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ON ADMISSIBLE RELAXATION FUNCTIONS IN THE THEORY OF LINEAR VISCO-ELASTICITY

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The mechanical behaviour of a visco-elastic body is defined by the relaxation function of the material. The specific mechanical work done in the process of deformation of the body is defined by means of Stieltjes' integral. The basic thermodynamical laws demand that this work should be non-negative. This condition defines a class of admissible relaxation functions. Two theorems giving necessary and sufficient conditions of admissibility are proved in the paper. Finally, several types of admissible relaxation functions are discussed, including some „paradoxical" relaxation functions.

1. INTRODUCTION

In the present paper the pure tension (or compression) of a linear visco-elestic body is discussed. The class of bodies to be discussed here is defined in Section 3 by a class of linear integral operators which transform the strain $\varepsilon(t)$ into the stress $\sigma(t)$, where t denotes time. Every operator is given by its kernel $\psi(t)$, which is the relaxation function of the material.

We confine our considerations to the time interval $0 \leqslant t < \infty$ and denote by $W(\varepsilon; T)$ the mechanical work per unit volume of the body done by the stress $\sigma(t)$ on the strain $\varepsilon(t)$ in the time interval $\langle 0, T \rangle$. It is assumed that the body is deformed isothermically. In this case, the basic thermodynamical laws imply the condition

$$W(\varepsilon; T) \geqslant 0, \tag{1.1}$$

which must be satisfied for every $\varepsilon(t)$ and every $T > 0$. The condition (1.1) imposes some limitations on the form of the relaxation functions, or, using an alternative formulation, it defines a class of admissible relaxation functions. The necessary and sufficient condition of admissibility is given in Sect. 4. It is shown that the class of admissible relaxation functions is substantially wider than the class usually considered in the theory of linear visco-elasticity. Some of the relaxation functions, which belong to the first class mentioned above but do not belong to the second, define the paradoxical behaviour of the respective materials.

Breuer and Onat [3] performed a similar investigation, starting from less general pre-suppositions about $\varepsilon(t)$ and using the more strict condition

$$W(\varepsilon; T) > 0, \tag{1.2}$$

which they demanded to be satisfied by every $\varepsilon(t)$ non-vanishing in $\langle 0, T \rangle$. They demonstrated that the condition (1.2) was implied by the uniqueness of solution of a class of boundary-value problems in the theory of linear visco-elasticity. The authors derived the following *sufficient* conditions of admissibility of the relaxation function in the sense of the relation (1.2): the function $\psi(t)$ should be continuous, positive, decreasing and convex from bellow for $0 \leqslant t < \infty$, with the consequence that $\lim \psi(t) \geqslant 0$ as $t \to \infty$. It was then demonstrated that the exponential relaxation functions are admissible in the sense of (1.2).

In the present paper, the condition (1.1) is used for two reasons as a starting point rather than the condition (1.2). First of all, in the case of a perfectly elastic body which is also involved into the class of bodies discussed here, there exist some $\varepsilon(t)$ non-vanishing in $\langle 0, T \rangle$ such that the condition (1.2) is not fulfilled. Moreover, the relation (1.1) enables us to give conditions which are not only *sufficient* but at the same time also *necessary* for the admissibility of the relaxation functions. The examples given in Sect. 4 show that the existence of „paradoxical" admissible relaxation functions is not connected with the difference between the relations (1.1) and (1.2). Some of the „paradoxical" relaxation functions are admissible from the point of view of both these relations although they do not fulfil the conditions given in [3].

2. MATHEMATICAL PREREQUISITES

We shall discuss first the function with a bounded variation and the integrals of Stieltjes.

Let $f(x)$ be a function defined and bounded on the interval $\langle a, b \rangle$. Consider a division

$$a = x_0 < x_1 < \ldots < x_{m-1} < x_m = b \tag{2.1}$$

of $\langle a, b \rangle$ and put

$$V = \sum_{i=0}^{m-1} |f(x_{i+1}) - f(x_i)| . \tag{2.2}$$

Definition 1. *The function $f(x)$ has a bounded variation on the interval $\langle a, b \rangle$ if there exists a constant C such that*

$$V < C \tag{2.3}$$

for all the divisions (2.1) *of* $\langle a, b \rangle$ [5, VIII § 3].

