THE N^*_{33} RESONANCE AND THE THREE—POLE N/D APPROXIMATION IN THE STATIC MODEL

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Within the framework of the N/D method in the static model the forces in the πN channel are usually approximated by two poles corresponding to the N and N^* exchanges. In the present paper a third pole is added and its parameters determined from the requirement of obtaining the correct mass and width for N^*_{33} in the direct channel. The third pole represents in a phenomenological way the forces different from the N and N^* exchanges or the corrections to their static approximation. The physical interpretation of parameters of the third pole obtained in the calculation is given. The sign of its residue indicates that the third pole represents a correction of the static model and not a resonance exchange contribution.

INTRODUCTION

The partial wave amplitudes of the πN scattering obey within the framework of the static model the following relations:

$$\mathrm{Re} f_l(\omega) = \frac{P}{\pi} \int\limits_L \frac{\mathrm{Im} f_l(\omega')\mathrm{d}\omega'}{\omega' - \omega} + \frac{P}{\pi} \int\limits_R \frac{\mathrm{Im} f_l(\omega')\mathrm{d}\omega'}{\omega' - \omega}; \tag{1}$$

where $L(R)$ denotes the integration over the left (right)-hand cut. The unitarity condition gives on the upper edge of the right-hand cut

$$\mathrm{Im} f_l(\omega) = q^3(\omega)|f_l(\omega)|^2; \tag{2}$$

where $q(\omega)$ is the kinematic factor. The left-hand cut is connected with the forces responsible for the scattering, i. e., with the particle systems that can be exchanged. The unitarity condition (2) is a non-linear equation, and in solving eqs. (1) and (2) it is customary to use the N/D method developed by Chew and Mandelstam, which reduces eqs. (1) and (2) to two coupled equations. We write the N/D decomposition as

$$f_l(\omega) = N_l(\omega)D_l(\omega)^{-1}. \tag{3}$$

N contains only the left-hand cut singularities and D only the righthand cut ones. In this way we obtain

$$N_l(\omega) = \frac{1}{\pi} \int_L \frac{\mathrm{Im} f_l(\omega')}{\omega' - \omega} \, d\omega' ; \tag{4}$$

and

$$D_l(\omega) = 1 - \frac{\omega}{\pi} \int_R \frac{d\omega' q^3(\omega') N_l(\omega')}{\omega'(\omega' - \omega)} ; \tag{5}$$

Here $D(\omega)$ is normalized conveniently to unity at $\omega = 0$. Such a normalization is, of course, permitted since both N and D may be multiplied by a common constant without changing the amplitude $f_l(\omega)$.

The dispersion integral in eq. (5) causes the well-known convergence troubles, which are frequently avoided by introducing a cut-off[1]). If we use a cut-off then the integral on the righthand side of eq. (5) is changed to the form

$$D(\omega) = 1 - \frac{\omega}{\pi} \int_1^{\Lambda} \frac{d\omega' q^3(\omega') N(\omega')}{\omega'(\omega' - \omega)} . \tag{5'}$$

The final results are, however, quite sensitive to the value of Λ which, besides, is always fixed up in a more or less arbitrary way.

A different method to avoid the convergence troubles has been proposed by Petráš [3]. His approach is based on potential theory, where N and D are easily defined in the terms of Jost functions $g(\omega, r)$. Petráš thus obtained a perfectly well defined and convergent system of equations of the type (4) and (5). In the limit $r \to 0$ these equations reduce to ordinary ones, requiring however in an explicit way the $N(\omega)$ function to behave like ω^{-3} for $\omega \to \infty$, which assures the convergence. If, as is frequently the case, we approximate the N function by a sum of pole terms:

$$N(\omega) = \sum_j \frac{a_j}{\omega + \omega_j} ,$$

the requirement $N(\omega) \to \omega^{-3}$ for $\omega \to \infty$ gives the following conditions on the pole parameters

$$\sum_i a_i = 0; \qquad \sum_i a_i \omega_i = 0. \tag{7}$$

1) The motivation of the cut-off introduction may be found in the original paper by Chew and Low in Phys. Rev. *101* (1956), 1570, based on the idea of an extended nucleon as a source of the meson field.

