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ON A CLASS OF RATIONAL FUNCTIONS CONNECTED WITH THE DYNAMIC INTERPRETATION OF CDD POLES

JÁN WEISS, Bratislava

INTRODUCTION

As it is well known elementary particles physics so far has been lacking an experimental proof for the existence of CDD (Castillejo, Dalitz, Dyson [1]) poles. Nevertheless, especially the problem of the ambiguity of partial wave dispersion relations, connected with the problem of CDD poles, belongs in relativistic S-matrix theory to one of the interesting topics. This is understandable, because the acceptance of the CDD poles means the admission of a new kind of independent particles and the break-down of one of the basic principles of strong interaction physics (the maximal analyticity of the second degree) founded in bootstrap dynamics.

The CDD ambiguity analogy can be also established in the non-relaticistic potential theory. In paper [2] this analogy is constructed within the framework of the non-standard inverse problem. As the non-relativistic analogy of the CDD ambiguity is considered the ambiguity in the determination of potentials for the given positions of the CDD poles in the complex momentum plane k. Results of [2] show that the CDD poles exert a characteristic influence on the interaction of elementary particles. They prove the fact that the CDD poles affect chiefly at relatively large distances, where the Yukawa short-range forces are already negligible. This circumstance appears in the behaviour of the potential characterizing the corresponding interaction of particles in a such way that in the presence of the CDD poles the asymptotics of the potential for $r \to \infty$ is already not exponential, but rational. In the present paper we want to discuss again the rational behaviour of the long-range part of the potential with a different intention than in [2], of course.

We shall deal with the exact solution of the system of non-linear differential equations from paper [2] (system (22)), by which the connection of the CDD poles with the long-range potential asymptotics was directly investigated. Whereas in [2] it was sufficient to solve the system (22) approximately for finding the corresponding asymptotics, it can be shown that the exact solution of the system brings also a new notion of a mathematical character for the physical properties of the potentials. We are thinking of the fact that the rationality of the behaviour of the long-range potential part is based on a certain class of rational functions which are exact solutions of the non-linear system (22) from [2]. Apart from being dependent on the relative distance of interacting particles these functions depend also on the integration constants and, as it follows from [2], the above-mentioned constants have the meaning of parameters of the CDD poles. As it will be shown later, the number of the integration constants increases proportionally to the number of the CDD poles in the origin of the complex plane k, but the degree of the denominator in the rational function connected with the potential does not increase proportionally. The rule determining the increase of the highest power of the denominator is expressed by the following formula: $N(N+1)/2$, where N is the number of the CDD poles. However, we are sorry to say that just this property, i. e. the rapid rise of the degree of the polynomial in the denominator of the corresponding rational function excludes on principle the possibility to show correctly how the presence of CDD poles secures the priority of the rational functions with respect to the transcendental functions in the solutions of system (22). On the other hand, as we shall see in the particular cases of solving system (22), there exists a certain property, common to all functions of the class, which enables to gain by means of an algebraic way the sought solution for the function associated with the potential in the case of the arbitrary number of the CDD poles.

The brief contents of the work: In Section 2 we shall quote preliminaries of our problem. The system from paper [2] will be solved for $N = 1, 2, 3$ and 4 in Section 3. Section 4 contains the treatment by which one can extend the class of the considered rational functions. Conclusions are in Section 5.

PRELIMINARIES

The starting-point of our considerations is the system of non-linear differential equations (system (22) from [2])

$$\begin{aligned} \beta_1''(r) - 2\beta_2'(r) &= -2\beta_1'(r)\,\beta_1(r), \\ \beta_2''(r) - 2\beta_3'(r) &= -2\beta_1'(r)\,\beta_2(r), \\ \beta_3''(r) - 2\beta_4'(r) &= -2\beta_1'(r)\,\beta_3(r), \\ &\;\;\vdots \\ \beta_{j-1}''(r) - 2\beta_j'(r) &= -2\beta_1'(r)\,\beta_{j-1}(r), \\ \beta_j''(r) &= -2\beta_1'(r)\,\beta_j(r). \qquad j=1,2,\ldots,N \end{aligned} \tag{1}$$