Let $f(x)$ and $g(x)$ be functions defined and bounded on the interval $\langle a, b \rangle$. Consider a division (2.1) and put

$$x_i \leqslant \xi_i \leqslant x_{i+1}; \qquad \lambda = \max (x_{i+1} - x_i). \tag{2.4}$$

Definition 2. *Stieltjes' integral of the function $g(x)$ with respect to the function $f(x)$ on the interval $\langle a, b \rangle$ is defined by*

$$\int_a^b g(x)\, \mathrm{d}f(x) = \lim_{\lambda \to 0} \sum_{i=0}^{m-1} g(\xi_i)[f(x_{i+1}) - f(x_i)] \tag{2.5}$$

if the limit on the right-hand side of (2.5) *exists and depends neither on the division* (2.1) *nor on the particular choice of* ξ_i.

Theorem 1. *The integral* (2.5) *exists if $g(x)$ is continuous on the interval $\langle a, b \rangle$ and if $f(x)$ has a bounded variation on the same interval* [5, VIII § 6]

The definitions and theorems presented above may be generalized in a simple way for the case $a = -\infty$, $b = \infty$ [5, XVII § 5].

Let $f(x, y)$ be a function defined and bounded on the rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$. Consider a division

$$a = x_0 < x_1 < \ldots < x_{m-1} < x_m = b \tag{2.6}$$

$$c = y_0 < y_1 < \ldots < y_{n-1} < y_n = d$$

of the rectangle and put

$$V = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j)| . \tag{2.7}$$

Definition 3. *The function $f(x, y)$ has a bounded variation on the rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ if there exists a constant C such that*

$$V < C \tag{2.8}$$

for all the divisions (2.6) *of the rectangle* [4, III § 59].

If $g(x)$ has a bounded variation on $\langle a, b \rangle$ and $h(y)$ has a bounded variation on $\langle c, d \rangle$, then $f(x, y) = g(x) \times h(y)$ has a bounded variation on the rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$. This statement follows from the relation

$$|f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{+j1}) + f(x_i, y_j)| = \tag{2.9}$$

$$= |g(x_{i+1}) - g(x_i)| \times |h(y_{j+1}) - h(y_j)| .$$

Let $f(x, y)$ and $g(x, y)$ be functions defined and bounded on the rectangle

$a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$. Consider a division (2.6) and put

$$x_i \leqslant \xi_i \leqslant x_{i+1}; \quad y_j \leqslant \eta_j \leqslant y_{j+1}; \tag{2.10}$$

$$\lambda = \max\,[(x_{i+1} - x_i), \quad (y_{j+1} - y_j)].$$

Definition 4. *Stieltjes' integral of the function $g(x, y)$ with respect to the function $f(x, y)$ on the rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ is defined by*

$$\int_a^b \int_c^d g(x, y)\, \mathrm{d}f(x, y) = \lim_{\lambda \to 0} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} g(\xi_i, \eta_j) \times$$

$$\times\, [f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j)] \tag{2.11}$$

if the limit on the right-hand side of (2.11) exists and does not depend on the division (2.6) and on the particular choice of the points ξ_i, η_j.

Theorem 2. *The integral (2.11) exists if $g(x, y)$ is continuous on the rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ and if $f(x, y)$ has a bounded variation on the same rectangle* [4, III § 59; 6, § 4].

We pass now to the positive definite functions. The following definitions and theorems were taken from [2, IV §§ 18—20]. We confine our considerations to the case of real functions.

Let $D(\alpha)$ be a function defined and bounded on the interval $(-\infty, \infty)$.

Definition 5. *The function $D(\alpha)$ is called distribution if it increases monotonically on the interval $(-\infty, \infty)$ and if the relation*

$$D(\alpha) = \tfrac{1}{2}[D(\alpha + 0) - D(\alpha - 0)] \tag{2.12}$$

holds for every α.

