the materials, the stress-strain relation of which is given by equation (1.1), generalized in the way mentioned above. These materials will be called geometrically non-linear materials. Our task will be similar to that of paper [4]. We shall investigate the conditions which should be fulfilled by the relaxation function $\psi(t)$ and the measure of deformation $\eta(\varepsilon)$ of a geometrically non-linear material, in order to render the specific work done by the stress $\sigma(t)$ on the strain $\varepsilon(t)$ in the time interval $\langle 0, T\rangle$ non-negative.

2. FORMULATION OF THE PROBLEM

Let the strain $\varepsilon(t)$ and its derivative $\varepsilon'(t)$ be continuous in the interval $\langle 0, T\rangle$ for every positive T and let $\varepsilon(0) = 0$. Let $\eta(\varepsilon)$; $\eta(0) = 0$ denote a single-valued function defined in the interval $(-1, \infty)$ in such a way that the function $\eta(t) = \eta[\varepsilon(t)]$ and its derivative $\eta'(t)$ are also continuous in the interval $\langle 0, T\rangle$ for every positive T.

Definition 1. *The material is called geometrically non-linear if the relation*

$$\sigma(t) = E\int_0^t \psi(t-\tau)\, \eta'(\tau)\, d\tau \tag{2.1}$$

between the stress $\sigma(t)$ and the strain $\varepsilon(t)$ holds for every non-negative t, provided η is a non-linear function of ε.

Similarly as in the theory of linear visco-elasticity, it is assumed that the relaxation function $\psi(t)$; $\psi(0) = 1$ is defined and bounded in the interval $\langle 0, \infty)$ and continuous at every point of this interval. It is assumed further that the modulus of elasticity E is a positive constant.

The specific work $W(\varepsilon; T)$ done by the stress $\sigma(t)$ on the strain $\varepsilon(t)$ in the time interval $\langle 0, T\rangle$ is given in the case of a geometrically non-linear material by the relation

$$W(\varepsilon; T) = \int_0^T \sigma(t)\, \varepsilon'(t)\, dt = E\int_0^T\int_0^t \psi(t-\tau)\, \eta'(\tau)\; \varepsilon'(t)\, d\tau\, dt. \tag{2.2}$$

Suppose the body is deformed isothermically. If follows then from basic thermodynamical principles that the work given by the equation (2.2) must be non-negative.

It seems to be very difficult to find out how this condition restricts the shape of the functions $\psi(t)$ and $\eta(\varepsilon)$ if we demand it to be satisfied for an arbitrary $\varepsilon(t)$, which is continuously differentiable in the interval $\langle 0, T\rangle$. We shall therefore confine our considerations to a more narrow class of the functions $\varepsilon(t)$ given by

Definition 2. *The function $\varepsilon(t)$ is called a deformation cycle closed in the interval $\langle 0, T\rangle$ if the equations*

$$\varepsilon(T) = \varepsilon(0) = 0 \tag{2.3}$$

are valid for $\varepsilon(t)$.

It is obvious that the relations (2.3) imply the same relations for the function $\eta(t)$.

Definition 3. *The stress-strain relation of the geometrically non-linear material is admissible if the relation*

$$W(\varepsilon; T) \geqslant 0 \tag{2.4}$$

holds for every positive T and for every deformation cycle closed in the interval $\langle 0, T\rangle$.

The task of this paper is to find the conditions restricting the shape of the functions $\psi(t)$ and $\eta(\varepsilon)$, which would ensure the admissibility of the relation (2.1). The answer will be given by Theorem 2 which will be presented and proved in the next paragraph.

3. THE CONDITIONS OF ADMISSIBILITY OF THE STRESS-STRAIN RELATIONS

We shall accomplish the proof of the resulting Theorem 2 in several steps. First of all we shall prove the following

Lemma 1. *Let $y(x)$; $y(0) = 0$ is a function defined, bounded and non-decreasing in the interval $\langle a, b\rangle$, $a < 0 < b$. Let us write $y(x_i) = y_i$ for the sake of brevity. Then the relation*

$$S = x_1y_1 - x_2y_1 + x_2y_2 - x_3y_2 + \ldots + x_ny_n \geqslant 0 \tag{3.1}$$

holds for arbitrary $x_i \in \langle a, b\rangle$; $i = 1, 2, \ldots, n$.

