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INVERSE SCATTERING PROBLEM INCLUDING ARBITRARY PARAMETERS

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The inverse problem is formulated for determining the class of potentials containing the exponential potentials and the long-range rational tail for an arbitrary partial wave. As an investigation approach the generalized $N(E, r)/D(E, r)$ method from [3] is used. A possible way of the interpretation of arbitrary parameters resulting from the inclusion of the long-range region is also discussed.

1. INTRODUCTION

The solution of scattering problems in relativistic formalism (e. g. within the framework of the N/D method) exhibits the so-called CDD ambiguity. CDD poles are always connected with some arbitrary parameters entering the problem. To avoid the CDD-pole ambiguity it is generally necessary to impose additional restrictions upon the solution. In most cases these restrictions are equivalent to the requirement that all forces in the problem are due to exchange of particles in crossed channels. The non-relativistic analogon of exchange forces are Yukawa potentials. It is therefore not quite surprising that in the inverse problem of the non-relativistic potential scattering one founds features similar to the above mentioned situation. Namely, if the scattering data are given one may found a unique potential within the class of superposition of Yukawa potentials. If one admits also potentials decreasing as a negative power of r, then the solution becomes ambigous. It appears therefore that the non-relativistic analogon of CDD poles are potentials with a „rational tail" (e. g. decreasing like r^{-n} for $r \to \infty$).

In paper [2] we tried to establish the non-relativistic CDD ambiguity analogy and to construct with the help of this analogy a close formula for the corresponding potentials using constant quantities which occur in solving the inverse problem*. The presence of such quantities was induced according to the method elaborated in [7] and [9] just by taking into account the long-range forces in a reaction of particles. However no simple result for the potentials was achieved, although we eliminated finally the region of the short-range

* As to the applicability of the inverse problem method in the various fields of physics the reader is referred to [1].

forces. Nevertheless it turns out that the complete solution of the inverse problem covering both mentioned interaction regions is possible and it gives again some interesting results of mathematical character.

The algorithm of the proposed inverse problem is formulated within the framework of the off-the-mass-shell N/D method from [3]. We consider here the case of non-relativistic scattering which produces the potentials decreasing exponentially at infinity (particularly of the Bargmann type [4]) plus the long-range potentials with the rational behaviour by de Alfaro and Regge [5] and the corresponding wave functions, of course. The range of the problems is not so wide, but despite this the inclusion of arbitrary parameters into the inverse problem calculations can help to clarify various questions connected particularly with the ambiguity in the N/D method of the S-matrix approach. It should be added that the inverse problem approach by [3] seems to be available for the illustration of the close continuity of generalized N/D equations with the usual quantum mechanics and treatments.

First we shall show the connection between the $D(E, r)$ function and the Jost solution $g(k, r)e^{-ikr}$ possessing the analytical properties needed for the required type of the potentials (Sect. 2). The $g(k, r)$ function consists of two parts corresponding to the potentials of the short- and long-range character and both obey a symmetrical system of differential equations. This system, written in k and r variables, is solved exactly in the case when the singularities in the complex k plane are the finite number of poles (Sect. 3). In what follows (Sect. 4), we investigate the convergence of functions occurring in the Jost solution and then we pass to the case of discontinuities along a cut by the generalization of pole case results. The short- and long-range parts of the function $g(k, r)$ being solutions of the above system are functions of arbitrary constants and these parameters determine, as a matter of fact, (see [2]) positions and residues of CDD poles. Therefore we give also (Sect. 5) the dependence of the parameters of CDD poles on these arbitrary parameters. We do not intend to deal with the character of wave functions in the present paper, however finally as an atypical example of the inverse problem we want to apply our results for the generalization of the Noyes-Wong equation [6] to higher partial waves (Sect. 6) using the left-hand singularities as the N/D input data.

2. OFF-THE-MASS-SHELL N AND D FUNCTIONS AND THE MULTI-POLES JOST SOLUTION $f(k, r)$

The basic equation of the Petráš off-the-mass-shell N/D method has the following form

$$N(E, r) = F(E)e^{2ikr}D(E, r), \tag{1}$$

where $F(E)$ is the scattering amplitude, $N(E, r)$ and $D(E, r)$ are the generalized N and D functions. The $D(E, r)$ function is determined by the equation

$$D(E, r) = 1 + \frac{1}{2\pi}\int_C \frac{k' - k}{E' - E} N(E', r)dE'. \tag{2}$$

In eq. (2) E and k denote the energy and momentum variables respectively and the integration contour C encloses in a suitable way the singularities of the integrand (see [3]).

Eq. (1) is the starting-point also for the non-standard way of solving the inverse problem. If namely in Eq. (1) $F(E)$ is given as an input, we can obtain the potentials and the wave functions. The potentials have to be calcuelated from the expression

$$u(r) = \frac{i}{\pi}\int_C N'(E, r)dE, \tag{3}$$

and the Jost solution $f(E, r)$, connected with the wave function, depends on the $N(E, r)$ by way of the $g(E, r)$ function as follows

$$g(E, r) = 1 + \frac{1}{2\pi}\int_C \frac{k' + k}{E' - E} N(E', r)dE', \tag{4}$$

hence

$$g(E, r) = e^{ikr}f(E, r). \tag{5}$$

In this paper many considerations are offered for working conveniently in the k plane.

To obtain the potentials decreasing exponentially for a large r, it is necessary to take $g(k, r)$ in the form [7]

$$g(k, r) = 1 + 2\int_{m/2}^{-\infty} \frac{\alpha(k', r)}{k' + ik} dk'. \tag{6}$$

It can be easily proved that $D(E, r)$ in Eq. (2) is equivalent to the $g(-k, r)$ function with a discontinuity on a cut along the imaginary axis in the complex k plane. $N(E, r)$ in Eq. (1) is a function having the left-hand as well right-hand cut in the E plane and it behaves as $\exp 2ikr$ for $r \to \infty$ and $\mathrm{Im}N(E, 0) = 0$ for $E > 0$. Let us deform the integration path enclosing now the left-hand cut of $N(E, r)$ into a straight line running along this cut. We get

$$D(E,r) = 1 + \frac{1}{\pi} \int_{-\infty}^{-m^2/4} \frac{\mathrm{Im}N(E',r)}{\sqrt{-E'}+\sqrt{-E}}\,\mathrm{d}E'. \tag{7}$$

Finally if we substitute into (7) $k'^2 = E'$ and we write for the discontinuity

$$\frac{1}{\pi}\,\mathrm{Im}N(E,r) = \frac{\alpha(\sqrt{-E},r)}{\sqrt{-E}},$$

where $\alpha(\sqrt{-E},r)$ is a real function with the property $\lim_{r\to\infty} \alpha(\sqrt{-E},r) = 0$, Eq. (7) yields (using the denotation $k' \to ik'$)

$$D(E,r) = 1 + 2\int_{m/2}^{\infty} \frac{\alpha(k'\,r)\mathrm{d}k'}{k' - ik}.$$

Hence

$$g(-k,r) = g_{\mathrm{II}}(E,r) = D(E,r),$$

and $g_{\mathrm{II}}(E,r)$ is the $g(E,r)$ function taken on the second sheet of the Riemann surface. For $g(k,r)$ then the following relation must hold

$$g(k,r) = g(E,r) = 1 + \frac{1}{2\pi}\int_C \frac{k'+k}{E'-E} N(E'r)\mathrm{d}E',$$

where k is taken on the physical sheet.

If we want to obtain from our calculations also the class of long-range potentials, it is sufficient to extend $g(-k,r)$ by adding pole terms of various orders at the origin of the k plane (see for instance [2])

$$g(-k,r) = 1 + 2\int_{m/2}^{\infty} \frac{\alpha^{(l)}(k',r)\mathrm{d}k'}{k'-ik} + \sum_{j=1}^{l} \frac{\beta_j^{(l)}(r)}{(-ik)^j}, \tag{8}$$

where the pole functions $\beta_j^{(l)}(r)$ depend preliminarily on r (it will further be seen that $\beta_j^{(l)}(r)$ are also functions of arbitrary constants.