It is not difficult to show that any distribution has a bounded variation on $(-\infty, \infty)$.

Let $f(x)$ be a function defined and bounded on the interval $(-\infty, \infty)$, continuous at every finite x and even: $f(-x) = f(x)$.

Definition 6. *The function $f(x)$ is positive definite if it satisfies the relation*

$$\sum_{i=1}^{m} \sum_{j=1}^{m} f(x_i - x_j)\varrho_i\varrho_j \geqslant 0 \tag{2.13}$$

for arbitrary points $x_1, x_2, \ldots, x_m$ and arbitrary real numbers $\varrho_1, \varrho_2, \ldots, \varrho_m$.

Theorem 3 (*Bochner's Theorem*). *The function $f(x)$ is positive definite if and only if it can be written in the form*

$$f(x) = \int_{-\infty}^{\infty} \cos(\alpha x)\, \mathrm{d}D(\alpha), \tag{2.14}$$

where $D(\alpha)$ is an odd distribution: $D(-\alpha) = -D(\alpha)$; $D(0) = 0$.
The relation (2.14) may be written in an alternative form

$$f(x) = 2 \int_0^{\infty} \cos(\alpha x)\, \mathrm{d}D(\alpha). \tag{2.15}$$

From (2.14) or (2.15) there follows

$$f(0) = D(\infty) - D(-\infty) = 2D(\infty). \tag{2.16}$$

Theorem 4. *The sum and product of two positive definite functions is a positive definite function.*

3. THE RELAXATION FUNCTION AND THE SPECIFIC WORK

Consider a homogenous body subjected to pure tension (or compression). Suppose that the strain $\varepsilon(t)$ of the body has a bounded variation on $\langle 0, T \rangle$ for every positive T and put $\varepsilon(0) = 0$.

Definition 7. *The body is said to be linear visco-elastic if the relation*

$$\sigma(t) = E \int_0^t \psi(t - \tau)\, \mathrm{d}\varepsilon(\tau); \qquad 0 \leqslant t \leqslant T \tag{3.1}$$

holds between the strain $\varepsilon(t)$ and the stress $\sigma(t)$ for every positive T. The function $\psi(t)$ is defined and bounded on the interval $0 \leqslant t < \infty$ and continuous at every non-negative finite t; E is a positive constant.

The function $\psi(t)$ is called the relaxation function of the material, the constant E is called the instantaneous modulus of elasticity of the material. We shall see later that $\psi(0) > 0$. Thus it is possible to choose E such that $\psi(0) = 1$, which will be supposed throughout the rest of the paper. The assumption that $\varepsilon(t)$ has a bounded variation is more appropriate to the real physical situation than the usual assumption that $\varepsilon(t)$ is continuously differentiable; it allows, e. g., an instantaneous finite deformation of the body. The condition $\varepsilon(0) = 0$ together with the zero lower limit of the Stieltjes integral in (3.1) and its consequence $\sigma(0) = 0$ corresponds with the familiar assumption that the body has been left undisturbed in the time interval $(-\infty, 0)$ (see, for instance, [1, I § 2]).

We shall discuss now the specific mechanical work $W(\varepsilon; T)$ done by the stress $\sigma(t)$ on the strain $\varepsilon(t)$ in the time interval $\langle 0, T \rangle$.

Let us investigate first the special case discussed in paper [3]. Consider that $\varepsilon(t)$ has a derivative $\varepsilon'(t) = d\varepsilon(t)/dt$ which is piecewise continuous on $\langle 0, T\rangle$. In this case, (3.1) is replaced by the relation

$$\sigma(t) = E\int_0^t \psi(t-\tau)\varepsilon'(\tau)\,d\tau \tag{3.2}$$

with the usual Riemann integral, and the specific work is given by

$$W(\varepsilon; T) = \int_0^T \sigma(t)\varepsilon'(t)\,dt = E\int\int_{\Omega'} \psi(t-\tau)\varepsilon'(t)\varepsilon'(\tau)\,dt\,d\tau. \tag{3.3}$$