Proof. Suppose there exists a natural number k, $1 < k < n$, such that the numbers x_{k-1}, x_k, x_{k+1} do not form a non-decreasing or non-increasing sequence of three numbers, so that the inequality

$$(x_k - x_{k-1})\,(x_{k+1} - x_k) < 0 \tag{3.2}$$

holds. The function $y(x)$ is non-decreasing, thus

$$(y_k - y_{k-1})\,(x_{k+1} - x_k) \leqslant 0, \tag{3.3}$$

or, after some re-arrangements

$$- x_ky_{k-1} + x_ky_k - x_{k+1}y_k \geqslant - x_{k+1}y_{k-1}. \tag{3.4}$$

The relation (3.4) yields the relation

$$S \geqslant S' = x_1'y_1' - x_2'y_1' + x_2'y_2' - x_3'y_2' + \ldots + x_m'y_m' \tag{3.5}$$

where $y_i' = y(x_i')$ and the equations

$$x_1' = x_1; \qquad x_m' = x_n \tag{3.6}$$

are valid. The numbers $x_1', x_2', \ldots x_m'$ form a non-decreasing or a non-increasing subsequence of the sequence $x_1, x_2, \ldots, x_n$. It is sufficient now to prove the non-negativity of the sum S'. We shall prove it starting from the assumption $x_1' > 0 > x_m'$. In the other cases the proof may be carried out in a similar way. In our case, there exists a natural number k, $1 \leqslant k < m$, such that the relations

$$x_k' \geqslant 0 \geqslant x_{k+1}'; \qquad y_k' \geqslant 0 \geqslant y_{k+1}' \tag{3.7}$$

are valid. The sum S' can be divided into three parts in the following manner:

$$S' = S_1' + S_2' + S_3' \tag{3.8}$$

where

$$\begin{aligned} S_1' &= x_1'y_1' - x_2'y_1' + \ldots + x_{k-1}'y_{k-1}' - x_k'y_{k-1}' \\ S_2' &= x_k'y_k' - x_{k+1}'y_k' + x_{k+1}'y_{k+1}' \\ S_3' &= - x_{k+2}'y_{k+1}' + x_{k+2}'y_{k+2}' - \ldots - x_m'y_{m-1}' + x_m'y_m'. \end{aligned} \tag{3.9}$$

It follows from the conditions of Lemma 1 that $x_i'y_i' \geqslant 0$; $i = 1, 2, \ldots, m$. This relation and the relation (3.7) yield the inequality $S_2' \geqslant 0$. The sequence $x_1', x_2', \ldots, x_k'$ is a non-increasing sequence of non-negative numbers. The same holds for the sequence $y_1', y_2', \ldots, y_k'$. This implies the relation $x_i'y_i'$ –

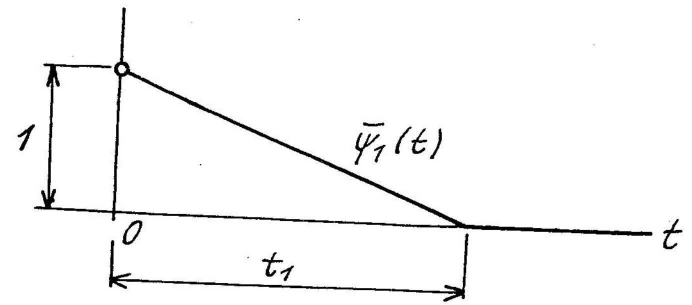

Fig. 1. Shape of the function $\psi_1(t)$.

$- x'_{i+1} y'_i \geqslant 0$, with the consequence $S'_1 \geqslant 0$. The inequality $S'_3 \geqslant 0$ may be proved in the same way. Equation (3.8) yields then $S' \geqslant 0$. Q. E. D.

Consider now a relaxation function $\bar{\psi}_1(t)$ given by

$$\bar{\psi}_1(t) = \begin{cases} 1 - t/t_1 & \text{for } 0 \leqslant t \leqslant t_1 \\ 0 & \text{for } t > t_1 \end{cases} \tag{3.10}$$

where t_1 is a positive number. The shape of the function is given in Fig. 1. We shall prove the following

Theorem 1. *The relation (2.1) is admissible if $\psi(t) = \bar{\psi}_1(t)$ and if $\eta(\varepsilon)$ is a non-decreasing function satisfying the condition $\eta(0) = 0$.*

Proof. The relation (2.1) may be re-written by means of integration per partes in the following way:

$$\sigma(t) = E\left\{[\psi(t-\tau)\,\eta(\tau)]_0^t - \int_0^t \frac{\partial\psi(t-\tau)}{\partial\tau}\,\eta(\tau)\,d\tau\right\} \tag{3.11}$$