The determination of $g(k,r)$ is equivalent, one can say, to the determination of the scattering amplitude, because the singularities of the Jost solution are simply related to the singularities of the scattering amplitude and we know that (5) holds, i. e.

$$g(k,r) = \mathrm{e}^{ikr} f(k,r).$$

The multi-pole $g(-k,r)$ function in the representation given by Eq. (8) is the fundamental function for our solving the inverse problem with arbitrary parameters.

3. SHORT- AND LONG-RANGE PARTS OF THE FUNCTION $g(k,r)$

Even though the analytical properties of $g(k,r)$ are known on account of (8), we do not yet know explicitly $\alpha^{(l)}(k,r)$ and $\beta_j^{(l)}(r)$. The function $g(k,r)$ in complete analogy with $g(E,r)$ in [3] obeys the equation of the radial Schrödinger equation type

$$g''(k,r) - 2ikg'(k,r) = u(r)g(k,r), \tag{10}$$

with the boundary condition $\lim_{r\to\infty} g(k,r) = 1$ and with $u(r)$ as the potential. Whence it follows that the functions $\alpha^{(l)}$ and $\beta_j^{(l)}$ must satisfy the system

$$\alpha^{(l)\prime\prime}(k,r) + k\alpha^{(l)\prime}(k,r) = u^{(l)}(r)\alpha^{(l)}(k,r), \tag{11}$$

$$\beta_j^{(l)\prime\prime}(r) - 2\beta_{j+1}^{(l)\prime} = u^{(l)}(r)\beta_j^{(l)}(r), \quad j = 1,2,\ldots l \tag{12}$$

where

$$u^{(l)}(r) = -2[\beta_1^{(l)\prime}(r) + \int_m^{\infty} \alpha^{(l)\prime}(k',r)\mathrm{d}k'], \tag{13}$$

and the boundary condition has to be fulfilled

$$\lim_{r\to\infty} \alpha^{(l)}(k,r) = \lim_{r\to\infty} \beta_j^{(l)}(r) = 0.$$

Relation (13) corresponds to (3) for the considered class of potentials. It should be noted that the system of Eqs. (11) and (12) was solved in particular cases: when $\beta_j^{(l)}(r) = 0$ in paper [7] and when $\alpha^{(l)}(k,r) \to 0$ for large r in paper [8].

If we wish to obtain the complete solution of system (11) and (12), it will be convenient to find at first its solution in the case when the singularities of $g(k,r)$ are the poles only. The generalization of the achieved results for the case where this function exhibits the discontinuity along a cut will be straightforward.

Thus let us consider $g(k,r)$ with n poles on the positive part of the imaginary axis in the points $\frac{1}{2}ik_i$ and with l poles in the point $k = 0$

$$g(k,r) = 1 + 2\sum_{\mu=1,2\ldots}^{n} \frac{\alpha_\mu^{(l)}(k_i,r)}{2ik + k_\mu} + \sum_{j=1}^{l} \frac{\beta_j^{(l)}(r)}{(ik)^j}, \tag{14}$$

the index $i = 1, 2, 3, \ldots, n$

(For the sake of convenience we suppress the symbol $\sum_{\mu=1}^{n}$ at the second term of $g(k, r)$ in formula (14). In all the following calculations it will be necessary to carry out the summation from 1 to n in every expression containing the index μ. Fraction expressions have to contain this summation before the fraction line.)

The form of the equations that the individual $\alpha_\mu^{(l)}$ and $\beta_j^{(l)}$ obey in this case is the same as in Eq. (11) and Eq. (12).

$$\alpha_i^{(l)\prime\prime}(k_i, r) + k_i \alpha_i^{(l)\prime}(k_i, r) = u^{(l)}(r)\alpha_i^{(l)}(k_i, r), \tag{11'}$$

$$\beta_j^{(l)\prime\prime}(r) - 2\beta_{j+1}^{(l)}(r) = u^{(l)}(r)\beta_j^{(l)}(r), \tag{12'}$$

and the potentials are determined by the relation

$$u^{(l)}(r) = -2(\beta_1^{(l)\prime}(r) + \alpha_\mu^{(l)\prime}(k_i, r)). \tag{15}$$

Now the above-mentioned system has the following solution (the way of its solving is briefly indicated in the appendix i, ii)

$$\alpha_i^{(l)}(k_i, r) = A_i^{(l)} k_i e^{-k_i r}\left(1 + 2\frac{\alpha_\mu^{(l)}(k_i, r)}{k_\mu + k_i} + \sum_{j=1}^{l}\left(\frac{2}{k_i}\right)^j \beta_j^{(l)}(r)\right), \tag{16}$$

$$i = 1, 2, \ldots, n$$

$$\beta_1^{(l)}(r) = \sum_{\sigma = \begin{cases}2\tau \text{ for an even } l\\ 2\tau - 1 \text{ for an odd } l\end{cases}}^{l-2} (\{s_{\sigma+1}^{(0)}(r)\} - \{s_{\sigma-1}^{(0)}(r)\})\left(1 + A_{\sigma+1}^{(\nu)}(k_\mu^{(\nu)}, r)\alpha_\mu^{(l)}(k_i, r)\right), \tag{17}$$

$$\tau = 0, 1, 2, \ldots$$
$$\nu = 0, 1, 2, \ldots, \sigma + 2$$

$$\beta_j^{(l)}(r) = {}^{z_0}_{z_1}\sum_{\sigma = \begin{cases}2\tau \text{ for an even } l\\ 2\tau + 1 \text{ for an odd } l\end{cases}}^{l-2} \{s_\sigma^{(j-2)}(r)\}(\beta_1^{(\sigma+2)} - \beta_1^{(\sigma)}), \tag{18}$$

$$j = 2, 3, 4, \ldots, l$$
$$\tau = 0, 1, 2, \ldots$$

with z_0 when l and j are either even or odd simultaneously and with z_1 when l and j are even and odd mutually (these rules hold only as long as the indices of s are not $-1, 0$), where

$$\{s_\sigma^{(0)}(r)\} = \sum_{\nu = \begin{cases}2\tau \text{ for an even } \sigma\\ 2\tau + 1 \text{ for an odd } \sigma\end{cases}}^{\sigma} (2\nu + 1)\frac{z_\nu^2(r)}{z_{\nu-1}(r)z_{\nu+1}(r)}, \tag{19}$$

$$\tau = 0, 1, 2, \ldots$$

$$\{s_\sigma^{(j-2)}(r)\} = \sum_{\nu = \begin{cases}2\tau \text{ for an even } \sigma\\ 2\tau + 1 \text{ for an odd } \sigma\end{cases}}^{\sigma} (2\nu + 1)\frac{z_\nu(r)}{z_{\nu+1}(r)} z_1^{\omega}(r)\left[\frac{(\nu - 1)(\nu - 2)}{2} \times \right.$$

$$\left. \times \frac{z_{\nu-3}(r)z_\nu(r)}{z_{\nu-2}(r)z_{\nu-1}(r)} + \sum_{\lambda = \nu-1, \nu-2, \ldots}^{\nu-j+2 \geq 2} (2\lambda + 1)\frac{z_{\lambda-3}(r)\, z_\lambda(r)}{z_{\lambda-2}(r)z_{\lambda-1}(r)}\right], \tag{20}$$

$$\omega = \nu - j = 2, 3, 4, \ldots$$

$$A_\sigma^{(\nu)}(k_\mu^{(\nu)} r) = \sum_{\nu=0,1,2,\ldots}^{\sigma+1} \left(\frac{2}{k_\mu}\right)^{\nu+1} {}^{z_0}_{z_1}\{s_\sigma^{(\nu-1)}(r)\}, \tag{21}$$

with similar rules as for as z_0 and z_1 and the indices of s as above, $\{s_\sigma^{(j-2)}(r)\} \equiv 0$ for $\sigma < j - 2$; the expression in square brackets is equal to 1 for $j > \nu$. In Eqs. (19—21) $z_\nu(r)$ are polynomials fulfilling the recurrent equation

$$z_{\nu-1}(r)z_\nu''(r) - 2z_{\nu-1}'(r)z_\nu'(r) + z_{\nu-1}''(r)z_\nu(r) = 0, \tag{22}$$

with the initial polynomials $z_0 = 1$ and $z_1 = r + c_1$, c_1 being an arbitrary constant.