The symbol Ω' denotes a triangle in the plane $\{t, \tau\}$ given by $0 \leqslant t \leqslant T$; $0 \leqslant \tau \leqslant t$. If we define now the relaxation function $\psi(t)$ also for the negative values of its argument by $\psi(-t) = \psi(t)$, (3.3) may be re-written in the alternative form

$$W(\varepsilon; T) = \tfrac{1}{2}E\int\int_{\Omega} \Psi(t-\tau)\varepsilon'(t)\varepsilon'(\tau)\,dt\,d\tau, \tag{3.4}$$

where Ω denotes a square given by $0 \leqslant t \leqslant T$, $0 \leqslant \tau \leqslant T$.

Let us return now to the original assumption about the function $\varepsilon(t)$. It follows from the results of the Definition 3 that the function $f(t, \tau) = \varepsilon(t)\varepsilon(\tau)$ has a bounded variation on the square $0 \leqslant t \leqslant T$, $0 \leqslant \tau \leqslant T$. Further, it is obvious that the assumption $\psi(-t) = \psi(t)$ and the continuity of $\psi(t)$ imply that $\psi(t-\tau)$ is continuous on the same square. It is thus possible to define the specific work by the following

Definition 8. *The specific mechanical work $W(\varepsilon; T)$ done by the stress $\sigma(t)$ on the strain $\varepsilon(t)$ in the time interval $\langle 0, T\rangle$ is given by the Stieltjes integral*

$$W(\varepsilon; T) = \tfrac{1}{2}E\int\int_{\Omega} \psi(t-\tau)\,d[\varepsilon(t)\varepsilon(\tau)]. \tag{3.5}$$

In order to demonstrate the physical relevance of this definition, we shall show first that (3.5) is reduced to (3.4) if $\varepsilon'(t)$ exists and is piecewise continuous on $\langle 0, T\rangle$. Consider a division

$$0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = T \tag{3.6}$$

of $\langle 0, T\rangle$ and put

$$t_k \leqslant \bar{t}_k \leqslant t_{k+1}; \quad \lambda = \max\,(t_{k+1} - t_k). \tag{3.7}$$

The relation (3.5) and Definition 4 yield

$$W(\varepsilon; T) = \tfrac{1}{2}E\lim_{\lambda\to 0}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\psi(\bar{t}_i - \bar{t}_j)\,[\varepsilon(t_{i+1}) - \varepsilon(t_i)]\,[\varepsilon(t_{j+1}) - \varepsilon(t_j)], \tag{3.8}$$

where the limit depends neither on the division (3.6) nor on the particular

choice of $\bar{t}_k$. Thus we can choose the division (3.6) and the points $\bar{t}_k$ such that, according to the theorem on the mean value

$$\varepsilon(t_{k+1}) - \varepsilon(t_k) = \varepsilon'(\bar{t}_k)\,(t_{k+1} - t_k). \tag{3.9}$$

After the substitution of (3.9) into (3.8) we obtain

$$W = \tfrac{1}{2}E\lim_{\lambda\to 0}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\psi(\bar{t}_i - \bar{t}_j)\varepsilon'(\bar{t}_i)\varepsilon'(\bar{t}_j)\,(t_{i+1} - t_i)\,(t_{j+1} - t_j), \tag{3.10}$$

which is the definition formula of Riemannian integral in (3.4).

We shall present now the resulting formulae for the specific work, obtained from the relation (3.5) for two types of the function $\varepsilon(t)$ which are not piecewise continuously differentiable on $\langle 0, T\rangle$.

