ANOTHER DERIVATION OF THE KINETIC EQUATION FOR STRONG ELECTRIC FIELDS

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In the present paper the kinetic equation for the electron distribution function in a strong electric field is derived in a different way than in the author's recent paper. Generalization to non-equilibrium phonons is suggested and an application given.

INTRODUCTION

Recently the author, assuming the equilibrium of phonons, has derived the kinetic equation for the electron distribution function in cases where a rectangular semiconductor sample with dominating electron-phonon scattering is exposed to a strong electric field, the intensity $E_x = E$ of which is limited only by the effective mass approximation [1]. In the mentioned paper the standard way of solving the density matrix equation [2] was modified so as to leave in the iteration process the series in powers of the electron-phonon interaction only. In order to reach the steady state, the interaction of the system with the environment, tending to preserve the steady state once obtained, was taken into account.

This paper aims to demonstrate that the same equation can be obtained for the ring shaped sample suggested in Appendix A of paper [3]. This treatment is chosen because of the fact that for the derivation of the mentioned equation

$$\sum_{\sigma_y,\sigma_z(\sigma_x=0;\vec{\sigma}\neq 0)} [\{\varepsilon(\vec{k}+\vec{\sigma}) - \varepsilon(\vec{k}) - \hbar\omega_\sigma^*\}^{-2} |V_\sigma^*|^2 \bar{N}_\sigma^* \{f(\vec{k}+\vec{\sigma}) - f(k)\} +$$

$$+ \{\varepsilon(\vec{k}-\vec{\sigma}) - \varepsilon(\vec{k}) + \hbar\omega_\sigma^*\}^{-2} |V_\sigma^*|^2 (\bar{N}_\sigma^* + 1)\{f(\vec{k}-\vec{\sigma}) - f(\vec{k})\}] +$$

$$+ \frac{\pi m^*}{2eE\hbar^2} \sum_{\vec{\sigma}(\sigma_x\neq 0;\vec{\sigma}\neq 0)} |\sigma_x|^{-1} |V_\sigma^*|^2 [\bar{N}_\sigma^* \{f(\vec{k}+\vec{\sigma}) - f(\vec{k})\} +$$

$$+ (\bar{N}_\sigma^* + 1)\{f(\vec{k}-\vec{\sigma}) - f(\vec{k})\}] = 0 \qquad (1)$$

with $\vec{k}$ and $\vec{\sigma}$ being electron and phonon wave vectors respectively, $V_{\vec{\sigma}}$ standing for the coupling parameter of the electron-phonon interaction, $\omega_{\vec{\sigma}}$ denoting the phonon frequency and $\bar{N}_{\vec{\sigma}}$ being the number of phonons of the wave vector $\vec{\sigma}$ in thermal equilibrium, the elements of the operator calculus derived in paper [4] are — i contrast to paper [1] — not used.

FORMULATION OF THE PROBLEM

In the treatment of paper [3] the cylindrical coordinates r, ϑ, z were used and the cylindrical ring with r_1 the inner and r_2 the outer radii was limited by the planes $z = 0$, $z = L_z$ and located in a homogeneous but time dependent magnetic field parallel to the z-axis and given by

$$\mathscr{H} = -\frac{2E^\circ c}{s\bar{r}} \exp st$$

with r denoting the average radius of the ring. The magnetic field is chosen so as to give rise to an electric field of the magnitude

$$E = \frac{E^\circ r}{\bar{r}} \exp st$$

and pointing in the direction of an increasing ϑ. For a thin enough ring $((r_2 - r_1)/r \ll 1)$ this field is almost homogeneous and for s sufficiently small it can be considered stationary.

In the effective mass approximation for the ring thin enough the electron Hamiltonian can be written in the form [3]:

$$H_0 = \frac{1}{2m^*}(P_\xi^2 + P_\eta^2 + P_\zeta^2)^2$$

in which

$$P_\xi = \frac{1}{i\bar{r}}\frac{\partial}{\partial\vartheta}, \quad P_\eta = \frac{1}{i}\frac{\partial}{\partial r}, \quad P_\zeta = \frac{1}{i}\frac{\partial}{\partial z}$$

are momentum components corresponding to the new variables

$$\xi = \bar{r}\vartheta, \quad \eta = r - r_1, \quad \zeta = z.$$

For the thin ring the term linear in E° of the Hamiltonian

$$H = \frac{1}{2m^*}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$$

with

$$A_r = A_z = 0, \quad A_\vartheta = -\frac{cE^\circ r}{s\bar{r}} \exp st$$

can be written in the form

$$H_F = \frac{eE^\circ}{m^* s} P_\xi \exp st.$$

The other terms, having their origin in

$$H_2 = \frac{e^2A^2}{2m^*c^2} + \frac{i\hbar e}{2m^*c} \operatorname{div} \vec{A}$$

need not be considered since $\operatorname{div} \vec{A} = -(2cE^\circ/sr) \exp st$ and for the ring thin enough also $A^2 \doteq [(cE^\circ/s)\exp st]^2$ exhibit the non-operator behaviour owing to which the exponential factors $\exp\left(+\frac{i}{\hbar}H_2 t\right)$ and $\exp\left(-\frac{i}{\hbar}H_2 t\right)$ are eliminated from the density matrix equation expressed in the interaction representation.