Consequently, the solutions $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ of system (11') and (12') are mutually associated according to Eqs. (16—18) by means of linear nonhomogenous equations in which the polynomials defined by (22) play an important role. The structure of these equations is likewise remarkable. The partial sums of the $z_\nu(r)$ polynomials appear everywhere. In these partial sums the polynomials obey a certain occupation rule in accordance with Eqs. (19) and (20).

Although we found only Eqs. (16—18) for $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ by solving system (11') and (12'), it is not difficult — due to the character of expression (21) — to calculate from the above both series of the sought functions. The terms of $A_{\sigma+1}^{(\nu)}(k_\mu^{(\nu)}, r)$ with $\nu \geqslant 1$ are composed symmetrically of the same polynomial configurations which occur in $\beta_j^{(l)}(r)$ for $j \geqslant 2$, respectively. The properties of the polynomials $z_\nu(r)$ were investigated in [8]. The most important property among them expressing simultaneously the symmetry of Eqs. (12') is related to (22). From Eqs. (17) and (18) it follows that the number of the polynomials and from Eq. (22) that the number of the arbitrary constants increases proportionally with an increasing l, beginning from z_1 for $l = 1$. The increase of the polynomial degree is defined by the rule $\frac{1}{2}\nu(\nu + 1)$. For the sake of comprehensibility of the obtained results we quote in the appendix iii Eqs. (17) and (18) for two concrete values of l as a useful example. The polynomials $z_\nu(r)$, as it will be obvious further, form for the individual partial waves a repulsive centrifugal potential barrier and a long-range potential tail, but with respect to Eqs. (15) and (16) they affect the short-range potentials too.

Let us quote as an example the function $g(k, r)$ with one pole on the imaginary axis in the point $k = ik_1/2$ and with l poles in the point $k = 0$. In this case

$$\alpha_1^{(l)}(k_1, r) = A_1^{(l)} k_1 e^{-k_1 r}\left(1 + \frac{\alpha_1^{(l)}}{k_1} + \sum_{j=1}^{l}\left(\frac{2}{k_1}\right)^j \beta_j^{(l)}(r)\right), \tag{23}$$

$$\beta_1^{(l)}(r) = \sum_{\sigma = \begin{cases} 2\tau \text{ f. a. e. } l \\ 2\tau - 1 \text{ f. a. o. } l \end{cases}}^{l-2} (\{s_{\sigma+1}^{(0)}\} - \{s_{\sigma-1}^{(0)}\})\,(1 + A_{\sigma+1}^{(\nu)}(k_1^{(\nu)} r)\alpha_1^{(l)}(k_1, r)), \tag{24}$$

$$\tau = 0, 1, 2, \dots$$
$$\nu = 0, 1, 2, \dots, \sigma + 2$$

and $\beta_j^{(l)}(j \geqslant 2)$ and $A_{\sigma+1}^{(\nu)}(k_1^{(\nu)}, r)$ are determined equally as in Eqs. (18) and (21). The sum in Eq. (23) can be easily calculated, when we realize that the solutions (17) and (18) have such a character that the differencies $\beta_1^{(\sigma+2)} - \beta_1^{(\sigma)}$ yield just

$$\beta_1^{(\sigma+2)} - \beta_1^{(\sigma)} = (\{s_{\sigma+1}^{(0)}\} - \{s_{\sigma-1}^{(0)}\})\,(1 + A_{\sigma+1}^{(\nu)}(k_1^{(\nu)}, r)\alpha_1^{(l)}(k_1, r)).$$

Hence

$$\sum_{j=1}^{l}\left(\frac{2}{k_1}\right)^j \beta_j^{(l)}(r) = \sum_{\sigma = \begin{cases} 2\tau \text{ f. a. e. } l \\ 2\tau - 1 \text{ f. a. o. } l \end{cases}}^{l-2} A_{\sigma+1}^{(\nu)}(k_1^{(\nu)}, r)\,(\beta_1^{(\sigma+2)} - \beta_1^{(\sigma)}) =$$

$$= \sum_{\sigma = \begin{cases} 2\tau \text{ f. a. e. } l \\ 2\tau - 1 \text{ f. a. o. } l \end{cases}}^{l-2} (\{s_{\sigma+1}^{(0)}\} - \{s_{\sigma-1}^{(0)}\}) A_{\sigma+1}^{(\nu)}(k_1^{(\nu)}, r)\,[1 + A_{\sigma+1}^{(\nu)}(k_1^{(\nu)}, r)\alpha_1^{(l)}(k_1, r)].$$

Thus we get for $\alpha_1^{(l)}(k_1, r)$ the result

$$\alpha_1^{(l)}(k_1, r) = \frac{A_1^{(l)} k_1 e^{-k_1 r}\left\{1 + \sum_{\sigma = \begin{cases} 2\tau \text{ f. a. e. } l \\ 2\tau - 1 \text{ f. a. o. } l \end{cases}}^{-2} A_{\sigma+1}^{(\nu)}(k_1^{(\nu)}, r)\,[\{s_{\sigma+1}^{(0)}\} - \{s_{\sigma-1}^{(0)}\}]\right\}}{\left\{1 - [1 + \sum_{\sigma = \begin{cases} 2\tau \text{ f. a. e. } l. \\ 2\tau - 1 \text{ f. a. o. } l \end{cases}}^{l-2} (A_{\sigma+1}^{(\nu)}(k_1^{(\nu)}, r))^2 (\{s_{\sigma+1}^{(0)}\} - \{s_{\sigma-1}^{(0)}\})] A_1 e^{-k_1 r}\right\}} \tag{25}$$

We note that we shall return to the case of solving Eqs. (11) and (12) later.

4. ASYMPTOTIC BEHAVIOUR OF $\alpha_i^{(l)}$ AND $\beta_j^{(l)}$

In the previous section we convinced ourselves that the functions $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ are determined by Eqs. (16—18) and that both series of the considered functions are connected with the partial sums of the $z_\nu(r)$ polynomials. Before investigating the asymptotical form of $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ let us ask the question how the $\{s_\sigma^{(j-1)}(r)\}$ behave in the region of the large and small r. To answer it, let us write at least some first polynomials from [8] (it turns out to be convenient to write r instead of $r + c_1$ — this is allowed with respect to the invariability of Eqs. (11′) and (12′) under such a transformation)

$$z_0 = 1,$$
$$z_1 = r,$$
$$z_2 = r^3 + 3c_2,$$
$$z_3 = r^6 + 15c_2 r^3 + 5c_3 r - 45c_2^2,$$
$$z_4 = r^{10} + 45c_2 r^7 + 35c_3 r^5 + 7c_4 r^3 - 525c_2 c_3 r^2 + 4725c_2^3 r - (175/3)c_3^2 + 21c_2 c_4.$$

The behaviour of $\{s_\sigma^{(j-1)}(r)\}$ for large r and finite c_j ($j = 1, 2, \dots, l$) depends only on the upper index. It seems to be very difficult to find in general the value $\lim_{\substack{r\to\infty \\ c_j \text{ finite}}} \{s^{(j-1)}(r)\}$, because it is altogether difficult to get an analytical expression valid for arbitrary $z_\nu(r)$. However, the mentioned limit for the individual cases exists and is finite. The results of the limiting procedure can be obtained also in the way used in [2], where the solution of (12′) was directly sought by expanding $\beta_j^{(l)}$ into a power series of $1/r$.