Consider first a single-step function

$$\varepsilon_1(t) = \begin{cases} 0 \text{ for } 0 \leqslant t \leqslant t_1 \\ C \text{ for } t_1 < t \leqslant T, \end{cases} \tag{3.11}$$

where $0 < t_1 < T$ and C is a constant. The equation (3.5) yields

$$W(\varepsilon; T) = \tfrac{1}{2}E\,C^2, \tag{3.12}$$

i. e. the specific work in this case is equal to the specific work for an absolutely elastic body. The same fact follows from that part of the theory of linear visco-elasticity (the part less general from our point of view), which is based on the rheological models instead of the relaxation functions [1, I § 14]. This also confirms the correctness of our definition of the specific work.

Finally, consider a multiple-step function

$$\varepsilon_2(t) = \begin{cases} 0 \text{ for } 0 \leqslant t \leqslant t_1 \\ C_k \text{ for } t_k < t \leqslant t_{k+1}, \end{cases} \tag{3.13}$$

where $0 = t_0 < \ldots < t_m < t_{m+1} = T$ and C_k are constants. The equation (3.5) yields in this case

$$\begin{aligned} W(\varepsilon_2; T) = \tfrac{1}{2}E\{&C_1^2 + C_2^2 + C_3^2 + \ldots + C_m^2 + \\ &+ 2[C_1C_2\psi(t_2 - t_1) + C_1C_3\psi(t_3 - t_1) + \ldots + C_1C_m\psi(t_m - t_1) + \\ &+ C_2C_3\psi(t_3 - t_2) + \ldots + C_2C_m\psi(t_m - t_2) + \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ C_{m-1}C_m\psi(t_m - t_{m-1})]\} = \sum_{i=1}^{m}\sum_{j=1}^{m}\tfrac{1}{2}E\,\psi(t_i - t_j)C_iC_j. \end{aligned} \tag{3.14}$$

4. ADMISSIBLE RELAXATION FUNCTIONS

The considerations presented in Sect. 1 form the basis for

Definition 9. *The relaxation function $\psi(t)$ defined for negative values of its argument by $\psi(t) = \psi(-t)$, is admissible if the relation*

$$W(\varepsilon; T) \geqslant 0 \tag{4.1}$$

is valid for every $\varepsilon(t)$ with a bounded variation on $\langle 0, T\rangle$ and every $T > 0$.

We shall demonstrate that the following theorem holds for the admissible relaxation functions:

Theorem 5. *The relaxation function is admissible if and only if it is positively definite.*

Proof. Sufficiency: Consider a division (3.6) of $\langle 0, T\rangle$ and put

$$\varepsilon(t_k) - \varepsilon(t_{k-1}) = \varrho_k; \quad t_{k-1} \leqslant \bar{t}_k \leqslant t_k; \tag{4.2}$$

$$\lambda = \max\,(t_k - t_{k-1}).$$

The relation (3.8) yields

$$W(\varepsilon; T) = \tfrac{1}{2}E \lim_{\lambda \to 0} \sum_{i=1}^{m} \sum_{j=1}^{m} \psi(\bar{t}_i - \bar{t}_j)\varrho_i\varrho_j. \tag{4.3}$$

If the relaxation function is positive definite, it follows from the Definition 6 that the sum on the right-hand side of (4.3) is non-negative for every division (3.6) and arbitrary choice of $\bar{t}_k$. Passing to the limit, we obtain (4.1). Q. E. D.
Necessity: Suppose that the relaxation function is not positively definite, i. e. that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \psi(t_i - t_j)\varrho_i\varrho_j < 0 \tag{4.4}$$

is valid for certain points t_k and numbers ϱ_k ($k = 1, 2, \ldots, m$). We may assume without loss of generality that

$$0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T. \tag{4.5}$$

Let us choose now a multiple-step function $\varepsilon_2(t)$ given by (3.13) such that $C_k = \varrho_k$. The relation (3.14) then yields

$$W(\varepsilon_2; T) = \tfrac{1}{2}E \sum_{i=1}^{m} \sum_{j=1}^{m} \psi(t_i - t_j)\varrho_i\varrho_j < 0, \tag{4.6}$$

which is contradictory to the condition (4.1). Q. E. D.