Equations (7) are easily derived from eq. (6) if one makes use of the identity

$$\frac{1}{\omega + \omega i} = \frac{1}{\omega} - \frac{\omega_i}{\omega^2} + \frac{\omega_i^2}{\omega^2(\omega + \omega_i)} .$$

The purpose of the present paper is to use the three pole approximation to the left-hand cut, where two poles are given by the N and N^* exchanges. The third pole is chosen to fulfil the relations (7). The possibility of obtaining the N^* resonance with the experimental values of mass and width is then examined, and the calculated behaviour of the phase shift δ_{33} is compared to the experimental data.

The paper is organised as follows: the first part introduces the P wave dispersion relations. In the second part some details of the three-pole calculations are given. The third part presents results which are summarized and commented on in the last part.

P WAVE DISPERSION RELATIONS

There are four P partial wave amplitudes of πN scattering which correspond to four different quantum numbers I and J. In the present paper we deal with the N_{33}^* resonance which is in the P_{33} ($l = 1$, $I = 3/2$, $J = 3/2$) partial wave at the kinetic energy 159 MeV in the centre-of-mass system (195 MeV in the laboratory system).

The dispersion relations for P partial wave amplitudes are given by the following equation [1]:

$$\mathrm{Re} f_i(\omega) = \frac{\lambda_i}{\omega} + \frac{P}{\pi} \int_1^{\infty} \left\{ \frac{\mathrm{Im} f_i(\omega')}{\omega' - \omega} + \frac{A_{ij} \mathrm{Im} f_j(\omega')}{\omega' + \omega} \right\} d\omega' ; \tag{8}$$

where $f_i(\omega) = \dfrac{e^{i\delta_i} \sin \delta_i}{q^3}$; the index i represents an amplitude where $2I$, $2J$ is successively equal to (1, 1); (1, 3); (3, 1); (3, 3); ω is the total meson energy, λ_i and the crossing matrix A_{ij} are:

$$\boldsymbol{\lambda} = \frac{\alpha f^2}{3} \begin{pmatrix} -4 \\ -1 \\ -1 \\ 2 \end{pmatrix} ; \qquad \mathbf{A} = \frac{1}{9} \begin{pmatrix} 1 & -4 & -4 & 16 \\ -2 & -1 & 8 & 4 \\ -2 & 8 & -1 & 4 \\ 4 & 2 & 2 & 1 \end{pmatrix} ; \tag{9}$$

$f^2 \doteq 0.08$ is the coupling constant.

The two particles unitarity condition for these amplitudes is:

$$\mathrm{Im} f_i(\omega) = q^3(\omega) |f_i(\omega)|^2 . \tag{10}$$

In considering the system of equations (8), (10), we neglect the inelastic processes. In the opposite case eq. (10) is not valid and there would remain only four equations (8) for eight unknown functions (real and imaginary parts of $f_i(\omega)$). Dispersion relations (8) are derived from dispersion relations for forward scattering by the static approach $\left(\frac{\mu}{M} \to 0\right)$. The kinematic factor in this case is

$$q(\omega) = (\omega^2 - 1)^{\frac{1}{2}}; \tag{11}$$

For the P_{33} partial wave amplitude we have

$$\mathrm{Re} f_{33} = \frac{4}{3}\frac{f^2}{\omega} + \frac{P}{\pi}\int_1^\infty \left\{\frac{\mathrm{Im} f_{33}}{\omega' - \omega} + \frac{\mathrm{Im}(4f_{11} + 2f_{13} + 2f_{31} + f_{33})}{9(\omega' + \omega)}\right\} d\omega' \tag{12}$$

Since there is a resonance in the P_{33} partial wave, we shall suppose that the contribution from this partial wave dominates over the contributions from other partial waves and determines the value of the dispersion integral [2]. Consequently we shall neglect other partial waves in the left-hand cut[2]) of eq. (12).

If we suppose that the width of the resonance is small enough ($\Gamma \to 0$) we can substitute the imaginary part of f_{33} (ω) by the δ-function and after performing the integration we obtain

$$\mathrm{Re} f_{33}(\omega) = \frac{4}{3}\frac{f^2}{\omega} + \frac{1}{9}\frac{\Gamma}{2}\frac{1}{q^3(\omega_{33})}\frac{1}{\omega_{33} + \omega} + \frac{P}{\pi}\int_1^\infty \frac{\mathrm{Im} f_{33}(\omega')}{\omega' - \omega} d\omega'; \tag{13}$$

where ω_{33} is the resonance energy.