For $c_1 \neq 0$, $c_2 = c_3 = \dots = c_1 \to 0$ and $r \gg 1$ we have

$$\{s_\sigma^{(j-1)}\}_{r\gg1} = \frac{B_j^{(l)}}{(r + c_1)^j}, \tag{26}$$

$$j = 1, 2, 3, \dots, l$$
$$\sigma = 0, 1, 2, \dots, l$$

where

$$B_j^{(l)} = \frac{(l + j)!}{2^j j!(l - j)!}.$$

Since according to Eqs. (17) and (18)

$$\sum_{j=1}^{l}\left(\frac{2}{k_i}\right)^j \beta_j^{(l)}(r) = \sum_{\sigma = \begin{cases} 2\tau \text{ f. a. e. } l \\ 2\tau - 1 \text{ f. a. o.} \end{cases}}^{l-2} (\{s_{\sigma+1}^{(0)}\} - \{s_{\sigma-1}^{(0)}\}) A_{\sigma+1}^{(\nu)}(k_i^{(\nu)}, r) \times$$

$$\times [1 + A_{\sigma+1}^{(\nu)}(k_\mu^{(\nu)}, r)\alpha_\mu^{(l)}] \tag{27}$$

$$i = 1, 2, \ldots, n$$
$$\tau = 0, 1, 2, \ldots$$

and consequently $\alpha_i^{(l)}$ in Eq. (16) is

$$\alpha_i^{(l)} = A_i^{(l)} k_i e^{-k_i r}\Bigg(1 + \frac{2\alpha_\mu^{(l)}(k_i, r)}{k_\mu + k_i} + \sum_{\sigma = \left\{ \begin{smallmatrix} 2\tau \text{ f. a. e. } l \\ 2\tau - 1 \text{ f. a. o. } l \end{smallmatrix} \right.}^{l-2} (\{s_{\sigma+1}^{(0)}\} -$$

$$- \{s_{\sigma-1}^{(0)}\}) A_{\sigma+1}^{(\nu)}(k_i^{(\nu)}, r)\, [1 + A_{\sigma+1}^{(\nu)}(k_\mu^{(\nu)}, r)\alpha_\mu^{(l)}]\Bigg), \tag{27'}$$

we see that for $r \gg 1$ the $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ behave as

$$\alpha_i^{(l)} = A_i^{(l)} k_i e^{-k_i r}, \tag{28}$$

$$\beta_1^{(l)} = \sum_{\sigma = \left\{ \begin{smallmatrix} 2\tau \text{ f. a. e. } l \\ 2\tau - 1 \text{ f. a. o. } l \end{smallmatrix} \right.}^{l-2} (\{s_{\sigma+1}^{(0)}\} - \{s_{\sigma-1}^{(0)}\}), \tag{29}$$

$$\beta_j^{(l)} = \overset{z_0}{\underset{z_1}{}} \sum_{\sigma = \left\{ \begin{smallmatrix} 2\tau \text{ f. a. e. } l \\ 2\tau + 1 \text{ f. a. o. } l \end{smallmatrix} \right.}^{l-2} \{s_\sigma^{(j-2)}\} (\{s_{\sigma+1}^{(0)}\} - \{s_{(0)}^{\sigma-1}\}). \tag{30}$$

$$j = 2, 3, 4, \ldots, l$$
$$\tau = 0, 1, 2, \ldots$$

with the same rule for the use of z_0 and z_1 as in formula (21). The corresponding potentials for the various partial waves follow from (15)

$$u^{(l)}(\tilde{k}, r) = -2 \sum_{\sigma = \left\{ \begin{smallmatrix} 2\tau \text{ f. a. e. } l \\ 2\tau - 1 \text{ f. a. o. } l \end{smallmatrix} \right.}^{l-2} (\{s_{\sigma+1}^{(0)}\} - \{s_{\sigma-1}^{(0)}\})' + 2A_\mu^{(l)} k_\mu^2 e^{-k_\mu r}. \tag{31}$$

Here $\tilde{k}$ stands for $k_1, k_2, \ldots, k_i, \ldots, k_n$.

In the simplest case of $c_1 \neq 0$ and $c_j = 0$ for $j \geqslant 2$ with respect to Eq. (26) we get

$$u^{(l)}(\tilde{k}, r) = \frac{l(l+1)}{(r + c_1)^2} + 2A_\mu^{(l)} k_\mu^2 e^{-k_\mu r}. \tag{32}$$

Relations (31) and (32) for the sufficiently large r indicate that $u^{(l)}(\tilde{k}, r)$ consists of the superposition of the exponentially decreasing potentials and the potentials of the long-range forces decreasing in the limit $r \to \infty$ rationally ($\sim r^{-h}$, $h \geqslant 2$). So we obtain for $r \to \infty$, omitting the exponential terms and the terms decreasing faster than r^{-2}, the centrifugal barrier potential corresponding to the appropriate angular momentum l.

For $r \to 0$ the potential $u^{(l)}(\tilde{k}, r)$ is lacking already in the l-wave asymptotics. In the point $r = 0$ the functions $\alpha_i^{(l)}$, $\beta_1^{(l)}$, $\beta_j^{(l)}$ and thus also the potentials are regular. The regularity of the potentials in $r = 0$ may be interpreted by making use of the conception on CDD poles (Sect. 5). On the other hand one can respect the asymptotic behaviour as it is the rule (i. e. the singularity of $u^{(l)}(\tilde{k}, r)$ in $r = 0$), but then we are obliged to admit some supplementary conditions on $\alpha_i^{(l)}$ and $\beta_j^{(l)}$. For the region of the short distances it is then necessary to ask for satisfying the formulae

$$\alpha_i^{(l)} = \sum_{\nu=1}^{l} \frac{a_i^{(\nu)}}{r^\nu}, \qquad \beta_j^{(l)} = \sum_{\nu=1}^{l} \frac{b_j^{(\nu)}}{r^\nu}, \tag{33}$$

where $a_i^{(\nu)}$ and $b_j^{(\nu)}$ are some constants. Conditions (33) can be fulfilled by a suitable choice of the integration constants c_j in the solution of Eqs. (11') and (12'). This result was verified in the considerably restricted case of solving Eqs. (11') and (12') in [9], but it is correct also in the present general case.

5. ARBITRARY PARAMETERS IN $g(k, r)$ AND PARAMETERS OF CDD POLES

Now we can discuss the solution of Eqs. (11) and (12). In Sect. 4 we saw that in the pure pole case, taking finite c_j, no divergency difficulties arise in the r variable using the representation of the function $g(k, r)$ from Eq. (14). In the variable k this function is analytical with the exception of the considered poles. For these reasons the results we have achieved for the function $g(k, r)$ with n poles on the imaginary axis of k may be extended to the singularities of the type of discontinuities along the cut of the imaginary axis from $\frac{1}{2}m$ to ∞. Therefore we obtain the solution of Eqs. (11) and (12) straightforwardly by substituting the sums in relations (16 — 18) by the corresponding integrals. Since on account of (19—21) both formulae (17) and (18) may be expressed as one (denote next $\sum_{\sigma = \left\{ \begin{smallmatrix} 2\tau \text{ f. a. e. } l \\ 2\tau - 1 \text{ f. a. o. } l \end{smallmatrix} \right.}^{l-2} (\{s_{\sigma+1}^{(0)}\} - \{s_{\sigma+1}^{(0)}\})$ as $\boldsymbol{S}_{\sigma+-}^{(0)}(r)$)

$$\beta_j^{(l)}(r) = \overset{z_0}{\underset{z_1}{}} \boldsymbol{S}_{\sigma+-}^{(0)}(r)\, \{s_\sigma^{(j-2)}(r)\}\, (1 + A_{\sigma+1}^{(\nu)}(k_\mu^{(\nu)}, r)\alpha_\mu^{(l)}(\tilde{k}, r)), \tag{34}$$

and since for the sum over j in Eq. (16) we have formula (27), we get immediately

$$\alpha^{(l)}(k, r) = A^{(l)}(k) e^{-kr}[1 + \boldsymbol{S}_{\sigma+-}^{(0)}(r) A_{\sigma+1}^{(\nu)}(k^{(\nu)}, r) +$$

$$+\int_m^\infty \left(\frac{2}{k+k'} + S^{(0)}_{\sigma+-}(r)(r)\, A^{(\nu)}_{\sigma+1}(k^{(\nu)}, r) A^{(\nu)}_{\sigma+1}(k'^{(\nu)}, r)\right)\alpha^{(l)}(k', r)\, dk'], \quad (35)$$

$$\beta^{(l)}_j(r) = {}^{z_0}_{z_1}S^{(0)}_{\sigma+-}(r)\,\{s^{(j-2)}_\sigma(r)\}\,(1 + \int_m^\infty A^{(\nu)}_{\sigma+1}(k'^{(\nu)}, r)\alpha^{(l)}(k', r) dk'). \quad (36)$$

$$\nu = 0, 1, 2, \ldots, \sigma + 1$$

In Eq. (34) k represents the set $k_1, k_2, \ldots, k_i, \ldots, k_n$. For $\alpha^{(l)}(k, r)$ and $\beta^{(l)}_j(r)$ we have thus linear nonhomogenous integral equations. If the functions $A^{(l)}(k)$ or $\alpha^{(l)}(k, 0)$ are known, Eqs. (35) and (36) enable us to evaluate $\alpha^{(l)}(k, r)$, $\beta^{(l)}_j(r)$ and with their help also the potential according to Eq. (13). The functions $A^{(l)}(k)$ are connected with the discontinuity of the S-matrix along the cut in the following way

$$\frac{1}{4\pi i}[S_l(ik/2 - \varepsilon) - S_l(ik/2 + \varepsilon)] = A^{(l)}(k). \quad (37)$$