Let us set $\psi(t) = \bar{\psi}_1(t)$ and take into account the condition $\eta(0) = 0$. We obtain

$$\sigma(t) = E[\eta(t) - M_\eta(t)], \tag{3.12}$$

where the symbol $M_\eta(t)$ denotes the average value of the function $\eta(\tau)$ in the interval $\langle t - t_1, t\rangle$, which is given by

$$M_\eta(t) = \frac{1}{t_1}\int_{t-t_1}^{t} \eta(\tau)\,d\tau. \tag{3.13}$$

It follows from the assumptions introduced in paragraph 1 that $\eta(t) = 0$ for $t \leqslant 0$. According to the relations $(2.2)_1$ and (3.12), the equation

$$W(\varepsilon; T) = E\{\int_0^T \eta(t)\,\varepsilon'(t)\,dt - \int_0^T M_\eta(t)\,\varepsilon'(t)\,dt\} \tag{3.14}$$

is then valid for the specific work. The second integral on the right-hand side of this relation may be modified again by means of integration per partes so that we obtain

$$W(\varepsilon; T) = E\left\{\int_0^T \eta(t)\,\varepsilon'(t)\,dt - [M_\eta(t)\,\varepsilon(t)]_0^T + \int_0^T \frac{dM_\eta(t)}{dt}\,\varepsilon(t)\,dt\right\}. \tag{3.15}$$

Let us suppose now in conformity with the conditions of Definition 3 that $\varepsilon(t)$ is a deformation cycle closed in the interval $\langle 0, T\rangle$. The first two terms on the right-hand side of (3.15) vanish in this case. Furthermore, we may write

$$\frac{dM_\eta(t)}{dt} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}[M_\eta(t+\Delta t) - M_\eta(t)] = \tag{3.16}$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{M_\eta(t) + \frac{1}{t_1}[\eta(t)\,\Delta t - \eta(t-t_1)\,\Delta t] - M_\eta(t)\right\} =$$

$$= \frac{1}{t_1}[\eta(t) - \eta(t-t_1)].$$

Finally, we obtain

$$W(\varepsilon; T) = \frac{E}{t_1}\int_0^T [\eta(t) - \eta(t-t_1)]\,\varepsilon(t)\,dt. \tag{3.17}$$

Since E and t_1 are positive constants, it is sufficient to prove that the relation

$$I = \int_0^T [\eta(t) - \eta(t-t_1)]\,\varepsilon(t)\,dt \geqslant 0 \tag{3.18}$$

is valid for every $\varepsilon(t)$ closed in the interval $\langle 0, T\rangle$.

Proof of the inequality (3.18) will be based on Lemma 1. Suppose that $t_1 < T$. (In the reverse case, the validity of the relation (3.18) is obvious because $\eta(t-t_1) = 0$ and $\operatorname{sgn}\eta(t) = \operatorname{sgn}\varepsilon(t)$). Choose a natural number k and set $\Delta t = t_1/k$. Then there exists a natural number $n \geqslant k$ such that

$$n\Delta t \leqslant T \leqslant (n+1)\,\Delta t. \tag{3.19}$$

Let us divide the interval $\langle 0, T\rangle$ in the following way:

$$t_0 \leqslant 0 \leqslant t_1 < t_2 < \ldots < t_n = T, \tag{3.20}$$

where

$$t_i = t_{i-1} + \Delta t; \qquad i = 1, 2, \ldots, n. \tag{3.21}$$

It has been already mentioned that the function $\varepsilon(t)$ and $\eta(t)$ vanish in the interval $\langle t_0, 0\rangle$. Let us consider now the respective integral sum I_n instead of the integral I. For this sum the relation

$$I_n = \Delta t \sum_{i=1}^{n} (\varepsilon_i\eta_i - \varepsilon_i\eta_{i-k}) \tag{3.22}$$

is valid, where the notation $\eta_i = \eta(t_i)$; $\varepsilon_i = \varepsilon(t_i)$ is used for the sake of brevity and where $\eta_j = 0$ for $j = 0, -1, \ldots, 1-k$. The sum I_n may be rewritten in the alternate form

$$I_n = \Delta t \sum_{r=n-k+1}^{n} (\varepsilon_r \eta_r - \varepsilon_r \eta_{r-k} + \varepsilon_{r-k}\, \eta_{r-k} - \varepsilon_{r-k}\, \eta_{r-2k} + \ldots + \varepsilon_{r-\alpha k}\, \eta_{r-\alpha k}). \quad (3.23)$$

In this relation, α is a natural number for which

$$(\alpha - 1)\, t_1 \leqslant T \leqslant \alpha t_1 \quad (3.24)$$

holds. The sums written behind the summation symbol in (3.23) satisfy the conditions of Lemma 1 because the functions $\varepsilon(t)$ and $\eta(t)$ are continuous in the interval $\langle 0, T\rangle$. Consequently, these sums are non-negative, as well as the integral sum I_n for every natural n. Passint to the limit $n \to \infty$, which is possible thanks to the (R)-integrability of the integrand in (3.18), we obtain $I \geqslant 0$. Q. E. D.