THE THREE-POLE APPROACH

We can see from the last equation that the amplitude f_{33} contains a pole at $\omega = 0$ with the residue $C_1 = \frac{4}{3} f^2 = \frac{4}{9}\gamma_{11}$ and a pole at $\omega = -\omega_{33}$ with the residue $C_2 = \frac{1}{9}\frac{\Gamma}{2}(\omega_{33}^2 - 1)^{-3/2} = \frac{1}{9}\gamma_{33}$; and the right-hand cut from 1 to infinity.

Both poles are associated with exchanges of lowest mass particles with quantum numbers of the channel

$$\pi + N \to \pi + N,$$

i. e., the nucleon and the N^*_{33} resonance. The mass of N^*_{33} is $M + \omega_{33}$.

In further calculations we shall make use of Petráš's results [3]. Petráš expressed the scattering amplitude in the so-called N/D form

$$f(\omega) = N(\omega)D(\omega)^{-1} \tag{14}$$

and from the assumption that $N(\omega)$ is

$$N(\omega) = \sum_j \frac{a_j}{\omega + \omega_j}; \tag{15}$$

where a_j are some constants which have to be determined and $-\omega_j$ the known positions of the poles, $D(\omega)$ is

$$D(\omega) = b_0 + \omega b_1 + \frac{\omega^2 - 1}{\pi}\sum_j \frac{a_j}{\omega + \omega_j}\{\sqrt{\omega^2 - 1}[\ln(\omega + \sqrt{\omega^2 - 1}) - i\pi - \\ - \sqrt{(-\omega_j)^2 - 1}[\ln(-\omega_j + \sqrt{(-\omega_j)^2 - 1}) - i\pi]\}, \tag{16}$$

and the following conditions are valid:

$$\sum_i a_i = 0; \tag{17}$$

$$1 + \sum_i a_i \omega_i = 0; \tag{18}$$

Eqs. (14) and (15) give

$$a_j = C_j D(-\omega_j) \tag{19}$$

where C_j are the known residues of the poles of the scattering amplitude at $\omega = -\omega_j$. The term $[(-\omega_j)^2 - 1]^{1/2}$ means that the function $(\omega^2 - 1)^{1/2}$ should be analytically continued from the upper side of the right-hand cut up to the point $\omega = -\omega_j$. Eqs. (17) (18) (19) are sufficient for the full determination of the constants a_j and b_0, b_1. Condition (18) has no deeper physical meaning, it leads only to a normalization of the N and D functions which can be chosen arbitrarily without affecting the scattering amplitude.

The three-pole approach consists in the addition of a third pole to the left-hand cut. The position of this pole is not fixed beforehand and will approximately be determined later by the comparison of the calculated results and the data. If we choose instead of (18) the normalization $D(1) = 1$, we obtain the following system of equations:

$$a_1 + a_2 + a_3 = 0; \tag{20}$$

$$b_0 + b_1 = 1; \tag{21}$$

[2]) Note however that the nucleon-exchange contribution is included.

$$a_1 = C_1 D(0); \tag{22}$$
$$a_2 = C_2 D(-2.13); \tag{23}$$
$$a_3 = C_3 D(-\omega_3); \tag{24}$$

where a_1, a_2, a_3, b_0, b_1, the position of the third pole (denoted by $-\omega_3$) and its residue C_3 are unknown parameters. To determine these seven constants we need two more equations. We can actually formulate them, because we know the position (ω_{33}) of the N^*_{33} resonance and its reduced half-width γ_{33} from experimental data.

Within the N/D method a resonance corresponds to zero of $\mathrm{Re}D(\omega_{res})$ and the reduced half-width is given by [4]

$$\gamma = -\frac{N(\omega_{res})}{\mathrm{Re}D'(\omega_{res})};$$

hence the following must hold:

$$\mathrm{Re}D(2.13) = 0; \tag{25}$$

$$\gamma_{33} = -\frac{N(2.13)}{\mathrm{Re}D'(2.13)}; \tag{26}$$

The correct values for γ_{33} and γ_{11} are according to [5]:

$$\gamma_{11} = 0.246 \pm 0.006; \qquad \gamma_{33} = 0.12 \pm 0.01;$$

hence the corresponding residues are

$$C_1 \doteq 0.109; \qquad C_2 \doteq 0.013.$$

The function $N(\omega)$ has in the three-pole approach the following form:

$$N(\omega) = \frac{a_1}{\omega} + \frac{a_2}{\omega + 2.13} + \frac{a_3}{\omega + \omega_3}; \tag{27}$$

