TRANSIENT ELECTROMAGNETIC FIELD DUE TO A SINGLE TRAPEZIUM CURRENT PULSE

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Suppose a vertical electric dipole is imbedded in a homogenous, isotropic and infinite space, characterized by an electric conductivity σ and magnetic permeability μ. Suppose furthermore this dipole produces in the above mentioned space a current pulse the shape of which is a non-isosceles trapezoid, (Fig. 1).

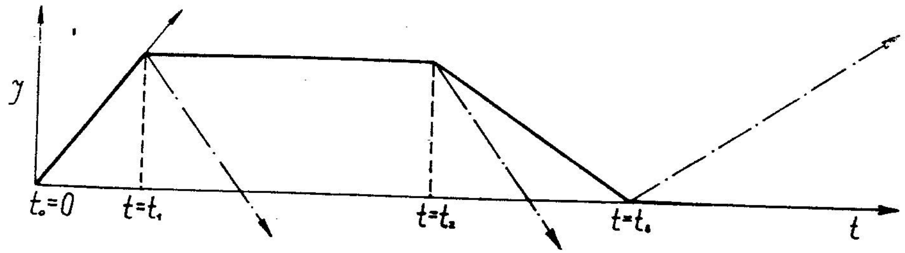

Fig. 1.

The current trapezoid pulse can be produced as follows:

1. at the moment $t_0 = 0$ we switch on into the circuit a current pulse the time character of which is defined by the function t/t_1, where t_1 is the building-up period during which the current reaches its maximum value.

2. after a given time-lag interval $t = t_1$ we switch on the second current pulse defined by the function $-t/t_1$ the character of which is decreasing one; consequently the dipole intensity during the following time interval will be constant.

3. at the next moment $t = t_2$ we switch on the third current pulse defined by the function $-t/t_1$ in consequence of which the dipole intensity will be — during the time interval $t = t_3$ — zero.

4. at the moment $t = t_3$ we switch on a current pulse defined by the function t/t_3 in consequence of which the current intensity will be in $t > t_3$ all the time zero.

We are to determine the time—space distribution of the electromagnetic field for the above mentioned current pulse, i. e. to derive the theoretical formulae for its non-zero components as the functions of the time and space coordinates.

We shall solve this problem in spherical coordinates with the centre of the electric dipole located in the origin of the coordinates, (Fig. 2). First of all using the Laplace transformation we must determine the image of the current pulse.

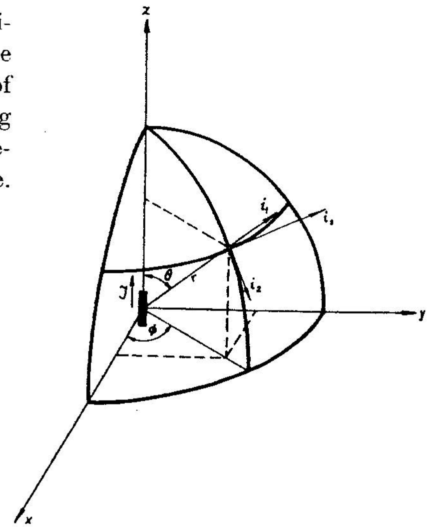

Fig. 2.

First we get:

$$\frac{t}{t_1} \risingdotseq \frac{1}{t_1}\frac{1}{p^2} \tag{1}$$

for the first (increasing) current pulse. The second (decreasing) pulse is switched on with a time lag t_1. thus for its image we have:

$$-\frac{t}{t_1} \risingdotseq \frac{1}{t_1}\frac{e^{-pt_1}}{p^2}. \tag{2}$$

The third (decreasing) pulse is switched on with a time lag t_2. thus we get for its image:

$$-\frac{t}{t_1} \risingdotseq -\frac{1}{t_1}\frac{e^{-pt_2}}{p^2}. \tag{3}$$

The last (increasing) pulse has a time lag t_3 and its image is:

$$\frac{t}{t_1} \risingdotseq \frac{1}{t_3}\frac{e^{-pt_3}}{p^2}. \tag{4}$$

By adding equations (1), (2), (3), (4) we obtain for the image of the pulse, the shape of which is a non isosceles trapezoid, the formula:

$$i(p) = \frac{1}{t_1}\left(\frac{1}{p^2} - \frac{e^{-pt_1}}{p^2} - \frac{e^{-pt_2}}{p^2}\right) + \frac{1}{t_3}\frac{e^{-pt_3}}{p^2} \tag{5}$$