The functions $\alpha^{(l)}(k, r)$ and $\beta^{(l)}_j(r)$ are also the functions of the arbitrary constants, as we know. Their number is l. Due to these constants, included into our calculations in an unaffected way, of course, $\alpha^{(l)}(k, 0)$ and $\beta^{(l)}_j(0)$ are regular. In paper [2] it was proved that $\beta^{(l)}_j(0)$ — the residues of the poles of the function $g(k, r)$ given by Eq. (8) for $r \to 0$ and $k > 0$ — are in connection with parameters (positions and residues) of CDD poles. If $-E_\varrho$ are the positions of the CDD poles, situated in the gap between right and left cuts of the energy plane, and Γ_ϱ are their residues, the following has to hold (according to [2])

$$\sum_{j=1}^{l/2} \frac{\beta^{(l)}_{2j-1}(0)}{(-1)^j(-E_\varrho)^j} = \int_{m^2/4}^\infty \frac{\alpha^{(l)}(\sqrt{E'}, 0) dE'}{\sqrt{E'}(E' - E_\varrho)}, \quad (38)$$

$$\Gamma_\varrho = \frac{(-E_\varrho)^{l/2}}{\prod\limits_{j=1 \neq \varrho}^{l/2}(E_j - E_\varrho)}\left(1 + \sum_{j=1}^{l/2}\frac{\beta^{(l)}_{2j}(0)}{E^j_\varrho} + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\mathscr{D}(E', 0)\, dE'}{E' + E_\varrho}\right), \quad (39)$$

$$\varrho = 1, 2, \ldots, l/2$$

where the function $\mathscr{D}(E, 0)$ expressed now within the framework of the functions $N(E, r)$ and $D(E, r)$ has the structure

$$\mathscr{D}(E, 0) = \frac{\prod\limits_{j=1}^{l/2}(E + E_j)}{E^{l/2}}\left(1 + \sum_{j=1}^{l/2}\frac{\Gamma_j}{E + E_j} - \frac{1}{\pi}\int_0^\infty \frac{k' N(E', 0)}{E' - E} dE'\right). \quad (40)$$

The residues $\beta^{(l)}_j(0)$ ($j = 1, 2, \ldots, l$) occurring in Eqs. (38) and (39) are determined in a general case by (36), thus

$$\beta^{(l)}_j(0) = {}^{z_0}_{z_1}S^{(0)}_{\sigma+-}(0)\,\{s^{(j-2)}_\sigma(0)\}\left(1 + \int_{m^2}^\infty A^{(\nu)}_{\sigma+1}(\sqrt{E'^{(\nu)}}, 0)\alpha^{(l)}(\sqrt{E'}, 0)\, dE'\right). \quad (41)$$

In such a way Eqs. (39) and (40) attest that the arbitrary parameters, achieved by our method, correspond to the parameters of the CDD poles and consequently, one can say, they represent the subtraction constants of the N/D method. Hence there exists the explanation why the potentials in formula (13) exhibit the regular behaviour in $r = 0$. The regularity of the potentials in the origin is affected by the participation of the CDD poles in the interaction of particles. It is obvious from Eqs. (35) and (36) that this influence of the CDD poles upon the iteration of elementary particles appears at long and short distances, as well. The iteration procedure applied for instance on Eqs. (35) and (36) results in the superposition of the exponential potentials in (13) which suppress the rational potential terms at short distances. On the other hand in the region $r \to \infty$ the exponential terms may be in turn neglected and so only in this asymptotic region the contribution of the CDD poles can be unambigously tested. It is determined by formula (29).

Let us consider an example illustrating the connection of the constants c_j with the parameters of the CDD poles. For one CDD pole in the plane E with the position $-E_1$ and the residuum Γ_1 we have to take two functions $\beta^{(2)}_1(r)$ and $\beta^{(2)}_2(r)$. According to Eq. (41) their residues are

$$\beta^{(2)}_1(0) = \frac{3c_1^2}{c_1^3 + 3c_2}\left[1 + \int_{m^2}^\infty \left(\frac{1}{E'} + \frac{2}{c_1}\frac{\sqrt{E'}}{E'^2}\right)\alpha^{(2)}(\sqrt{E'}, 0)\, dE'\right], \quad (42)$$

$$\beta^{(2)}_2(0) = \frac{3c_1}{c_1^3 + 3c_2}\left[1 + \int_{m^2}^\infty \left(\frac{1}{E'} + \frac{2}{c_1}\frac{\sqrt{E'}}{E'^2}\right)\alpha^{(2)}(\sqrt{E'}, 0)\, dE'\right].$$

The integration constants c_1 and c_2 from Eq. (42) correspond to the parameters of this CDD pole on the basis of the equations

$$\beta^{(2)}_1(0) = E_1 \int_{m^2/4}^\infty \frac{\alpha^{(2)}(\sqrt{E'}, 0)\, dE'}{\sqrt{E'}(E' - E_1)}, \quad (43)$$

$$\beta_2^{(2)}(0) = -\left[\Gamma_1 + E_1\left(1 + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\mathscr{D}(E', 0)}{E' + E_1}\,\mathrm{d}E'\right)\right].$$

For the ratio of both residues $\beta_1^{(2)}(0)$ and $\beta_2^{(2)}(0)$ from Eq. (42) it follows

$$\frac{\beta_1^{(2)}(0)}{\beta_2^{(2)}(0)} = c_1 .$$

6. GENERALIZED NOYES-WONG EQUATION

So far we have taken into account the analytical properties of the Jost solution (more exactly of $g(k, r)$) as the necessary input in our considerations and the indicated method permitted to find the potentials which are consistent with these properties and consequently with the required analytical properties of the scattering amplitude. If we formulate the problem inversely — in a way usual in the dispersion relations — then we have to determine the scattering amplitude from the left-hand singularities. The Noyes-Wong equation for instance fulfils this purpose.

The Noyes-Wong equation for an arbitrary partial wave may be derived in our approach without knowing the functions $\alpha_i^{(l)}(k, r)$ (in the n poles case). Let us ussume that $\alpha_i^{(l)}(k, r)$ and $\beta_j^{(l)}(r)$ for $r \to 0$ behave according to Eq. (33). Then putting $a_i^{(l)} = \lim_{r\to 0} r^l \alpha_i^{(l)}(k, r)$ into Eq. (27′), we obtain a homogenous system of equations for the constants $a_i^{(l)}$

$$a_i^{(l)} = A_i^{(l)} k_i \left(\frac{2}{k_\mu + k_i} + \boldsymbol{S}_{\sigma+-}^{(0)}(0) \Lambda_{\sigma+1}^{(\nu)}(\bar{k}^{(\nu)}, 0) \Lambda_{\sigma+1}^{(\nu)}(k_\mu^{(\nu)}, 0)\right) a_\mu^{(l)} , \qquad (45)$$

where $A_i^{(l)}$ are constants, as well, and the partial sums and $\Lambda_{\sigma+1}^{(\nu)}$ are defined by Eqs. (19—21).