and the function $D(\omega)$ is

$$D(\omega) = b_0 + \omega b_1 + \frac{\omega^2 - 1}{\pi}\left\{\frac{a_1}{\omega}\left[\sqrt{\omega^2 - 1}(\ln(\omega + \sqrt{\omega^2 - 1}) - i\pi) - \frac{\pi}{2}\right] + \right.$$
$$+ \frac{a_2}{\omega + 2.13}[\sqrt{\omega^2 - 1}(\ln(\omega + \sqrt{\omega^2 - 1}) - i\pi) + 1.1347] +$$
$$+ \frac{a_3}{\omega + \omega_3}\cdot[\sqrt{\omega^2 - 1}(\ln(\omega + \sqrt{\omega^2 - 1}) - i\pi) +$$
$$\left. + \sqrt{\omega^2 - 1}(\ln(-\omega_3 - \sqrt{\omega_3^2 - 1}) - i\pi)]\right\}; \tag{28}$$

thus

$$\mathrm{Re}D(\omega) = b_0 + \omega b_1 + \frac{\omega^2 - 1}{\pi}\left\{\frac{a_1}{\omega}\left(\Omega - \frac{\pi}{2}\right) + \right.$$
$$\left. + \frac{a_2}{\omega + 2.13}(\Omega + 1.1347) + \frac{a_3}{\omega + \omega_3}(\Omega + \Omega_3)\right\}, \tag{29}$$

where

$$\Omega = \sqrt{\omega^2 - 1}\ \ln(\omega + \sqrt{\omega^2 - 1}); \tag{30}$$

and the term Ω_3 means the value of Ω for $\omega = \omega_3$.

CALCULATIONS AND RESULTS

After the substitution of $N(\omega)$ and $D(\omega)$ into (22—25) we have to solve the following system of equations:

$$a_1 + a_2 + a_3 = 0; \tag{31}$$

$$b_0 + b_1 = 1; \tag{32}$$

$$a_1 = C_1\left\{b_0 - 0.4043a_2 - \frac{a_3}{\omega_3}\frac{1}{\pi}\left(\frac{\pi}{2} + \Omega_3\right)\right\}; \tag{33}$$

$$a_2 = C_2\left\{b_0 - 2.13b_1 + 1.43a_1 + \frac{1.1258}{\omega_3 - 2.13}a_3(-1.1347 + \Omega_3)\right\} \tag{34}$$

Table 1

ω_3	a_1	a_2	a_3	b_0	b_1	γ_{33}	C_3
6.0	0.18966	0.03960	−0.22926	1.65407	−0.65407	0.1077	−0.02522
9.0	0.18644	0.03779	−0.22423	1.61914	−0.61914	0.1100	−0.01561
12.0	0.18407	0.03642	−0.22049	1.59264	−0.59264	0.1132	−0.00922
15.0	0.18230	0.03537	−0,21767	1.57235	−0.57235	0.1146	−0.00626
18.0	0.18091	0.03452	−0.21543	1.55612	−0.55612	0.1151	−0.00448
21.0	0.17977	0.03381	−0.21358	1.54262	−0.54262	0.1153	−0.00334
30.0	0.17725	0.03220	−0.20945	1.51226	−0.51226	0.1131	−0.00168
50.0	0.17395	0.02999	−0.20394	1.47130	−0.47130	0.1125	−0.00057

$$a_3 = C_3 \left\{ b_0 - \omega_3 b_1 + \frac{\omega_3^2 - 1}{\pi} \left[\frac{a_1}{\omega_3} \left(\Omega_3 + \frac{\pi}{2} \right) + \frac{a_2}{2.13 - \omega_3} (-\Omega_3 + 1.1347) \right] \right\}; \quad (35)$$

$$b_0 + 2.13 b_1 - 0.2305 a_1 + 0.5998 a_2 + \frac{1.1258}{2.13 + \omega_3} a_3 (1.1347 + \Omega_3) = 0; \quad (36)$$

and eq. (26).

The solution of this system has been obtained numerically. The results of the numerical calculations are in Table 1, where a_1, a_2, a_3, b_0, b_1, C_3, γ_{33} are calculated for the following values of ω_3: 6, 9, 12, 15, 18, 21, 30, 50. We can see that for the values of ω_3 the values of γ_{33} are within the region 0.12 ± 0.01, which is the experimental value of γ_{33}. It is therefore not possible to fix in a unique way the position of the third pole, because our system of equations may be satisfied by any ω_3 from 9 up to the 50. At the same time residues C_3 of the third pole are changed from -0.0146 to -0.0006, when the third pole is shifted from the value $\omega_3 = 9$ to the value $\omega_3 = 50$.