The formulae for the images of the non-zero components of the electromagnetic field have however been derived in the papers [1], [2] in the following forms:

$$e_r(p) = \frac{i(p)\mathrm{d}l}{2\pi\sigma r^3}(1 + \alpha p^{\frac{1}{2}}) \cos\Theta \exp(-\alpha p^{\frac{1}{2}}), \tag{6}$$

$$e_\Theta(p) = \frac{i(p)\mathrm{d}l}{4\pi\sigma r^3}(1 + \alpha p^{\frac{1}{2}} + \alpha^2 p) \sin\Theta \exp(-\alpha p^{\frac{1}{2}}) \tag{7}$$

$$h_\Phi(p) = \frac{i(p)\mathrm{d}l}{4\pi r^2}(1 + \alpha p^{\frac{1}{2}}) \sin\Theta \exp(-\alpha p^{\frac{1}{2}}) \tag{8}$$

where: $\alpha^2 = \sigma\mu r^2$, $\mathrm{d}l$ = length of the elementary dipole, $i(p)$ is the image of the current pulse defined in our case by the equation (5).

Let us denote for short:

$$\frac{\mathrm{d}l}{2\pi\sigma r^3}\cos\Theta = A_r \tag{9}$$

$$\frac{\mathrm{d}l}{4\pi\sigma r^3}\sin\Theta = A_\Theta \tag{10}$$

$$\frac{\mathrm{d}l}{4\pi r^2}\sin\Theta = A_\Phi \tag{11}$$

and transcribe equations (6), (7), (8) with regard to equation (5) and we obtain:

$$e_r(p) = A_r\left\{\left[\frac{1}{t_1}\left(\frac{1}{p^2} - \frac{e^{-pt_1}}{p^2} - \frac{e^{-pt_2}}{p^2}\right) + \frac{1}{t_3}\frac{e^{-pt_3}}{p^2}\right][1 + \alpha p^{\frac{1}{2}}]e^{-\alpha p^{1/2}}\right\} =$$

$$= A_r\left\{\frac{1}{t_1}\left[\frac{e^{-\alpha p^{1/2}}}{p^2} - \frac{e^{-pt_1-\alpha p^{1/2}}}{p^2} - \frac{e^{-pt^2-\alpha p^{1/2}}}{p^2} + \frac{\alpha e^{-\alpha p^{1/2}}}{p^{\frac{3}{2}}} - \frac{\alpha e^{-pt_1-\alpha p^{1/2}}}{p^{\frac{3}{2}}} - \right.\right.$$

$$\left.\left. - \frac{\alpha e^{-pt_2-\alpha p^{1/2}}}{p^{\frac{3}{2}}}\right] + \frac{1}{t_3}\left[\frac{e^{-pt_3-\alpha p^{1/2}}}{p^2} + \frac{\alpha e^{-pt_3-\alpha p^{1/2}}}{p^{\frac{3}{2}}}\right]\right\}, \tag{6a}$$

$$e_\Theta(p) = A_\Theta\left\{\left[\frac{1}{t_1}\left(\frac{1}{p^2} - \frac{e^{-pt_1}}{p^2} - \frac{e^{-pt_2}}{p^2}\right) + \frac{1}{t_3}\frac{e^{-pt_3}}{p^2}\right][1 + \alpha p^{\frac{1}{2}} + \alpha^2 p]e^{-\alpha p^{1/2}}\right\} =$$

$$= A_\Theta\left\{\frac{1}{t_1}\left[\frac{e^{-\alpha p^{1/2}}}{p^2} - \frac{e^{-pt_1-\alpha p^{1/2}}}{p^2} - \frac{e^{-pt_2-\alpha p^{1/2}}}{p^2} + \frac{\alpha e^{-\alpha p^{1/2}}}{p^{\frac{3}{2}}} - \frac{\alpha e^{-pt_1-\alpha p^{1/2}}}{p^{\frac{3}{2}}} - \right.\right.$$

$$\left.\left. - \frac{\alpha e^{-pt_2-\alpha p^{1/2}}}{p^{\frac{3}{2}}} + \frac{\alpha^2 e^{-\alpha p^{1/2}}}{p} - \frac{\alpha^2 e^{-pt_1-\alpha p^{1/2}}}{p} - \frac{\alpha^2 e^{-pt_2-\alpha p^{1/2}}}{p}\right]\right\} +$$

$$+ A_\Theta\left\{\frac{1}{t_3}\left[\frac{e^{-pt_3-\alpha p^{1/2}}}{p^2} + \frac{\alpha e^{-pt_3-\alpha p^{1/2}}}{p^{\frac{3}{2}}} + \frac{\alpha^2 e^{-pt_3-\alpha p^{1/2}}}{p}\right]\right\}. \tag{7a}$$

$$h_\Phi(p) = A_\Phi\left\{\left[\frac{1}{t_1}\left(\frac{1}{p^2} - \frac{e^{-pt_1}}{p^2} - \frac{e^{-pt_2}}{p^2}\right) + \frac{1}{t_3}\frac{e^{-pt_3}}{p^2}\right][1 + \alpha p^{\frac{1}{2}}]e^{-\alpha p^{1/2}}\right\} =$$