If instead n poles on the imaginary axis k one deals with the discontinuity along the cut, then in complete analogy with Eq. (35), Eq. (45) passes to the integral form

$$a^{(l)}(k) = A^{(l)}(k) \int_m^\infty \left[\frac{2}{k' + k} + \boldsymbol{S}_{\sigma+-}^{(0)}(0)\, \Lambda_{\sigma+1}^{(\nu)}(k^{(\nu)}, 0)\, \Lambda_{\sigma+1}^{(\nu)}(k'^{(\nu)}, 0)\right] a^{(l)}(k')\,\mathrm{d}k' . \qquad (46)$$

Here $A^{(l)}(k)$ represent the left-hand cut discontinuities as the input quantities. Eqs. (45) and (46) are the generalized forms of the Noyes-Wong equation for two current types of singularities. In the particular cases of P and D waves eq. (46) has the form in agreement with [9]

$$a^{(1)}(k) = A^{(1)}(k) \int_m^\infty \left(\frac{2}{k + k'} + \frac{4}{c_1 k k'}\right) a^{(1)}(k')\,\mathrm{d}k' ,$$

$$a^{(2)}(k) = A^{(2)}(k) \int_m^\infty \left[\frac{2}{k + k'} + \frac{3c_2^2}{c_2^3 + 3c_2}\left(\frac{2}{k} + \frac{4}{c_1 k^2}\right)\left(\frac{2}{k'} + \frac{4}{c_1 k'^2}\right) a^{(2)}(k')\,\mathrm{d}k'\right].$$

Eq. (46) is the homogenous integral equation (it can be proved that if all constants $c_j = 0$, the equation becomes nonhomogenous) for the determination of the functions $a^{(l)}(k)$ by way of which one can construct the S-matrix.

7. CONCLUSIONS

We have applied the non-relativistic inverse problem approach formulated in [3] in the terms of the generalized N/D method to determine a class of potentials including besides the superposition of exponential potentials the long-range rational tail. The extension of the problem to the rational terms is based on adding the poles of the various orders in the point $k = 0$ in the function $g(k, r)$ defined by Eq. (6). In our considerations the function $g(k, r)$ has taken the role of the $D(E, r)$ function in the k and r variables. It is to be regarded as the main contribution of this method that the rationality of potentials including also the centrifugal barrier is here determined by means of a set of the polynomials $z_\nu(r)$. Among the properties which these polynomials possess (see [8] in detail) one property is particularly remarkable — the recurrence expressed by Eq. (22) —

$$z_{\nu-1}(r) z_\nu''(r) - 2z_{\nu-1}'(r) z_{\nu-1}'(r) + z_{\nu-1}''(r) z_\nu(r) = 0 .$$

Resulting from the symmetrical Eqs. (12′) (when $\alpha_i^{(l)} \to 0$) and being similarly a symmetrical equation, the mentioned equation can be written as

$$z_\nu(r) = z_{\nu-2}(r)\left[(2\nu - 1)\int \left(\frac{z_{\nu-1}(r)}{z_{\nu-2}(r)}\right)^2 \mathrm{d}r + c_\nu\right],$$

and in this form it represents in a sense a sufficient condition when an integral of a squared rational function is a rational function. The arbitrary parameters, as the logical origin of which the recurrent equation may be considered and the number of which is equal to the orbital quantum number l, were interpreted in this work as the parameters of the CDD poles.

Our considerations concerned the determination of the mentioned class of the local potentials. In the following we shall try to use the obtained results for the investigation of wave functions. The connection between the described method and the standard inverse problem method of Gel'fand-Levitan-Marchenko is dealt with in [10].

APPENDIX

(i) In the derivation of formula (16) we proceed as follows. Let us introduce instead of the function $g(k, r)$ the function

$$g(k, r) = e^{2ikr}h(-k, r). \tag{A.1}$$

Substituting (A.1) into (10) one gets

$$h''(-k, r) + 2ikh'(-k, r) = u^{(l)}(r)h(-k, r). \tag{A.2}$$

Eq. (A.2) has to be fulfilled for every k, thus also for $k = -ik_i/2$ (the index $i = 1, 2, \ldots, n$). Therefore, we have

$$[h''(-k, r)]_{k=-ik_i/2} + k_i[h'(-k, r)]_{k=-ik_i/2} = u^{(l)}(r)[h(-k, r)]_{k=-ik_i/2}. \tag{A.3}$$

Compare Eq. (A.3) with Eqs. (11') for $\alpha_i^{(l)}$. We see that the following relation must be satisfied

$$\alpha_i^{(l)}(k_i, r) = A_i^{(l)}k_i[h(-k, r)]_{k=-ik_i/2}, \tag{A.4}$$

where $A_i^{(l)}$ is an arbitrary constant and the multiplication factor $A_i^{(l)}k_i$ is taken for dimensional reasons. Owing to Eqs. (14), (A.3) and (A.4) the solution of (11') has the form

$$\alpha_i^{(l)}(k, r) = A_i^{(l)}k_i e^{-k_i r}\left[1 + 2\frac{\alpha_\mu^{(l)}(k, r)}{k_\mu + k_i} + \sum_{j=1}^{l}\left(\frac{2}{k_i}\right)^j \beta_j^{(l)}(r)\right], \tag{A.5}$$

where k stands for the set $k_1, k_2, \ldots, k_i, \ldots, k_n$.

(ii) The solution of Eqs. (11') and (12') cannot be directly found for an arbitrary l, because the solutions for a given l are associated with the solutions of the individual foregoing cases with smaller values of l. We want here to show one particular case of solving Eqs. (11') and (12') for $l = 3$. System (11') and (12') has the following form*

$$\alpha_i'' + k_i\alpha_i' + 2(\beta_1' + \alpha_\mu')\alpha_i = 0, \tag{A.6}$$

$$\beta_1'' - 2\beta_2' + 2(\beta_1' + \alpha_\mu')\beta_1 = 0, \tag{A.7}$$

$$\beta_2'' - 2\beta_3' + 2(\beta_1' + \alpha_\mu')\beta_2 = 0, \tag{A.8}$$

$$\beta_3'' + 2(\beta_1' + \alpha_\mu')\beta_3 = 0. \tag{A.9}$$

* Up to Eq. (A. 17) in the calculations we suppress the upper index of $\alpha_i^{(l)}$ and $\beta_j^{(l)}$. The index i runs from 1 to n; α_μ' means $\sum_{\mu=1}^{n} \alpha_\mu'$.

Let us multiply Eq. (A.8) by β_3 and Eq. (A.9) by $-\beta_2$ and add. We have

$$\beta_2''\beta_3 - \beta_3''\beta_2 - 2\beta_3'\beta_3 = 0.$$

This equation may be integrated. Since the functions α_i and β_j must fulfill the boundary conditions

$$\lim_{r\to\infty} \alpha_i = \lim_{r\to\infty} \beta_j = 0, \tag{A.10}$$

the integration constant is to be taken zero. We get

$$\beta_2'\beta_3 - \beta_3'\beta_2 - \beta_3^2 = 0,$$

or

$$\beta_2' - \frac{\beta_3'}{\beta_3}\beta_2 = \beta_3. \tag{A.11}$$

Eq. (A.11) has the solution

$$\beta_2 = e^{\ln\beta_3}\left(\int \beta_3 e^{-\ln\beta_3}\,dr + c_1\right),$$

which leads to the relation

$$\beta_2 = \beta_3(r + c_1), \tag{A.12}$$

with the arbitrary constant c_1. Further it appears to be convenient to write shortly $r + c_1 \to r$. Hence (A.12) is

$$\beta_2 = r\beta_3. \tag{A.12'}$$

Similarly if we multiply Eq. (A.7) by β_2 and Eq. (A.8) by $-\beta_1$ and add, we obtain

$$\beta_1''\beta_2 - \beta_2''\beta_1 - 2\beta_2'\beta_2 + 2\beta_3'\beta_1 = 0. \tag{A.13}$$

Next add to Eq. (A.13) Eq. (A.9)

$$\beta_1''\beta_2 - \beta_2''\beta_1 - 2\beta_2'\beta_2 + 2(\beta_1'\beta_3 + \beta_3'\beta_1) + \beta_3'' + 2\alpha_\mu'\beta_3 = 0.$$

This equation will have the form suitable for integrating if we substitute its last term by an expression which we construct by the analogical canonic combination of Eqs. (A.6) and (A.9) as above, thus

$$\beta_1''\beta_2 - \beta_2''\beta_1 - 2\beta_2'\beta_2 + 2(\beta_1'\beta_3 + \beta_3'\beta_1) + \beta_3'' + \tag{A.14}$$

$$+ 2\frac{\alpha_\mu\beta_3'' - \alpha_\mu''\beta_3}{k_\mu} = 0.$$

Since with regard to (A.10) one can take as zero also the further integration constant, after integrating and some arranging, Eq. (A.14) leads to the result

$$\beta_1' + \left(\frac{2}{r} - \frac{\beta_2'}{\beta_2}\right)\beta_1 = \left(\frac{1}{r^2} - \frac{1}{r}\frac{\beta_2'}{\beta_2}\right)\left(1 + 2\frac{\alpha_\mu}{k_\mu}\right) + \beta_2 + \frac{2}{r}\frac{\alpha_\mu'}{k_\mu}.$$

Whence for β_1 we have

$$\beta_1 = \frac{\beta_2}{r}\left\{\int \frac{d}{dr}\left[\frac{r}{\beta_2}\left(1 + 2\frac{\alpha_\mu}{k_\mu}\right)\right] dr + \int r^2\, dr + c_2\right\},$$

and finally

$$\beta_1 = \frac{1}{r}\left(1 + 2\frac{\alpha_\mu}{k_\mu}\right) + \frac{\beta_2}{3r^2}(r^3 + 3c_2), \qquad \text{(A.15)}$$

c_2 being the new integration constant.