We shall use the representation in which the Hamiltonian of electron and phonons is diagonal. The characteristic functions of H_0 are defined by the relation

$$\Psi_k = \Omega^{-\frac{1}{2}} \exp[i(k_1\xi + k_2\eta + k_3\zeta)] \equiv \Omega^{-\frac{1}{2}} \exp(ik \,.\, r)$$

with k standing for (k_1, k_2, k_3) and r standing for (ξ, η, ζ). The functions Ψ_k satisfy the boundary conditions

$$\Psi(\xi + L_1, \eta, \zeta) = \Psi(\xi, \eta, \zeta)$$

$$\Psi(\xi, L_2, \zeta) = \Psi(\xi, 0, \zeta), \left(\frac{\partial\Psi}{\partial\eta}\right)_{\xi, L_2, \zeta} = \left(\frac{\partial\Psi}{\partial\eta}\right)_{\xi, 0, \zeta}$$

$$\Psi(\xi, \eta, L_3) = \Psi(\xi, \eta, 0), \left(\frac{\partial\Psi}{\partial\zeta}\right)_{\xi, \eta, L_3} = \left(\frac{\partial\Psi}{\partial\zeta}\right)_{\xi, \eta, 0}$$

in which $L_1 = 2\pi r$, $L_2 = r_2 - r_1$, $L_3 = L_z$ and $L_1L_2L_3 = \Omega$. Since in the new coordinates $\exp(ik \,.\, r) \equiv \exp[i(k_1\xi + k_2\eta + k_3\zeta)]$ corresponds to the expression $\exp(i\vec{k} \,.\, \vec{r})$, the electron-phonon interaction can be written in the form

$$H_i = \sum_q V_q a_q \exp(iq \,.\, r) + V_q^* a_q^* \exp(-iq \,.\, r)$$

with $q \,.\, r$ denoting $q_1\xi + q_2\eta + q_3\zeta$, V_q standing for the coupling parameter and a_q^+, a_q signifying the creation and annihilation operators of the phonon with the frequency ω_q. The phonon Hamiltonian can be written in the form $H_L = \sum(a_q^+ a_q + \frac{1}{2})\hbar\omega_q$ and its characteristic functions $|\ldots N_q \ldots>$ represent the numbers of phonons having the corresponding frequencies.

Though H_F is time dependent, the density matrix equation can be, as a consequence of the fact that H_F commutes with H_0, expressed in a representation similar to that used in paper [1]. If a new density operator ϱ'' is defined by the relation

$$\varrho = QR\varrho''R^{-1}Q^{-1}$$

with

$$Q \equiv \exp\left\{-\frac{i}{\hbar}\left[H_0 t + \int^t H_F(t')\mathrm{d}t'\right]\right\} = \exp\left\{-\frac{i}{\hbar}\left[H_0 t + \frac{eE^\circ P_\xi}{m^* s^2}\exp st\right]\right\}$$

$$R = \exp\left\{-\frac{i}{\hbar}H_L t\right\}$$

then the density matrix equation takes the form

$$\frac{\partial\varrho''}{\partial t} = \frac{i}{\hbar}[\varrho'', H_i''] \tag{2}$$

in which H_i'' is defined in the same way as ϱ''. If ϱ'' is expressed in powers of H_i'' and only the terms up to the second order are retained (2) gives the equations

$$\frac{\partial\varrho_1''(t_0, t)}{\partial t} = \frac{i}{\hbar}[\varrho''(t_0), H_i''(t)] \tag{3}$$

$$\frac{\partial\varrho_2''(t_0, t)}{\partial t} = \frac{i}{\hbar}[\varrho_1''(t_0, t), H_i''(t)]$$

in which ϱ_1'' and ϱ_2'' stand for the first and second order terms of the expansion

$$\varrho''(t) = \varrho''(t_0) + \varrho_1''(t_0, t) + \varrho_2''(t_0, t) + \ldots$$

Since for a convenient s we assume the realization of the steady state we can choose $t_0 = 0$ and the solutions of the equations (3) can be written in the form

$$\varrho_1''(t) = \frac{i}{\hbar}\int_0^t [\varrho(\mathrm{o}), H_i''(t')]\,\mathrm{d}t'$$

$$\varrho_2''(t) = \left(\frac{i}{\hbar}\right)^2\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''[[\varrho(\mathrm{o}), H_i''(t'')], H_i''(t')].$$

In accordance with paper [1] we assume $H_i = H_{int}\exp(-t/\tau_0)$, with H_{int} denoting the time independent electron-phonon interaction and τ_0 standing for the damping parameter characterizing the process affesting electron-phonon collisions after the time corresponding to the expected steady state. This damping is equivalent to the presence of the additional term $[\varrho''(t) - \varrho''(0)]/\tau_0$ in equation (2). Since we are interested only in the diagonal matrix elements of the density operator, we can say this additional term is similar to the term suggested by Lax [5] representing the interaction of the system with the environment. The only difference between these two terms consists in the fact that in our case, owing to the fact that $\varrho(0)$ is the steady state density matrix, the interaction tends to preserve the steady state once obtained, while in Lax's case the interaction tends to return the system into a thermal equilibrium.