In system (1) $\beta_j(r)$ are functions characterizing the pole terms of the Jost solution in the origin of the complex momentum plane k, r is the relative distance of two interacting particles and N is the number of the CDD poles at the point $k = 0$.[1])

The functions $\beta_j(r)$ must satisfy the boundary conditions

$$\lim_{r\to\infty} \beta_j(r) = 0. \tag{2}$$

For a given N between the functions $\beta_j(r)$ it is the function $\beta_1(r)$ which has the physical importance for its connection with the long-range potential $u(r)$. The corresponding relation is

$$u(r) = -2\beta_1'(r). \tag{3}$$

It will therefore be the aim of our calculations to find the function $\beta_1(r)$ for an arbitrary number of the CDD poles.

EXACT SOLUTION

The solution of (1) cannot be directly found for an arbitrary number of the CDD poles. The explanation for this is to be found in a fact that the solutions for a given N are associated with the solutions of the separate foregoing cases (from 1 into $N-1$). Let us solve system (1) for the particular cases with a succesively increasing N.[2])

A. $N = 1$

System (1) is reduced to one simple differential equation of the second order

$$\beta_1''(r) = -2\beta_1'(r)\,\beta_1(r), \tag{4}$$

which can be easily integrated. We have

$$\beta_1' + \beta_1^2 = \text{const.}$$

One can put the integration constant in (5) equal to zero on account of the condition

$$\lim_{r\to\infty} \beta_1(r) = 0.$$

[1]) Two functions $\beta_j(r)$ in the plane k correspond to one CDD pole in the complex energy plane E. For this reason in [2] the index j gains the values 1, 2, ..., $2s$, where s is the number of the CDD poles in the plane E.

[2]) In other connections and by other methods in [3] and [4] a more general type of equations than one (1) was solved for $j = 1$ and 2.

The Bernoulli equation

$$\beta_1' + \beta_1^2 = 0$$

which arises thus, has the solution

$$\beta_1(r) = \frac{1}{r + c_1}, \tag{6}$$

where c_1 is an integration constant and $r \neq -c_1$. Using the substitution $z_1(r) = r + c_1$ and assuming that $z_1(r) > 0$, we can express (6) in the form

$$\beta_1(r) = [\ln z_1(r)]'.$$

B. $N = 2$

In this case system (1) has the form

$$\beta_1''(r) - 2\beta_2'(r) = -2\beta_1'(r)\beta_1(r), \tag{7}$$

$$\beta_2''(r) = -2\beta_1'(r)\beta_2(r) \tag{8}$$

and the functions $\beta_1(r)$ and $\beta_2(r)$ fulfill the boundary conditions

$$\lim_{r\to\infty} \beta_1(r) = \lim_{r\to\infty} \beta_2(r) = 0. \tag{9}$$

Equation (7) can be integrated and the corresponding integration constant put to zero with regard to (9). We obtain the equation

$$\beta_1' - 2\beta_2 + \beta_1^2 = 0. \tag{10}$$

Let us multiply (7) by the function β_2, (8) by the function β_1 and subtract at last the mentioned equations (the second from the first). We get the equation

$$\beta_1''\beta_2 - \beta_2''\beta_1 - 2\beta_2'\beta_2 = 0.$$

The first integral of this equation, in which according to (9) the integration constant should be equal to zero, is

$$\beta_1'\beta_2 - \beta_2'\beta_1 - \beta_2^2 = 0,$$

or after a simple arrangement

$$\beta_1' - (\ln \beta_2)'\beta_1 = \beta_2, \tag{11}$$

where $\beta_2(r) > 0$. Equation (11) has the following solution

$$\beta_1 = e^{\ln\beta_2}[\int \beta_2 e^{-\ln\beta_2} dr + c_1].$$

which leads to the relation

$$\beta_1 = \beta_2(r + c_1) \tag{12}$$

with the arbitrary constant c_1. Eliminating β_2 from (10) and (12) we get the Bernoulli equation

$$\beta_1' - \frac{2}{r + c_1}\beta_1 + \beta_1^2 = 0. \tag{13}$$

By substituting $\beta_1 = 1/\gamma$, where $\gamma \neq 0$, (13) may be rewritten in the form

$$\gamma' + \frac{2}{r + c_1}\gamma = 1,$$

from where, supposing that $r + c_1 > 0$, it follows

$$\gamma = e^{-2\ln(r+c_1)}[\int e^{2\ln(r+c_1)} dr + c_2]$$

and after integration

$$\gamma = \frac{r + c_1}{3} - \frac{c_2}{(r + c_1)^2},$$

where c_2 is the arbitrary constant.