$$= A_\Phi\left\{\frac{1}{t_1}\left[\frac{e^{-\alpha p^{1/2}}}{p^2} - \frac{e^{-pt_1-\alpha p^{1/2}}}{p^2} - \frac{e^{-pt_2-\alpha p^{1/2}}}{p^2} + \frac{\alpha e^{-\alpha p^{1/2}}}{p^{\frac{3}{2}}} - \frac{\alpha e^{-pt_1-\alpha p^{1/2}}}{p^{\frac{3}{2}}} - \right.\right.$$

$$\left.\left. - \frac{\alpha e^{-pt_2-\alpha p^{1/2}}}{p^{\frac{3}{2}}} + \frac{1}{t_3}\left[\frac{e^{-pt_3-\alpha p^{1/2}}}{p^2} + \frac{\alpha e^{-pt_3-\alpha p^{1/2}}}{p^{\frac{3}{2}}}\right]\right]\right\}. \tag{8a}$$

Since equations (6a), (7a), (8a) are expressed for Laplace's transformation and we shall seek the originals of the baove quoted functions for Carson—Laplace's transformation, [4], we have to multiply the above mentioned equations by the term p, obtaining:

$$e_r(p) = A_r\left\{\frac{1}{t_1}\left[\frac{e^{-\alpha p^{1/2}}}{p} - \frac{e^{-pt_1-\alpha p^{1/2}}}{p} - \frac{e^{-pt_2-\alpha p^{1/2}}}{p} + \frac{\alpha e^{-\alpha p^{1/2}}}{p^{\frac{1}{2}}} - \frac{\alpha e^{-pt_1-\alpha p^{1/2}}}{p^{\frac{1}{2}}} - \right.\right.$$

$$\left.\left. - \frac{\alpha e^{-pt_2-\alpha p^{1/2}}}{p^{\frac{1}{2}}}\right] + \frac{1}{t_3}\left[\frac{e^{-pt_3-\alpha p^{1/2}}}{p} + \frac{\alpha e^{-pt_3-\alpha p^{1/2}}}{p^{\frac{1}{2}}}\right]\right\}, \tag{6b}$$

taking the formulae for the two further components of the electromagnetic field into consideration later.

Let us now determine the originals for the individual terms of equation (6b). First of all we get:

$$\frac{e^{-\alpha p^{1/2}}}{p} \doteqdot \left(t + \frac{\alpha^2}{2}\right)\operatorname{erfc}\frac{\alpha}{2\sqrt{t}} - \alpha t\,\varkappa(\alpha, t) \tag{6,1}$$

We cannot find the original for the term $\frac{e^{-\alpha p^{1/2}}}{p^{\frac{1}{2}}}$ directly in [4] but for its determination we can use the theorem [3]:

$$\bar{f}(p)\bar{g}(p) \doteqdot \frac{d}{dt}\int_0^t f(t-\xi)g(\xi)d\xi \tag{12}$$

In our case we have:

$$\bar{f}(p) = e^{-\alpha p^{1/2}} \doteqdot \operatorname{erfc}\frac{\alpha}{2\sqrt{t}};\ \bar{g}(p) = \frac{1}{p^{\frac{1}{2}}} \doteqdot \frac{2}{\sqrt{\pi}}\sqrt{t},$$

so that we may write:

$$\frac{e^{-\alpha p^{1/2}}}{\sqrt{p}} \doteqdot \frac{d}{dt}\int_0^t \operatorname{erfc}\frac{\alpha}{2\sqrt{t-\xi}}\frac{2}{\sqrt{\pi}}\sqrt{\xi}d\xi = -\int_0^t \operatorname{erfc}'\frac{\alpha}{2\sqrt{t-\xi}}\frac{\alpha\sqrt{\xi}}{2\sqrt{\pi(t-\xi)^{\frac{3}{2}}}}d\xi + \\ + \frac{2\sqrt{t}}{\sqrt{\pi}}\operatorname{erfc}(\infty) = 0. \tag{13}$$

We have yet to determine the originals for the term $[\exp(-pt_1)]/p$ which will be determined by the theorem:

$$e^{-ap}\bar{f}(p) \doteqdot \begin{cases} 0 & (t < a) \\ \varphi(t-a) & (t > a) \end{cases} \tag{14}$$