Take now only Eq. (A.7) into our calculations. Combining the remaining equations of the system so that it is possible to eliminate the corresponding terms $2\alpha_\mu'\beta_1$, $2\alpha_\mu'\beta_2$, $2\alpha_\mu'\beta_3$, one can write

$$\beta_1'' - 2\beta_2' + 2\beta_1'\beta_1 + 2\frac{\alpha_\mu\beta_1'' - \alpha_\mu''\beta_1}{k_\mu} - 4\left(\beta_2\frac{\alpha_\mu'}{k_\mu} + \beta_2'\frac{\alpha_\mu}{k_\mu}\right) + \qquad \text{(A.16)}$$

$$4\frac{\alpha_\mu\beta_2'' - \alpha_\mu''\beta_2}{k_\mu^2} - 8\left(\beta_3\frac{\alpha_\mu'}{k_\mu} + \beta_3'\frac{\alpha_\mu}{k_\mu}\right) + 8\frac{\alpha_\mu\beta_3'' - \alpha_\mu''\beta_3}{k_\mu^3} = 0.$$

If we integrate (A.16) taking the zero integration constant owing to (A.10) and substitute

$$\frac{1}{r} = \frac{z_1'}{z_1}; \qquad \frac{3r^2}{r^3 + 3c_2} = \frac{z_2'}{z_2},$$

we obtain

$$\left(1 + 2\frac{\alpha_\mu}{k_\mu}\right)\beta_1' - 2\frac{\alpha_\mu'}{k_\mu}\beta_1 + \beta_1^2 + 4\left(\frac{\alpha_\mu}{k_\mu^2} + 2\frac{z_1'}{z_1}\frac{\alpha_\mu}{k_\mu^3}\right)\beta_2' -$$

$$-2\left[1 + 2\frac{\alpha_\mu}{k_\mu} + 2\frac{\alpha_\mu'}{k_\mu^2} + 4\frac{z_1'}{z_1}\left(\frac{\alpha_\mu}{k_\mu^2} + \frac{\alpha_\mu'}{k_\mu^3}\right) + 4\frac{z_1'}{z_1}\frac{\alpha_\mu}{k_\mu^3}\right]\beta_2 = 0.$$

Using (A.15) for β_2 the last equation will already be dependent on β_1 only

$$\Lambda_3\beta_1^{(3)\prime} - 2\left(\frac{\alpha_\mu'}{k_\mu} + \Theta_1 + \Theta_2\right)\beta_1^{(3)} + \beta_1^{(3)2} - \qquad \text{(A.17)}$$

$$-4\frac{z_2'}{z_2}\left(\frac{\alpha_\mu}{k_\mu^2} + 2\frac{z_1'}{z_1}\frac{\alpha_\mu}{k_\mu^3}\right)\beta_1^{(1)\prime} + 2(\Theta_1 + \Theta_2)\beta_1^{(1)} = 0,$$

where

$$\Lambda_3 = 1 + 2\frac{\alpha_\mu}{k_\mu} + 2^2\frac{z_2'}{z_2}\frac{\alpha_\mu}{k_\mu^2} + 2^3\frac{z_1'}{z_1}\frac{z_2'}{z_2}\frac{\alpha_\mu}{k_\mu^3},$$

$$\Theta_1 = \frac{\alpha_\mu'}{k_\mu} + \frac{z_2'}{z_2}\left[1 + 2\frac{\alpha_\mu}{k_\mu} + 2\frac{\alpha_\mu'}{k_\mu^2} + 2^2\frac{z_1'}{z_1}\left(\frac{\alpha_\mu}{k_\mu^2} + \frac{\alpha_\mu'}{k_\mu^3}\right) + 2^2\frac{z_1'}{z_1^2}\frac{\alpha_\mu}{k_\mu^3}\right],$$

$$\Theta_2 = -2\left(\frac{z_2'}{z_2}\right)'\left(\frac{\alpha_\mu}{k_\mu^2} + 2\frac{z_1'}{z_1}\frac{\alpha_\mu}{k_\mu^3}\right),$$

$$\beta_1^{(1)} = \frac{z_1'}{z_1}\left(1 + 2\frac{\alpha_\mu}{k_\mu}\right). \qquad \text{(A.18)}$$

It is to be noted that (A.18) is the solution of the equation analogical to Eq. (A.17) when $l = 1$, i. e.

$$\left(1 + 2\frac{\alpha_\mu}{k_\mu}\right)\beta_1^{(1)\prime} - 2\frac{\alpha_\mu'}{k_\mu}\beta_1^{(1)} + \beta_1^{(1)2} = 0.$$

Eq. (A.17) is the Riccati differential equation for the function $\beta_1^{(3)}$. The particular integral of Eq. (A.17) is

$$\beta_1^{(3)} = \beta_1^{(1)}.$$

From the Riccati equation (A.17) we get the Bernoulli equation by the transformation

$$\beta_1^{(3)} = \beta_1^{(1)} + \varphi. \qquad \text{(A.19)}$$

Putting (A.19) into (A.17) we find the Bernoulli equation for the function φ

$$\Lambda_3\varphi' + \left[2\left(\frac{z_1'}{z_1} - \frac{z_2'}{z_2}\right)\Lambda_3 - \Lambda_3'\right]\varphi + \varphi^2 = 0. \qquad \text{(A.20)}$$

Taking

$$\gamma = \frac{1}{\varphi}, \qquad \varphi \neq 0, \tag{A.21}$$

for γ from (A.20) one obtains

$$\gamma' + \left[2\left(\frac{z_2'}{z_2} - \frac{z_1'}{z_1}\right) + \frac{\Lambda_3'}{\Lambda_3}\right]\gamma = \frac{1}{\Lambda_3}.$$

This implies

$$\gamma = e^{-2(\ln z_2 - \ln z_1 + \ln \Lambda_3)}\left[\int \Lambda^{-3} e^{2(\ln z_2 - \ln z_1 + \ln \Lambda_3)}\, dr + c_3\right],$$

where c_3 is the third integration constant of the case $l = 3$. Hence computing the corresponding integral we have

$$\gamma = \frac{z_1^2}{z_2^2}\Lambda_3\left(\frac{1}{5}r^5 + 3c_2r^2 - \frac{9c_2^2}{r} + c_3\right). \tag{A.22}$$

Thus by (A.18), (A.19), (A.21), (A.22) we obtain the result

$$\beta_1^{(3)} = \frac{1}{r}\left(1 + 2\frac{\alpha_\mu}{k_\mu}\right) + \frac{(r^3 + 3c_2)^2}{r^2}\,\frac{\left(1 + 2\frac{\alpha_\mu}{k_\mu} + 4\frac{z_2'}{z_2}\frac{\alpha_\mu}{k_\mu^2} + 8\frac{z_1'}{z_1}\frac{z_2'}{z_2}\frac{\alpha_\mu}{k_\mu^3}\right)}{\frac{1}{5}r^5 + 3c_2r^2 - \frac{9c_2^2}{r} + c_3}. \tag{A.23}$$

