Fyzikálny časopis 18 (1968), No. 2

A NOTE ON THE CONSTRUCTION OF GENERATORS OF THE SO(4,1) GROUP FOR THE HYDROGEN ATOM

JOZEF LÁNIK, Bratislava

INTRODUCTION

The symmetry of the SU(6) group used for a classification of strong-interacting particles is only approximate. Symmetry-breaking terms are introduced in a phenomenological way and are understood as perturbations in the mass-operator [1—4]. In order to explain the mass-splitting within the framework of the group theory, non-compact extensions of the original symmetry groups were suggested [5].

Since this was an attempt at a new method it had first to be tested on exactly solvable quantum-mechanical systems. One of those is the hydrogen atom (H-atom).

Fock [6] showed a long time ago that the H-atom has the SO(4) symmetry group of the Hamiltonian and that eigenstates corresponding to a given energy form a basis for unitary irreducible representations of this group. Recently the simple non-compact extension of the group SO(4), namely SO(4,1), has been suggested and studied by Barut et al. [7]. It was shown that the SO(4,1) group has an infinitely-dimensional unitary irreducible representation which contains all the bound-state levels of the H-atom. It has been also shown that the energy spectrum problem can be solved within the framework of this SO(4,1) group [8].

The purpose of this paper is to present a method for constructing generators of the SO(4,1) group for the H-atom in terms of Fock's variables $x_1, \ldots, x_4$ which is different from the methods of other authors [8—11]. Our method is based on the knowledge of the commutation relations of the generators of the SO(4,1) group and on Fock's results [6], which show that eigenfunctions of H-atom bound states in the variables x_μ ($\mu = 1, \ldots, 4$)

$$\mathbf{x} = \frac{2p_0}{p_0^2+p^2}\,\mathbf{p}, \quad x_4 = \frac{p_0^2 - p^2}{p_0^2 + p^2},$$

(where $\mathbf{p}$ is the momentum, $p_0 = \sqrt{-2mE}$ and $p^2 = p_1^2 + p_2^2 + p_3^2$, $E < 0$ the energy of bound states), are harmonic homogeneous polynomials of the variables x_μ of the $(n-1)$-th order (n being the principal quantum number).

THE GENERATORS OF SO (4,1) GROUP FOR THE H-ATOM

To construct the generators of the de Sitter SO(4,1) group for the H-atom in terms of Fock's variables x_μ ($\mu = 1, \ldots, 4$) we shall start from the known commutation relations of these generators M_{ij} [12]:

$$[M_{ij},\ M_{lk}] = -\mathrm{i}(g_{il}M_{jk} - g_{jl}M_{ik} + g_{ik}M_{lj} - g_{jk}M_{li}), \tag{1}$$

where $M_{ij} = -M_{ji}$, the Latin subscripts i, j, k, l run over the set $1, 2, \ldots, 5$ and $g_{11} = g_{22} = g_{33} = g_{44} = -g_{55} = -1$, $g_{ij} = 0$ for $i \neq j$. For our purpose it is useful to rewrite (1) as follows

$$\begin{aligned} &[M_{\mu\nu}, M_{\varrho\delta}] = \mathrm{i}(\delta_{\mu\varrho}M_{\nu\delta} - \delta_{\nu\varrho}M_{\mu\delta} + \delta_{\mu\delta}M_{\varrho\nu} - \delta_{\nu\delta}M_{\varrho\mu}),\\ &[M_{\mu\nu}, M_{\lambda 5}] = \mathrm{i}(\delta_{\mu\lambda}M_{\nu 5} - \delta_{\nu\lambda}M_{\mu 5}),\\ &[M_{\mu 5}, M_{\lambda 5}] = -\mathrm{i}\, M_{\mu\lambda}, \end{aligned} \tag{2}$$

where the Greek subscripts $\mu, \nu, \varrho, \delta, \lambda$ run over the values $1, \ldots, 4$ and $\delta_{\mu\nu} = 0$, except for $\delta_{11} = \delta_{22} = \delta_{33} = \delta_{44} = 1$.

From these relations it follows that the operators $M_{\mu\nu}$ are generators of the maximal compact subgroup SO(4) of the de Sitter group. Therefore they can be chosen in terms of Fock's variables in the following form

$$M_{\mu\nu} = -\mathrm{i}\left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}\right). \tag{3}$$

Now the *noncompact* generators $M_{\mu 5}$ have also to be constructed in terms of Fock's variables x_μ. To do this we shall suppose that $M_{\mu 5}$ are differential operators of the first order in terms of the variables x_μ (since the operators $M_{\mu\nu}$ are differential ones of the first order), i. e. let them be supposed in the following most general form

$$M_{\mu 5} = \mathrm{i}\left[a(x)\frac{\partial}{\partial x_\mu} + b(x)x_\mu x_\nu \frac{\partial}{\partial x_\nu} + c(x)x_\mu\right], \tag{4}$$

where $a(x)$, $b(x)$, $c(x)$ are unknown functions of the variable $x = \sqrt{x_1^2 + \ldots + x_4^2}$, summation over ν from 1 to 4 being understood.