For the functions β_1 and β_2 we thus have

$$\beta_1(r) = \frac{3(r + c_1)^2}{(r + c_1)^3 + 3c_2}, \tag{14}$$

$$\beta_2(r) = \frac{3(r + c_1)}{(r + c_1)^3 + 3c_2}. \tag{15}$$

In what follows we shall write instead of $r + c_1$ simply r. We are justified to do this, because system (1) is invariant under the transformation $r \to r + c$, where c is constant.

If we write $z_2(r) = r^3 + 3c_2$ in the denominator of (14) and (15) and supposing that $z_2(r) > 0$, we have

$$\beta_1(r) = \frac{3r^2}{r^3 + 3c_2} = [\ln z_2(r)]', \tag{16}$$

$$\beta_2(r) = \frac{3r}{r^3 + 3c_2} = \frac{1}{r}[\ln z_2(r)]'. \tag{17}$$

C. $N = 3$

For three functions of (1) we have now the following equations

$$\beta_1''(r) - 2\beta_2'(r) = -2\beta_1'(r)\beta_1(r), \tag{18}$$

$$\beta_2''(r) - 2\beta_3'(r) = -2\beta_1'(r)\beta_2(r), \tag{19}$$

$$\beta_3''(r) = -2\beta_1'(r)\beta_3(r) \tag{20}$$

and the boundary conditions

$$\lim_{r\to\infty} \beta_1(r) = \lim_{r\to\infty} \beta_2(r) = \lim_{r\to\infty} \beta_3(r) = 0. \tag{21}$$

The first integral of the equation (18) is

$$\beta_1' - 2\beta_2 + \beta_1^2 = 0, \tag{22}$$

where according to (21) the integration constant is taken to be zero.

Multiplying equation (19) by β_3 and equation (20) by β_2, subtracting the second equation from the first and finally integrating the thus obtained equation (the integration constant being zero again), we get

$$\beta_2'\beta_3 - \beta_3'\beta_2 - \beta_3^2 = 0,$$

or

$$\beta_2' - (\ln \beta_3)'\beta_2 = \beta_3, \tag{23}$$

where $\beta_3(r) > 0$. This is the same type of equation as equation (11). That is why its solution is

$$\beta_2 = \beta_3(r + c_1) = r\beta_3. \tag{24}$$

Next, let us multiply (18) by β_2 and (19) by β_1, subtract mutually the multiplied equations and add equation (20) to the resulting equation. We have

$$\beta_1''\beta_2 - \beta_2''\beta_1 - 2\beta_2'\beta_2 + 2(\beta_3'\beta_1 + \beta_1'\beta_3) + \beta_3'' = 0. \tag{25}$$

Equation (25) can be integrated. The conditions (21) will be useful also the third time for determining the physical meaning of the integration constant. We can easily convince ourselves that after the integration and some little arrangement we get from (25)

$$\beta_1' + \left[\frac{2}{r} - (\ln \beta_2)'\right]\beta_1 + \frac{\beta_3'}{\beta_2} - \beta_2 = 0,$$

where $\beta_2(r) > 0$. Hence

$$\beta_1 = e^{-2\ln r + \ln\beta_2}\left[\int\left(\beta_2 - \frac{\beta_3'}{\beta_2}\right)e^{2\ln r - \ln\beta_2}\,dr + c_2\right]. \tag{26}$$

The integration in (26), using (24), gives

$$\beta_1 = \frac{\beta_2}{r}\left(\frac{r^3}{3} + c_2\right) + \frac{1}{r}. \tag{27}$$