If we introduce into the polynomials $z_1 = r = r + c_1$, $z_2 = r^3 + 3c_2$ the (new polynomial $z_3 = r^6 + 15c_2r^3 + 5c_3r - 45c_2^2$ and if we substitute also $\Lambda_1 = 1 + 2\frac{\alpha_\mu}{k_\mu}$, we can write the result (A.23) more comprehensively

$$\beta_1^{(3)}(r) = \frac{z_0}{z_1}\Lambda_1 + 5\frac{z_2^2}{z_1z_3}\Lambda_3. \tag{A.24}$$

Now we find easily according to (A.12), (A.15) and (A.24) the expressions for the functions $\beta_2^{(3)}$ and $\beta_3^{(3)}$:

$$\beta_2^{(3)} = \frac{z_2'}{z_2}(\beta_1^{(3)} - \beta_1^{(1)}), \tag{A.25}$$

$$\beta_3^{(3)} = \frac{1}{z_1}\frac{z_2'}{z_2}(\beta_1^{(3)} - \beta_1^{(1)}). \tag{A.26}$$

(iii) Eqs. (17) and (18) for $l = 6, 7$ have the following form

$$\beta_1^{(6)} = 3\frac{z_1^2}{z_0z_2}(1 + \Lambda_2) + 7\frac{z_3^2}{z_2z_4}(1 + \Lambda_4) + 11\frac{z_5^2}{z_4z_6}(1 + \Lambda_6),$$

$$\beta_2^{(6)} = \frac{z_0}{z_1}\beta_1^{(2)} + \left(\frac{z_0^2}{z_1} + 5\frac{z_2^2}{z_1z_3}\right)(\beta_1^{(4)} - \beta_1^{(2)}) + \left(\frac{z_0^2}{z_1} + 5\frac{z_2^2}{z_1z_3} + 9\frac{z_4^2}{z_3z_5}\right)(\beta_1^{(6)} - \beta_1^{(4)}),$$

$$\beta_3^{(6)} = 5z_1\frac{z_2}{z_3}(\beta_1^{(4)} - \beta_1^{(2)}) + z_1\left[5\frac{z_2}{z_3} + 9\frac{z_4}{z_5}\left(3\frac{z_1z_4}{z_2z_3} + 7\frac{z_0z_3}{z_1z_2}\right)\right](\beta_1^{(6)} - \beta_1^{(4)}),$$

$$\beta_4^{(6)} = 5\frac{z_2}{z_3}(\beta_1^{(4)} - \beta_1^{(2)}) + \left[5\frac{z_2}{z_3} + 9\frac{z_4}{z_5}\left(3\frac{z_1z_4}{z_2z_3} + 7\frac{z_0z_3}{z_1z_2} + 5\frac{z_2}{z_0z_1}\right)\right](\beta_1^{(6)} - \beta_1^{(4)}),$$

$$\beta_5^{(6)} = 9z_1\frac{z_4}{z_5}(\beta_1^{(6)} - \beta_1^{(4)}),$$

$$\beta_6^{(6)} = 9\frac{z_4}{z_5}(\beta_1^{(6)} - \beta_1^{(4)}),$$

$$\beta_1^{(7)} = \frac{z_0^2}{z_1}(1 + \Lambda_1) + 5\frac{z_2^2}{z_1z_3}(1 + \Lambda_3) + 9\frac{z_4^2}{z_3z_5}(1 + \Lambda_5) + 13\frac{z_6^2}{z_5z_7}(1 + \Lambda_7),$$

$$\beta_2^{(7)} = 3\frac{z_1^2}{z_0z_2}(\beta_1^{(3)} - \beta_1^{(1)}) + \left(3\frac{z_1^2}{z_0z_2} + 7\frac{z_3^2}{z_2z_4}\right)(\beta_1^{(5)} - \beta_1^{(3)}) + $$
$$+ \left(3\frac{z_1^2}{z_0z_2} + 7\frac{z_3^2}{z_2z_4} + 11\frac{z_5^2}{z_4z_6}\right)(\beta_1^{(7)} - \beta_1^{(5)}),$$

$$\beta_3^{(7)} = 3\frac{z_1}{z_2}(\beta_1^{(3)} - \beta_1^{(1)}) + \left[3\frac{z_1}{z_2} + 7\frac{z_3}{z_4}\left(\frac{z_0z_3}{z_1z_2} + 5\frac{z_2}{z_0z_1}\right)\right](\beta_1^{(5)} - \beta_1^{(3)}) +$$
$$+ \left[3\frac{z_1}{z_2} + 7\frac{z_3}{z_4}\left(\frac{z_0z_3}{z_1z_2} + 5\frac{z_2}{z_0z_1}\right) + 11z_1^2\frac{z_5}{z_6}\left(6\frac{z_2z_5}{z_3z_4} + 9\frac{z_1z_4}{z_2z_3}\right)\right](\beta_1^{(7)} - \beta_1^{(5)}),$$

$$\beta_4^{(7)} = 7z_1\frac{z_3}{z_4}(\beta_1^{(5)} - \beta_1^{(3)}) + \left[7z_1\frac{z_3}{z_4} + 11z_1\frac{z_5}{z_6}\left(6\frac{z_2z_5}{z_3z_4} + 9\frac{z_1z_4}{z_2z_3} + 7\frac{z_0z_3}{z_1z_2}\right)\right] \times$$
$$\times (\beta_1^{(7)} - \beta_1^{(5)}),$$

$$\beta_5^{(7)} = 7\frac{z_3}{z_4}(\beta_1^{(5)} - \beta_1^{(3)}) + \left[7\frac{z_3}{z_4} + 11\frac{z_5}{z_6}\left(6\frac{z_2z_5}{z_3z_4} + 9\frac{z_1z_4}{z_2z_3} + 7\frac{z_0z_3}{z_1z_2} + 5\frac{z_2}{z_0z_1}\right)\right] \times$$
$$\times (\beta_1^{(7)} - \beta_1^{(5)}),$$

$$\beta_6^{(7)} = 11 z_1 \frac{z_5}{z_6} (\beta_1^{(7)} - \beta_1^{(5)}),$$

$$\beta_7^{(7)} = 11 \frac{z_5}{z_6} (\beta_1^{(7)} - \beta_1^{(5)}),$$

where

$$\Lambda_1 = 1 + 2 \frac{\alpha_\mu}{k_\mu},$$

$$\Lambda_2 = 1 + 2 \frac{\alpha_\mu}{k_\mu} + 2^2 \frac{1}{z_1} \frac{\alpha_\mu}{k_\mu^2},$$

$$\Lambda_3 = 1 + 2 \frac{\alpha_\mu}{k_\mu} + 2^2 \frac{3 z_1^2}{z_0 z_2} \frac{\alpha_\mu}{k_\mu^2} + 2^3 \frac{3 z_1}{z_0 z_2} \frac{\alpha_\mu}{k_\mu^3},$$

$$\Lambda_4 = 1 + 2 \frac{\alpha_\mu}{k_\mu} + 2^2 \left(\frac{z_0^2}{z_1} + 5 \frac{z_2^2}{z_1 z_3} \right) \frac{\alpha_\mu}{k_\mu^2} + 2^3 \,.\, 5 z_1 \frac{z_2}{z_3} \frac{\alpha_\mu}{k_\mu^3} + (2^4) \,.\, 5 \frac{z_2}{z_3} \frac{\alpha_\mu}{k_\mu^4},$$

$$\Lambda_5 = 1 + 2 \frac{\alpha_\mu}{k_\mu} + 2^2 \left(3 \frac{z_1^2}{z_0 z_2} + 7 \frac{z_3^2}{z_2 z_4} \right) \frac{\alpha_\mu}{k_\mu^2} + 2^3 \left[3 \frac{z_1}{z_2} + 7 \frac{z_3}{z_4} \left(\frac{z_0 z_3}{z_1 z_2} + \right.\right.$$

$$\left.\left. + 5 \frac{z_2}{z_0 z_1} \right) \right] \frac{\alpha_\mu}{k_\mu^3} + (2^4) \,.\, 7 z_1 \frac{z_3}{z_4} \frac{\alpha_\mu}{k_\mu^4} + 2^5 \,.\, 7 \frac{z_3}{z_4} \frac{\alpha_\mu}{k_\mu^5},$$

etc.

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