Substituting (4) into the commutation relations of (2) we get after some calculations the condition for the functions $a(x)$, $b(x)$ and $c(x)$ (see Appendix I)

$$\frac{a(x)}{x}\frac{\mathrm{d}a(x)}{\mathrm{d}x} - a(x)b(x) + x\, b(x)\frac{\mathrm{d}a(x)}{\mathrm{d}x} = 1. \tag{5}$$

Hence we can say that the operators (3) and (4) obey the commutation relations (2) only if (5) is fulfilled.

In this way we have obtained general forms of the generators of SO(4,1) group in terms of Fock's variables.

We shall obtain the special froms of these generators on the basis of the requirement that the irreducible (infinitely dimensional) representations of this group be based on eigenfunctions of H-atom bound states [7, 8], i. e. on harmonic homogeneous polynomials (HHP) in four Fock's variables x_μ according to the results of [6]. In other words, we shall require such forms of the generators $M_{\mu 5}$ that the space of infinite irreducible representation of the SO(4,1) group be the direct sum of SO(4) representation subspaces. It is well-known (see e. g. [13]) that the HHP $f_n(x_1, \ldots, x_4)$ of the n-th order is defined by the relations

$$\Delta f_n(x_1, \ldots, x_4) = 0, \quad x_\mu \frac{\partial f_n(x_1, \ldots, x_4)}{\partial x_\mu} = n f_n(x_1, \ldots, x_4), \tag{6}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_4^2}.$$

Thus, if $f_n \equiv f_n(x_1, \ldots, x_4)$ is HHP of the n-th order in four Fock's variables, then it follows from (4)

$$M_{\mu 5} f_n = \mathrm{i}A\frac{\partial f_n}{\partial x_\mu} + \mathrm{i}\left[(c(x) + nb(x))x_\mu f_n - a_1(x)\frac{\partial f_n}{\partial x_\mu}\right], \tag{7}$$

where we have put

$$a(x) = A - a_1(x), \tag{8}$$

with A constant.

Now, in order to fulfil the requirement stated above, it is sufficient to fulfil the following conditions for the form of $M_{\mu 5}$, or, more precisely for the form of functions $a_1(x)$, $b(x)$, $c(x)$: Let f_n be the HHP of the n-th order in four Fock's variables x_μ, then the functions

$$[c(x) + n\, b(x)]x_\mu f_n - a_1(x)\frac{\partial f_n}{\partial x_\mu}$$

are required to be HHP's of the $(n+1)$-th orders and this must hold for every $n = 0, 1, 2, \ldots$. It means (according to (6)) that the following equations

$$\Delta\left\{[c(x) + n\, b(x)]x_\mu f_n - a_1(x)\frac{\partial f_n}{\partial x_\mu}\right\} = 0 \tag{9}$$

and

$$x_\nu \frac{\partial}{\partial x_\nu}\left\{[c(x) + n\, b(x)]x_\mu f_n - a_1(x)\frac{\partial f_n}{\partial x_\mu}\right\} =$$

$$= (n+1)\left\{[c(x) + n\, b(x)]x_\mu f_n - a_1(x)\frac{\partial f_n}{\partial x_\mu}\right\} \tag{10}$$

have to be fulfilled for the functions $a_1(x)$, $b(x)$ and $c(x)$ and for every $n = 0, 1, 2, \ldots$ It is evident from (7) that if (9) and (10) are fulfilled (and since $\frac{\partial f_n}{\partial x_\mu}$ are HHP's of the $(n-1)$-th orders for every n), the space of representation of the SO(4,1) group based on the HHP's $f_n(x_1, \ldots, x_4)$, $(n = 0, 1, \ldots)$ is an irreducible infinitely dimensional one, i. e. the requirement above is fulfilled. This also means that the shifting among eigenstates corresponding to various eigenvalues E_N of the H-atom (N is the principal quantum number) is made possible by the generators $M_{\mu 5}$.