In our case we have: $a = t_1$; $\bar{f}(p) = \frac{1}{p} \doteqdot t$ so that we may write:

$$e^{-pt_1}/p^{\frac{1}{2}} \doteqdot t - t_1,\ (t > t_1) \tag{15}$$

Analogically, using the relation $f(p) = 1/p^{\frac{1}{2}} = 2\sqrt{t/\pi}$ we obtain:

$$e^{-pt_1}/p^{\frac{1}{2}} \doteqdot 2\sqrt{t - t_1/\pi},\ (t > t_1)$$

thus having all the needed originals to be able to write the resulting equation for the original (6b):

$$e_r(t) = A_r\left\{\frac{1}{t_1}\left(-\left[t + \frac{\alpha^2}{2}\right]\operatorname{erfc}\frac{\alpha}{2\sqrt{t}} + \alpha t\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{\alpha^2}{4t}\right) - t + t_1 - t + t_2 + \right.\right. \\ + \int_0^t \operatorname{erfc}'\frac{\alpha}{2\sqrt{t-\xi}}\frac{\alpha\sqrt{\xi}}{2\sqrt{\pi(t-\xi)^{\frac{3}{2}}}}d\xi - 2\alpha\sqrt{\frac{t-t_1}{\pi}} - 2\alpha\left.\sqrt{\frac{t-t_2}{\pi}}\right) + \\ + \frac{1}{t_3}\left(t - t_3 + \left[t + \frac{\alpha^2}{2}\operatorname{erfc}\frac{\alpha}{2\sqrt{t}} - \alpha t\frac{1}{\pi\sqrt{t}}\exp\left(-\frac{\alpha^2}{4t}\right) + 2\alpha\sqrt{\frac{t-t_1}{\pi}} - \right.\right.$$

$$\left. - \alpha\int_0^t \operatorname{erfc}'\frac{\alpha}{2\sqrt{t-\xi}} \times \frac{\alpha\sqrt{\xi}}{2\sqrt{\pi(t-\xi)^{\frac{3}{2}}}}d\xi\right)\right\} = \boldsymbol{F}_1(t) \tag{6c}$$

By comparing equations (6a), (7a), (8a) we find we have yet to determine the originals of the expressions

$$\frac{p}{t_1}\left\{\frac{\alpha^2 pe^{-\alpha p^{1/2}}}{p^2} - \frac{\alpha^2 pe^{-pt_1-\alpha p^{1/2}}}{p^2} - \frac{\alpha^2 pe^{-pt_2-\alpha p^{1/2}}}{p^2}\right\} + \frac{p}{t_3}\frac{\alpha^2 pe^{-pt_3-\alpha p^{1/2}}}{p^2},$$

to get a complete solution of the problem, i. e. — irrespective of the constants — to determine the originals of the expressions:

$$e^{-\alpha p^{1/2}};\quad e^{-pt_1};\quad e^{-pt_2},\quad e^{-pt_3} \tag{7b}$$

We obtain for the first expression directly:

$$e^{-\alpha p^{1/2}} \doteqdot \operatorname{erfc}\frac{\alpha}{2\sqrt{t}}, \tag{7c}$$

whereas we may quote for the next three expressions:

$$\left.\begin{array}{ll} e^{-pt_1} \doteqdot 1 & (t > t_1) \\ e^{-pt_2} \doteqdot 1 & (t > t_2) \\ e^{-pt_3} \doteqdot 1 & (t > t_3) \end{array}\right\} \tag{7d}$$

Referring to equations (7c) and (7d) we obtain after substituting into the above mentioned expression:

$$-\frac{\alpha^2}{t_1}\left(+\operatorname{erfc}\frac{\alpha}{2\sqrt{t}} + 1^* + 1^{**}\right) + \frac{\alpha^2}{t_3}\left(\operatorname{erfc}\frac{\alpha}{2\sqrt{t}} + 1^{***}\right) = \boldsymbol{F}_2(t) \tag{7e}$$

where the asterisks at the ones denote that this expression apllies only for the time intervals given in equations (7d). Referring now to equations (6a), (7a), (8a), (6c) we get the rewritten reduced form of the resulting formulae:

$$e_r(t) = A_r\boldsymbol{F}_1(t), \tag{17}$$
$$e_\Theta(t) = A_\Theta\boldsymbol{F}_1(t) + A_\Theta\boldsymbol{F}_2(t), \tag{18}$$
$$h_\Phi(t) = A_\Phi\boldsymbol{F}_1(t) \tag{19}$$

Considering the derived formulae, we see the transient electromagnetic field will depend on the electrical conductivity of the space in which the electrical dipole is imbedded, partly explicitly — through the expressions A_r, A_Θ, A_Φ partly implicitly — through the expression for α. An analogical assertion applies also for the dependence of the transient process of the electromagnetic field on the mutual position of the field source and the point in which we examine the transient process.

The dependence of the transient process of the electromagnetic field on the shape of the single trapezoid pulse is expressed by the functions $F_1(t)$, $F_2(t)$ respectively.

Concluding we may say that the derived formulae can be used as a basis for theoretical considerations concerning the transient process in connection with the electrical conductivity and mutual position of the source and the point in which we examine the field.

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Received April 7th, 1967

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