Now we shall use the calculated functions $N(\omega)$ and $D(\omega)$ to determine the dependence of the phase δ_{33} on the kinetic energy of π mesons in the laboratory system. The third pole is successively placed at $\omega_3 = 9$, 21, 50, and results are compared with the experimental phase shift [6]. The functions $N(\omega)$ and $D(\omega)$ for these values are known because we know the constants a_1, a_2, a_3, b_0, b_1. In the N/D method the phase is given by

$$\cot \delta(\omega) = \frac{1}{q^2(\omega)} \cdot \frac{\mathrm{Re} D(\omega)}{N(\omega)} \quad (37)$$

Calculated values are given in Table 2, experimental ones in Table 3, where the index l denotes the laboratory system. We can see (Fig. 1) that phases calculated in this way are slightly above the experimental data, but they are

Table 2

ω_π^l *kin* [MeV]	δ_{33} $\omega_3 = 9.0$	δ_{33} $\omega_3 = 21.0$	δ_{33} $\omega_3 = 50.0$
30.0	3°0′	3°0′	3°0′
83.5	16°10′	16°50′	16°40′
134.0	47°40′	49°0′	48°40′
171.5	75°30′	74°0′	76°50′

Table 3

ω_π^l *kin* [MeV]	21.5	24.8	31.0	37.0	41.5	58.0	83.0	98.0	113.0
δ_{33} [grad]	1.38	1.84	2.53	3.60	4.14	7.54	4.91	21.17	28.03
ω_π^l *kin* [MeV]	120.0	150.0	142.0	151.0	165.0	170.0	176.0	189.0	194.0
δ_{33} [grad]	31.93	47.86	45.85	54.30	65.94	69.25	74.44	85.13	89.45

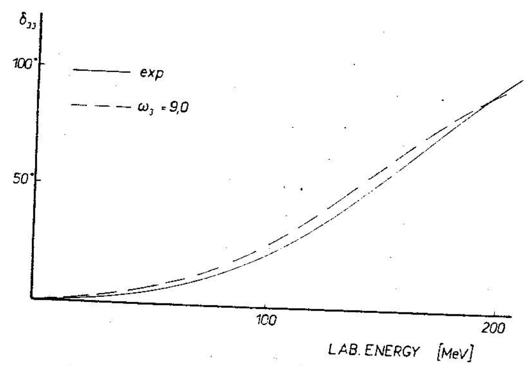

Fig. 1.

almost unchanged by a shift of the third pole. The value of γ_{33} is also insensitive to the third pole position ω_3.

SUMMARY OF RESULTS AND DISCUSSION

Dispersion relations for the $l = 1$ partial wave lead generally to the principal problems of convergence solved often by introducing a cut-off. In order to avoid these complications we used the method developed by Petráš [3], which adds further poles to the left-hand cut and the proper adjusting of residua assures the convergence of the integrals in question. The method was applied in the present paper to the calculation of the N_{33}^* resonance in the static model. The use of the static model naturally imposes some limits on the energy regions analysed, for instance in the case of the πN scattering the static model may be reasonable up to, say, 300 MeV for the kinetic energy of the incident meson.

The addition of the third pole to the left-hand cut in a partial wave is in principle much easier to accept than the introduction of the cut-off. The third pole may represent the contribution from the neglected resonance exchange, or the correction to approximations used in the static model. In particular, in calculating the resonance exchange contribution one frequently uses a zero width approximation. This might turn to be a crude approximation calling for a correction at least by the introduction of a further pole.

In the present paper we have introduced the third pole to the $I = J = \frac{3}{2}$, πN static model, and apart from the convergence we required the correct values of N^*_{33} mass and width. The result is not quite trivial since, as can be expected a priori (and is also shown in Table 1.), it is not clear beforehand if the correct mass and width of the N^*_{33} can be obtained within a three-pole formula of the type we use.

The third pole is evidently not connected with any resonance exchange. In fact all the elements of the crossing-matrix are positive in the $I = J = \frac{3}{2}$ channel. The residue of the third pole we obtained is negative and therefore it has nothing to do with a resonance exchange. Besides, the acceptable position of the third pole lies in the region where the static model is not reliable any more and therefore the most plausible interpretation of the third pole parameters is the following: The third pole takes care of the crude approximations of the static model which are necessary for both the avoiding of a cut-off and the giving of the N^* resonance with the correct value of the mass and width in the direct channel.

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