Now we turn to Eqs. (9) and (10). After some calculations they get the forms

$$\left[-\frac{d^2 a_1(x)}{dx^2} - \frac{2n+1}{x}\frac{da_1(x)}{dx} + 2(c(x) + nb(x))\right]\frac{\partial f_n}{\partial x_\mu} +$$
$$+\left[\frac{d^2}{dx^2}(c(x) + nb(x)) + \frac{2n+5}{x}\frac{d}{dx}(c(x) + nb(x))\right]x_\mu f_n = 0, \tag{11}$$
$$x\left[\frac{d}{dx}(c(x) + nb(x))\right]x_\mu f_n - \left[x\frac{da_1(x)}{dx} - 2a_1(x)\right]\frac{\partial f_n}{\partial x_\mu} = 0,$$

where $n = 0, 1, 2, \ldots$ Thus these equations represent infinitely many conditions for the functions $a_1(x)$, $b(x)$ and $c(x)$, and can be rewritten in the following algebraic equations (see Appendix II)

$$a_1(x) = Bx^2, \tag{12}$$

$$c(x) + nb(x) = 2(n+1)B,$$

where B is a constant and $n = 0, 1, 2, \ldots$ These infinitely many equations are

evidently linearly dependent. Their solutions exist and are of the forms

$$a_1(x) = Bx^2, \tag{13}$$

$$c(x) = b(x) = 2B.$$

These functions have also to obey Eq. (5) and with respect to (8) this is possible only if the following equality is fulfilled

$$A = -\frac{1}{4B}. \tag{14}$$

Hence, with respect to (3), (4), (8), (13) and (14) the generators of the de Sitter group in terms of Fock's variables x_μ (putting $2B = 1$)*) are

$$M_{\mu\nu} = -i\left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}\right), \tag{15}$$

$$M_{\mu 5} = -i\left[\frac{1+x^2}{2}\frac{\partial}{\partial x_\mu} - x_\mu\left(x_\nu \frac{\partial}{\partial x_\nu} + 1\right)\right].$$

These forms of generators are similar to those in [11].

With respect to Newton [12], the Casimir operators of this group are

$$Q = M_{\mu 5}M_{\mu 5} - \frac{1}{2}M_{\mu\nu}M_{\mu\nu} \tag{16}$$

and

$$W = -w_i w^i,$$

where $w^i = \frac{1}{8}\varepsilon^{ijklm} M_{jk}M_{lm}$, in which $\varepsilon^{12345} = 1$ and ε^{ijklm} is totally antisymmetric. It is easy to calculate these invariants in the space of harmonic homogeneous polynomials in Fock's variables on the unit sphere $(x^2 = 1)$ in the same way as in [14]

$$Q = 2, \tag{17}$$

$$W = 0.$$

*) It is clear that the Casimir operators of the group (see (16)) generally depend on the constant B. But their values in the space of the HHP's in Fock's variables on the unit sphere $(x^2 = 1)$ can easily be proved not to depend on the constant B. Therefore B is arbitrary.

The representation of SO(4,1) characterized by the values (17) of two invariants Q and W belongs to the class I of infinite unitary representations with respect to Newton's classification [12]. The space of the representation is the direct sum of SO(4) representation subspaces, each subspace being contained at most once. According to Fock's results [6] this means that such a representation of the SO(4,1) group contains all the bound-state levels of the H-atom.

CONCLUSION

In the paper the generators of the SO(4,1) group have been constructed in terms of Fock's variables $x_1, \ldots, x_4$ for the H-atom in a different way from the one mentioned in [8—11]. The method was based on the knowledge of the commutation relations of generators of the SO(4,1) group. According to this we have assumed a general form for the generators. The result (15) has been obtained under the conditions that the generators must obey the known commutation relations (1) and act on the harmonic homogeneous polynomials in Fock's variables $x_1, \ldots, x_4$, $(x^2 = 1)$. It has also been shown that the unitary irreducible representation (infinite) of an algebra of the generators (15) is characterized by the values (17) of two invariants, i. e., it belongs to Newton's class I.

In this connection it could be mentioned that generators of the de Sitter group for the H-atom were also calculated in terms of the variables $\boldsymbol{p}$ and $\boldsymbol{r}$ in classical [15] as well as in quantum-mechanical [16, 17] cases. The solution of the energy spectrum problem within the framework of this group was arrived at in [8].**)

APPENDIX I

The equation (5) will be deduced form (2), (3) and (4). Combining those the following relations are obtained after some simple but lengthy calculations

$$\left[x_\mu \frac{\partial a(x)}{\partial x_\nu} - x_\nu \frac{\partial a(x)}{\partial x_\mu}\right] \frac{\partial}{\partial x_\lambda} + \left[x_\mu \frac{\partial c(x)}{\partial x_\nu} - x_\nu \frac{\partial c(x)}{\partial x_\mu}\right] x_\lambda + \left[x_\mu \frac{\partial b(x)}{\partial x_\nu} - x_\nu \frac{\partial b(x)}{\partial x_\mu}\right] x_\lambda x_\varrho \frac{\partial}{\partial x_\varrho} = 0, \qquad \text{(I, 1)}$$

$$a(x) \frac{\partial a(x)}{\partial x_\mu} - x_\mu\, a(x)\, b(x) + x_\mu b(x)\, x_\nu \frac{\partial a(x)}{\partial x_\nu} = x_\mu, \qquad \text{(I, 2)}$$

$$x_\lambda \frac{\partial c(x)}{\partial x_\mu} - x_\mu \frac{\partial c(x)}{\partial x_\lambda} + \left[x_\lambda \frac{\partial b(x)}{\partial_\mu} - x_\mu \frac{\partial b(x)}{\partial x_\lambda}\right] x_\varrho \frac{\partial}{\partial x_\varrho} = 0, \qquad \text{(I, 3)}$$

where the Greek subscripts run over the values 1, ..., 4.