Insert for $\beta_2(r)$ into (22) the expression from (27). In such a way we obtain the Riccati differential equation for the function $\beta_1(r)$

$$\beta_1' - 2\frac{3r^2}{r^3 + 3c_2}\beta_1 + \beta_1^2 + 2\frac{3r}{r^3 + 3c_2} = 0. \tag{28}$$

In the relations (27) and (28) c_2 is an arbitrary constant, for which $r^3 \neq -3c_2$ must fulfilled. The particular integral of the equation (28) is

$$\beta_1^* = \frac{1}{r}. \tag{29}$$

We can transform equation (28) by

$$\beta_1 = \varphi + \beta_1^* \tag{30}$$

into the Bernoulli equation for the function φ

$$\varphi' + \left(\frac{2}{r} - \frac{6r^2}{r^3 + 3c_2}\right)\varphi + \varphi^2 = 0. \tag{31}$$

If we substitue in (31)

$$\gamma = \frac{1}{\varphi}, \quad \varphi \neq 0 \tag{32}$$

it is necessary to solve the equation

$$\gamma' - \left(\frac{2}{r} - \frac{6r^2}{r^3 + 3c_2}\right)\gamma = 1. \tag{33}$$

When $r^3 + 3c_2 > 0$, the solution of (33) is

$$\gamma = e^{2[\ln r - \ln(r^3+3c_2)]}\{\int e^{-2[\ln r - \ln(r^3+3c_2)]}\,dr + c_3\}, \tag{34}$$

where c_3 is the arbitrary constant. The integration in (34) yields

$$\gamma = \frac{r^2}{(r^3+3c_2)^2}\left(\frac{r^5}{5} + 3c_2r^2 - \frac{9c_2^2}{r} + c_3\right). \tag{35}$$

On the basis of (35), (32), (30) and (29) we can easily find the expression for the sought function

$$\beta_1(r) = \frac{1}{r} + \frac{(r^3 + 3c_2)^2}{r^2}\cdot\frac{1}{\frac{r^5}{5} + 3c_2r^2 + c_3 - \frac{9c_2^2}{r}}, \tag{36}$$

or in the adjusted form

$$\beta_1(r) = \frac{6r^5 + 45c_2r^2 + 5c_3}{r^6 + 15c_2r^3 + 5c_3r - 45c_2^2}. \tag{37}$$

We can again write the rational function (37) as follows

$$\beta_1(r) = [\ln z_3(r)]', \tag{38}$$

where the polynomial, characterizing the case $N = 3$, is

$$z_3(r) = r^6 + 3 \,.\, 5c_2r^3 + 5c_3r - 3^2 \,.\, 5c_2^2 \tag{39}$$

and with regard to (38) one must require $z_3(r) > 0$.

The functions $\beta_2(r)$ and $\beta_3(r)$ can now be calculated from formulae (24), (27) and the known function $\beta_1(r)$.

D. $N = 4$

Finally we start from the following system

$$\begin{aligned} \beta_1''(r) - 2\beta_2'(r) &= -2\beta_1'(r)\beta_1(r), & (40)\\ \beta_2''(r) - 2\beta_3'(r) &= -2\beta_1'(r)\beta_2(r), & (41)\\ \beta_3''(r) - 2\beta_4'(r) &= -2\beta_1'(r)\beta_3(r), & (42)\\ \beta_4''(r) \qquad\quad &= -2\beta_1'(r)\beta_4(r), & (43) \end{aligned}$$

in which the boundary conditions

$$\lim_{r\to\infty} \beta_1(r) = \lim_{r\to\infty} \beta_2(r) = \lim_{r\to\infty} \beta_3(r) = \lim_{r\to\infty} \beta_4(r) = 0 \tag{44}$$

hold for its functions.