Let us pass now to the resulting

Theorem 2. *The relation* (2.1) *is admissible if the function* $\psi(t)$ *is positive, decreasing and convex from below in the interval* $\langle 0, \infty)$ *and if the function* $\eta(\varepsilon)$ *is non-decreasing in the interval* $(-1, \infty)$ *provided that* $\eta(0) = 0$.

Proof. The functions $\varepsilon'(t)$ and $\eta'(t)$ are continuous in the interval $\langle 0, T\rangle$. Consequently, the function $f(t, \tau) = \varepsilon'(t)\, \eta'(\tau)$ is continuous on the triangle Ω which is situated in the plane $\{t, \tau\}$ and for the points of which the relations

$$0 \leqslant \tau \leqslant t; \qquad 0 \leqslant t \leqslant T \quad (3.25)$$

are valid. The relation (2.2) can be re-written in the following manner:

$$W(\varepsilon; T) = E \iint_{\Omega} \psi(t - \tau) f(t, \tau)\, dt\, d\tau. \quad (3.26)$$

From the point of view of functional analysis, the functions $f(t, \tau)$ and $\psi(t - \tau)$ may be considered as points of the space C_Ω of continuous functions defined on the area Ω with corresponding metrics (see, for instance, [6]). Let us choose a certain function $f(t, \tau) \in C_\Omega$. The specific work $W(\varepsilon; T)$ given by the relation (3. 26) may be then considered as a linear functional defined on the set $\{\psi\}$ of the relaxation functions, which can be expressed by the relation

$$W(\psi) = E \iint_{\Omega} \psi(t - \tau) f(t, \tau)\, dt\, d\tau. \quad (3.27)$$

The functional (3.27) is additive and continuous 6]. It follows from its additivity and from Theorem 1 that

$$W(\bar{\psi}) \geqslant 0, \quad (3.28)$$

where $\bar{\psi}$ is the sum

$$\bar{\psi}(t) = \sum_{i=1}^{n} \lambda_i \bar{\psi}_i(t); \qquad \lambda_i > 0; \qquad \sum_{i=1}^{n} \lambda_i = 1 \quad (3.29)$$

composed from the functions $\bar{\psi}_i(t)$, which are defined, similarly as the function $\bar{\psi}_1(t)$, by the relation (3.10), in which for t_1 a certain positive t_i is substituted.

Suppose now that $\psi(t)$ is a positive, continuous and decreasing function defined in the interval $\langle 0, T\rangle$, which is convex from below and for which $\psi(0) = 1$ is valid. Let us divide the interval $\langle 0, T\rangle$ into n pieces with the same length and denote by $\psi_0 = \psi(0)$, $\psi_1, \psi_2, \ldots, \psi_n = \psi(T)$ the values of this function at the points of division of this interval. Let us denote further

$$\Delta_i = \psi_i - \psi_{i-1}. \quad (3.30)$$

It is obvious that

$$\Delta_1 \geqslant \Delta_2 \geqslant \ldots \geqslant \Delta_n. \quad (3.31)$$

Let us substitute for the function $\psi(t)$ the function $\tilde{\psi}(t)$, which is composed of straight pieces and which coincides with $\psi(t)$ in the points of division: $\tilde{\psi}_i = \psi_i$. The function $\psi(t)$ is continuous and monotonous so that

$$\|\psi(t) - \tilde{\psi}(t)\| = \max_{t \in \langle 0, T\rangle} |\psi(t) - \tilde{\psi}(t)| \leqslant \Delta_1, \quad (3.32)$$

where the definition of metrics in C_Ω by means of norm was used. It may be easily proved that there allways exists a function $\tilde{\psi}(t)$ given by the relation (3.29) such that $\bar{\psi}(t) = \bar{\psi}(t)$ on $\langle 0, T\rangle$, provided the function $\psi(t)$ is convex from below. Consequently, the relation

$$W(\tilde{\psi}) \geqslant 0 \quad (3.33)$$

holds. Let us pass now to the limit $n \to \infty$. It follows from the continuity of the function $\psi(t)$ that $\Delta_1 \to 0$ and, consequently, $\tilde{\psi}(t) \to \psi(t)$ in the sense of the metric given by the relation (3.32). Since the functional (3.27) is continuous, we obtain $W(\tilde{\psi}) \to W(\psi)$. Finally, the relation (3.33) yields the relation

$$W(\psi) \geqslant 0. \quad (3.34)$$

Q. E. D.

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Katedra stavebnej mechaniky
Stavebnej fakulty SVŠT,
Bratislava