Since the functions $a(x)$, $b(x)$ and $c(x)$ depend only on the variable $x = \sqrt{x_1^2 + \ldots + x_4^2}$, i. e.

$$\frac{\partial a(x)}{\partial x_\mu} = \frac{x_\mu}{x} \frac{\mathrm{d}a(x)}{\mathrm{d}x},$$

$$\frac{\partial b(x)}{\partial x_\mu} = \frac{x_\mu}{x} \frac{\mathrm{d}b(x)}{\mathrm{d}x},$$

$$\frac{\partial c(x)}{\partial x_\mu} = \frac{x_\mu}{x} \frac{\mathrm{d}c(x)}{\mathrm{d}x},$$

the relations (I, 1) and (I, 3) are automatically fulfilled and from (I, 2) one gets

$$\frac{a(x)}{x} \frac{\mathrm{d}a(x)}{\mathrm{d}x} - a(x)\, b(x) + x\, b(x) \frac{\mathrm{d}a(x)}{\mathrm{d}x} = 1.$$

APPENDIX II

The following equations are obtained from (11), multiplying it by $x_\mu \neq 0$ and after summation over μ from 1 to 4 (since $f_n(x_1, \ldots, x_4) \neq 0$ for $n = 0, 1, 2, \ldots$)

$$n\left[-a_1''(x) - \frac{2n+1}{x} a_1'(x) + 2(c(x) + nb(x))\right] + x^2\left[c''(x) + nb''(x) + \frac{2n+5}{x}(c'(x) + nb'(x))\right] = 0, \qquad \text{(II, 1)}$$

$$x^3(c'(x) + nb'(x)) - n(xa_1'(x) - 2a_1(x)) = 0. \qquad \text{(II, 2)}$$

Combining these equations the relation

$$c(x) + nb(x) = \frac{2(n+1)}{x^2} a_1(x) \qquad \text{(II, 3)}$$

**) I am indebted very much to Dr. M. Petráš for reading the manuscript and for his valuable remarks and criticism. I also express my thanks to J. Černuch for valuable suggestions.

can be derived after some calculations. Then, from (II, 2) and (II, 3) we get

$$x a_1'(x) = 2 a_1(x)$$

which equation has a solution

$$a_1(x) = B\, x^2, \tag{II, 4}$$

where B is a constant of integration, and, from (II, 3) we get

$$c(x) + n\, b(x) = 2(n + 1)B, \qquad n = 0, 1, 2, \ldots$$

It can easily be seen that these functions obey Eqs. (11).

REFERENCES

[1] Gürsey F., Radicati L. A., Phys. Rev. Lett. *13* (1964), 173.
[2] Pais A., Phys. Rev. Lett. *13* (1964), 175.
[3] Kuo T. K., Yao T., Phys. Rev. Lett. *13* (1964), 415.
[4] Bég M. A., Singh V., Phys. Rev. Lett. *13* (1964), 418.
[5] Barut A. O., Phys. Rev. *135* (1964), B 839.
[6] Fock V., Zeitschrift f. Physik *98* (1935), 145.
[7] Barut A. O., Budini P., Fronsdal C., Proc. Roy. Soc. *A 291* (1966), 106.
[8] Budini P., Nuovo Cimento *44 A* (1966), 363.
[9] Han M. Y., Nuovo Cimento *42 B* (1966), 367.
[10] Castilho Alcarás J. A., Leal Ferreira P., Nuovo Cimento *46 B* (1966), 273.
[11] Малкин И. А., Манько В. И., ЖЕТФ — Письма в редакцию *2* (1965), 230.
[12] Newton T. D., Annals of Mathematics *51* (1950), 730.
[13] Виленкин Н. Я., *Специальные функции и теория представлений групп*, Изд. Наука, Москва 1965, 437.
[14] Böhm A., Trieste Preprint IC/65/82.
[15] Bacry H., Nuovo Cimento *41* (1966), 222.
[16] Musto R., Phys. Rev. *148* (1966), 1274.
[17] Pratt R. H., Jordan T. F., Phys. Rev. *148* (1966), 1276.

Received August 12th, 1967

Fyzikálny ústav SAV,
Bratislava