The integrated equation (40) with zero integration constant (due to (44)) has again the form

$$\beta_1' - 2\beta_2 + \beta_1^2 = 0. \tag{45}$$

The same operations we have made in the two last equations of the above cases analogically give for equations (42) and (43) the relation between β_3 and β_4

$$\beta_3 = r\beta_4. \tag{46}$$

In order to find the relation between these functions and the functions β_1 and β_2, let us multiply (41) by β_3, subtract from this equation the equation (42) multiplied by β_2, multiply then the resulting equation by -1 and add to the equation which we obtain by multiplying (40) by β_4 and (43) by $-\beta_1$ and by their mutual addition. The result of this procedure is

$$\beta_1''\beta_4 - \beta_4''\beta_1 - 2(\beta_2'\beta_4 + \beta_4'\beta_2) + \beta_3''\beta_2 - \beta_2''\beta_3 + 2\beta_3'\beta_3 = 0. \tag{47}$$

Let us now integrate the equation (47). Using the conditions of (44) for the determination of the integration constant and after some arranging we obtain

$$\beta_1' - \frac{\beta_4'}{\beta_4}\beta_1 - 2\beta_2 + \frac{\beta_3'}{\beta_4}\beta_2 - \frac{\beta_3}{\beta_4}\beta_2' + \frac{\beta_3^2}{\beta_4} = 0. \tag{48}$$

For β_1 this implies

$$\beta_1 = e^{\ln\beta_4}\left[\int\left(\beta_2 - r\frac{\beta_4'}{\beta_4}\beta_2 + r\beta_2' - r^2\beta_4\right)e^{-\ln\beta_4}dr - c_2\right], \tag{49}$$

$$\beta_4(r) > 0,$$

where we have used (46) and written the integration constant as $-c_2$. The integral in (49) can be computed. The integration leads to the next relation between the functions of our system

$$\beta_1 = r\beta_2 - \frac{r^3}{3}\beta_4 - c_2\beta_4. \tag{50}$$

Take now the following combinations of the equations (40—43): multiply (40) by β_2, (41) by $-\beta_1$, sum up the obtained equations and add (42) to the resulting equation. We get

$$\beta_1''\beta_2 - \beta_2''\beta_1 - 2\beta_2'\beta_2 + 2(\beta_3'\beta_1 + \beta_1'\beta_3) + \beta_3'' - 2\beta_4' = 0. \tag{51}$$

The first integral of this equation is

$$\beta_1'\beta_2 - \beta_2'\beta_1 - \beta_2^2 + 2\beta_1\beta_3 + \beta_3' - 2\beta_4 = 0 \tag{52}$$

with the zero integration constant on account of (44). When we exclude β_1 and β_4 from (52) according to (50) and (46), we obtain the differential equation of the first order for the function β_3

$$\beta_3'\left[1 - \left(\frac{r^3}{3} + \frac{c_2}{r}\right)\beta_2\right] + \beta_3\left[\left(\frac{4}{3}r + \frac{c_2}{r^2}\right)\beta_2 + \left(\frac{r^3}{3} + \frac{c_2}{r}\right)\beta_2' - \frac{2}{r}\right] - \beta_3^2\left(\frac{2}{3}r^2 + \frac{2c_2}{r}\right) = 0. \tag{53}$$

The equation (53) may be rewritten to the from

$$\beta_3' - \left[\left(\ln\frac{3r - (r^3 + 3c_2)\beta_2}{r}\right)' + \frac{2}{r}\right]\beta_3 = \frac{2r^3 + 6c_2}{3r - (r^3 + 3c_2)\beta_2}\beta_3^2, \tag{54}$$

or by subtituting $\Lambda = 1/\beta_3$ it may be transformed to the equation

$$\Lambda' + \left[\left(\ln\frac{3r - (r^3 + 3c_2)\beta_2}{r}\right)' + \frac{2}{r}\right]\Lambda = -\frac{2r^3 + 6c_2}{3r - (r^3 + 3c_2)\beta_2}. \tag{55}$$

We can write later the solution of (55) in the following way

$$\Lambda = e^{-\left[\ln \frac{3r-(r^3+3c_2)\beta_2}{r} + 2\ln r\right]} \left\{ \int \frac{-(2r^3 + 6c_2)}{3r - (r^3 + 3c_2)\beta_2} \cdot e^{\left[\ln \frac{3r-(r^3+3c_2)\beta_2}{r} + 2\ln r\right]} dr - \frac{1}{3} c_3 \right\}, \quad (56)$$

where the integration constant is indicated as $-1/3\, c_3$ and $3r - (r^3 + 3c_2)\beta_2 > 0$. After computing the integral in (56) there follows for β_3 the result

$$\beta_3 = -\frac{3r^2 - (r^4 + 3c_2r)\beta_2}{\frac{2}{5}r^5 + 3c_2r^2 + \frac{1}{3}c_3}. \quad (57)$$