Combining Theorem 3 and its consequences with Theorem 5 and taking into account the condition $\psi(0) = 1$ we obtain the

Theorem 6. *The relaxation function $\psi(t)$ is admissible if and only if it can be written in the form*

$$\psi(t) = \int_0^\infty \cos\,(\alpha t)\,\mathrm{d}D(\alpha), \tag{4.7}$$

where $D(\alpha)$ is a distribution for which the conditions

$$D(0) = 0; \quad D(\infty) = 1 \tag{4.8}$$

are valid.

We shall finish the paper by presenting several types of admissible relaxation functions. It is not difficult to realize the physical characteristics of the materials corresponding to these relaxation functions if one takes into account that the function $E\psi(t)$ is equal (except the point $t = 0$) to the stress $\sigma(t)$ induced by the strain

$$\varepsilon(t) = \begin{cases} 0 \text{ for } t = 0 \\ 1 \text{ for } t > 0. \end{cases} \tag{4.9}$$

Let us consider the distribution

$$D_1(\alpha) = \frac{2}{\pi} \operatorname{arctg} \frac{\alpha}{k}; \quad k = > 0. \tag{4.10}$$

Substituting this distribution into (4.7) we obtain the relaxation function

$$\psi_1(t) = \exp\,(-k|t|). \tag{4.11}$$

Similarly, the distribution

$$D_2(\alpha) = \frac{2}{\pi} \sum_{i=1}^{m} C_i \operatorname{arctg} \frac{\alpha}{k_i} \tag{4.12}$$

which fulfils the conditions

$$\sum_{i=1}^{m} C_i = 1; \quad C_i > 0; \quad k_i > 0, \tag{4.13}$$

gives the relaxation function

$$\psi_2(t) = \sum_{i=1}^{m} C_i \exp\,(-k_i|t|). \tag{4.14}$$

The relaxation functions of this kind are familiar in the theory of linear visco-elasticity. These functions satisfy also the more strict relation (1.2) and the conditions given in [3].

The distribution

$$D_3(\alpha) = \begin{cases} 0 \text{ for } \alpha = 0 \\ 1 \text{ for } \alpha > 0 \end{cases} \tag{4.15}$$

corresponds to the relaxation function

$$\psi_3(t) = 1\,. \tag{4.16}$$

It is the relaxation function of an absolutely elastic material. This function does not satisfy the relation (1.2) and consequently it does not fulfil the conditions given in [3].

The distribution

$$D_4(\alpha) = \begin{cases} 0 \text{ for } 0 \leqslant \alpha < k \\ i \text{ for } \alpha > k;\ k > 0 \end{cases} \tag{4.17}$$

corresponds to the relaxation function

$$\psi_4(t) = \cos kt\,. \tag{4.18}$$

This function does not satisfy the relation (1.2) and, moreover, it defines a paradoxical behaviour of the material. If we subject the body to the unit strain given by (4.9), the stress induced thereby in the body will be equal to $E \cos kt$, i. e., it will pulse about zero. It is also paradoxical that in this case a permanent strain can be obtained without applying any work. This could be done for instance by subjecting the body to the multiple-step strain

$$\varepsilon(t) = \begin{cases} 0 \text{ for } t = 0 \\ \delta/2 \text{ for } 0 < t \leqslant \pi/k \\ \delta \text{ for } t > \pi/k \end{cases} \tag{4.19}$$

where δ is the value of permanent strain. A relaxation function obtained from $\psi_4(t)$ by a process similar to that of deriving the function $\psi_2(t)$ from $\psi_1(t)$, has similar paradoxical features.

It is a consequence of Theorem 3 that the function

$$\psi_5(t) = \exp(-k_1|t|) \cos(k_2 t) \tag{4.20}$$

is an admissible relaxation function. The stress induced by the strain (4.9) pulses as in the previous case but its amplitude decreases with time. However, it could be proved in this case that the function $\psi_5(t)$ satisfies the relation (1.2). The strain $\varepsilon(t)$ non-vanishing on $\langle 0, T\rangle$ can be thus obtained only by applying positive specific work. Moreover, the function ψ_5 does not fulfil the conditions given in [3].

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