Since the electron distribution function $f(k)$ represents the diagonal elements of the matrix $\mathbf{f}$ defined by the relation

$$<k|\mathbf{f}|k> = \sum_N <kN|\varrho|kN>$$

and satisfying in the case of equilibrium phonons the equation

$$<kN|\varrho|k'N'> = <k|\mathbf{f}|k'>P(N)\delta_{NN},$$

with

$$P(N) = \exp\left\{-\frac{E_N}{k_0T}\right\}\left[\sum_N \exp\left\{-\frac{E_N}{k_0T}\right\}\right]^{-1},$$

we shall be interested in the diagonal elements of the matrix ϱ. Owing to the fact that the diagonal in $|N>$ matrix elements of $H_i''(t)$ and H_i equal to zero, the diagonal in $|N>$ matrix elements of $\varrho_1''(t)$ equal to zero too and the equation for the steady state can be obtained from the relation

$$\sum_N <kN|\varrho_2''(t)|kN> \equiv <k|f_2''(t)|k> = 0.$$

If in the expression for Q the function $\exp st$ is expanded in powers of st and s is assumed to satisfy the relation

$$s \ll \frac{1}{\tau}$$

in which τ is the relaxation time of the considered scattering process, the retention of the first three powers is sufficient and Q takes the form

$$Q = \exp\left\{-\frac{i}{\hbar}\left[[H_0 t + \frac{eE^\circ P_\xi}{m^*}\left(\frac{1}{s^2} + \frac{t}{s} + \frac{t^2}{2}\right)\right]\right\}.$$

The factor exp $\{-(iEP_\xi/2m^*\hbar)t^2\}$ is, except for the replacement $p_x \to P_\xi$, identical with the corresponding factor of paper [1]. Owing to its time independence the factor exp $\{-ieE^\circ P_\xi/\hbar m^* s^2\}$ will not play any role in expressing the diagonal matrix elements of ϱ_2'', since it will be compensated by the complex conjugated term. The factor exp $\{-(ieE^\circ P_\xi/m^*\hbar s)t\}$, contrary to paper [1], results in the appearance of the energy shift. To the electron energy $\varepsilon(k)$ the value $eE^\circ k_\xi/m^* s$ must be added now. But it must be mentioned that for the parameter s only the upper limit was given. If a convenient lower limit is chosen for s (e. g. in accordance with paper [3]: $s \gg \Delta\varepsilon/\hbar$ where $\Delta\varepsilon$ is of the order of the spacing of the translational electronic levels), the energy shift is nonessential and the same kinetic equation for the electron distribution function as in paper [1] can be obtained. The mentioned limit for s is responsible for P_ξ independent part of the velocity operator v- being negligible and P_ξ/m^* can be taken for v_ξ.

DISCUSSION

The solution of the derived kinetic equation represents a rather complicated problem. Nevertheless it can be used with advantage when electron-electron interaction is strong enough to cause the distribution of electrons in the form of a Maxwell distribution with the electron temperature and the momentum shifted in the direction of the applied electric field. The electron temperature and the momentum shift are determined in such a case from the conservation laws for energy and wave vector respectively. In such an approach the standard Boltzmann equation is applied even in a case of a strong electric field [6]. Since the Boltzmann equation is well substantiated only in those cases in which the distribution function can be approximated by a finite number of terms of the series in the powers of the applied field [7], the derived kinetic equation is for the shifted Maxwell distribution more convenient than the Boltzmann one.

Though the mentioned equation was derived with the assumption of equilibrium phonons one can generalize this approach even for a case of non-equilibrium phonons. When doing so we must use the general property of diagonal singularity pointed out by van Hove [8], securing negligibility of the terms containing nondiagonal elements of the density matrix in comparison with the terms containing its diagonal elements. Van Hove showed that from the matrix elements

$$<\alpha|V A_1 V \ldots A_n V|\alpha'>$$

(in the matrix element at least two factors V, representing the interaction between the systems, occur and A_i are operators diagonal in the considered

representation -- in our case their role is performed by the exponential factors having their origin in the transfer to interaction representation) those with $\alpha = \alpha'$ are larger than those with $\alpha \neq \alpha'$ and their rate increases with the increase of the size of the system.

With this in mind one can derive from the relation $\sum_n <\vec{k}N|\varrho_2|\vec{k}N> = 0$ the kinetic equation which differs from (1) in the fact that the phonon distribution is not the equilibrium one. Since the effect of a strong electric field on phonons (via electron-phonon interaction) is weaker than its effect on electrons we can expect the phonon distribution to satisfy the standard kinetic equation even in case of a strong electric field. We have thus linked equations for electron and phonon distributions leading to the generalization of Nakamura's paper [9] for the case of hot electrons. In paper [9] the electron system is assumed to be in the uniform motion with the drift velocity and the electron distribution function is given by the equilibrium one with the momentum shifted in the direction of the applied field. To this purpose the derived kinetic equation can be used even in the case of a strong electric field.

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