From the relations (57) and (50) we are able to determine β_2 as a function of β_1

$$\beta_2 = \frac{-(3r^4 + 9c_2r) + \left(\frac{6}{5}r^5 + 9c_2r^2 + c_3\right)\beta_1}{\frac{1}{5}r^6 + 3c_2r^3 + c_3r - 9c_2^2}. \quad (58)$$

Inserting (58) into (45), we obtain finally the Riccati equation only for the function β_1

$$\beta_1' - 2\frac{6r^5 + 45c_2r^2 + 5c_3}{r^6 + 15c_2r^3 + 5c_3r - 45c_2^2}\beta_1 + \beta_1^2 + 2\frac{15r^4 + 45c_2r}{r^6 + 15c_2r^3 + 5c_3r - 45c_2^2} = 0. \quad (59)$$

We transform the equation (59) into the Bernoulli equation by way of the relation

$$\beta_1 = \varphi + \beta_1^*, \quad (60)$$

where β_1^* is a particular integral of equation (59). It can be shown that the particular solution of the Riccati equation (59) is

$$\beta_1^* = \frac{3r^2}{r^3 + 3c_2}. \quad (61)$$

The corresponding Bernoulli equation has the form

$$\varphi' + 2\left(\frac{3r^2}{r^3 + 3c_2} - \frac{6r^5 + 45c_2r^2 + 5c_3}{r^6 + 15c_2r^3 + 5c_3r - 45c_2^2}\right)\varphi + \varphi^2 = 0. \quad (62)$$

From the equation, into which (62) is transformed by substituting

$$\gamma = \frac{1}{\varphi}, \quad \varphi \neq 0 \quad (63)$$

the solution for γ follows:

$$\gamma = \frac{(r^3 + 3c_2)^2}{(r^6 + 15c_2r^3 + 5c_3r - 45c_2^2)^2}\left[\int \frac{(r^6 + 15c_2r^3 + 5c_3r - 45c_2^2)^2}{(r^3 + 3c_2)^2} dr + c_4\right]. \quad (64)$$

Here c_4 is the arbitrary constant and $r^6 + 15c_2r^3 + 5c_3r - 45c_2^2 > 0$. The integral in (54) has a remarkable property: all transcendental functions, which arise at its computation, cancel out. Thus the result is again the rational function

$$\beta_1(r) = \frac{3r^2}{r^3 + 3c_2} + \frac{(r^6 + 15c_2r^3 + 5c_3r - 45c_2^2)^2}{(r^3 + 3c_2)^2}\left[\frac{r^7}{7} + 6c_2r^4 + 5c_3r^2 - 18c_2^2r - \frac{25}{3}c_3^2\frac{1}{r^3 + 3c_2} - \frac{9c_2(10c_3r^2 - 81c_2^2r)}{r^3 + 3c_2} + c_4\right]^{-1}, \quad (65)$$

or in the simpler form

$$\beta_1(r) = \frac{10r^9 + 315c_2r^6 + 175c_3r^4 + 21c_4r^2 - 1050c_2c_3r + 4725c_2^3}{r^{10} + 45c_2r^7 + 35c_3r^5 + 7c_4r^3 - 525c_2c_3r^2 + 4725c_2^3r - \frac{175}{3}c_3^2 + 21c_2c_4}. \quad (66)$$

The result (66) offers once more the denotation in the form of the logarithmical derivation of the polynomial characterizing the case $N = 4$

$$\beta_1(r) = [\ln z_4(r)]', \quad (67)$$

where

$$z_4(r) = r^{10} + 3^2 . 5c_2r^7 + 5 . 7c_3r^5 + 7c_4r^3 - 3 . 5^2 . 7c_2c_3r^2 + + 3^3 . 5^2 . 7c_2^3r - \frac{5^2 . 7}{3}c_3^2 + 3 . 7c_2c_4 \quad (68)$$

and $z_4(r) > 0$. The remaining three functions can be determined by means of (58), (57), (46) and the known function $\beta_1(r)$.

EXTENSION OF THE CLASS

Particular cases of solving system (1), which have been dealt with in section 3, lead to the conclusion that the functions $\beta_1(r)$ form a certain class of the rational functions with the following properties:

a) Every function $\beta_1^N(r)$ can be expressed as the logarithmical derivative of a certain polynomial

$$\beta_1^N(r) = \frac{\mathrm{d}}{\mathrm{d}r}\,[\ln z_N(r)], \tag{69}$$

where $N = 1, 2, 3, \ldots$ and $z_N(r) > 0$.

b) The functions $\beta_1^N(r)$ are the solutions of the Riccati differential equation

$$(\beta_1^N(r))' - 2\,\frac{z'_{N-1}(r)}{z_{N-1}(r)}\,\beta_1^N(r) + (\beta_1^N(r))^2 + \frac{z''_{N-1}(r)}{z_{N-1}(r)} = 0. \tag{70}$$

c) From the properties a) and b) there follows the reccurent equation between the polynomial $z_{N-1}(r)$ and $z_N(r)$

$$z_{N-1}(r)z''_N(r) - 2z'_{N-1}(r)z'_N(r) + z''_{N-1}(r)z_N(r) = 0. \tag{71}$$

d) The polynomial $z_N(r)$ may be determined from the creation relation

$$z_N(r) = \frac{z_0}{z_1}\prod_{j=\substack{2\\3}}^{N}\left[(2j-1)\int\left(\frac{z_{j-1}(r)}{z_{j-2}(r)}\right)^2 \mathrm{d}r + c_j\right] \quad \begin{matrix}\text{even } j \\ \text{for an even } N \\ \text{odd } j \\ \text{for an odd } N,\end{matrix} \tag{72}$$

where $z_0(r) = 1$, $z_1(r) = r$ and $c_2, c_3, \ldots, c_N$ are the arbitrary constants. However, the constants of the polynomials $z_2, z_3, \ldots, z_{N-2}$ have to be equal to the constants $c_2, c_3, \ldots, c_{N-2}$ of the polynomial z_N. As regards the constant c_1 in z_1 we do not write it, according to our argeement (see section 3, B).

Between the polynomials $z_N(r)$, $z_{N-1}(r)$ and $z_{N-2}(r)$ there exists the following dependence

$$z_N(r) = z_{N-2}(r)\left[(2N-1)\int\left(\frac{z_{N-1}(r)}{z_{N-2}(r)}\right)^2 \mathrm{d}r + c_N\right]. \tag{73}$$

e) The degree of the $z_N(r)$-th polynomial is defined by the rule

$$m = \frac{N(N+1)}{2},$$

where m is the highest power of the polynomial.

The relation (72) gives thus the structure of the polynomials by means of which one defines the functions β_1^N with a direct physical meaning. As expression (72) shows, the polynomials of particular cases are bound to the polynomials of other and other cases starting from z_0 for an even N and z_1 for an odd N. This connection between the polynomials is realized on the basis of the subordination of integration constants: of $(N-1)$ arbitrary integration constants, occuring in the polynomial $z_N(r)$, one requires for the constants of structural polynomails $z_1, \ldots, z_{N-2}$ to be equal to the constants $c_2, \ldots, c_{N-2}$.

It remians an unsolved problem, connected with an interesting feature of the representation of (72), namely that the integral of the squared ratio of a following and foregoing polynomial yields the rational function. Surveying the calculations of the functions $\beta_1^N(r)$ from Section 3, we see that the first three cases do not explain this problem, because in them the transcendental functions do not occur yet at the integration. The case of $N = 4$, in which the transcendental functions already appear, but finally cancel out, can lead at most to the assumption that real roots of the polynomial $z_{N-2}(r)$ play a decisive role at the vanishing of the transcendental terms. If we want to get information from the case of $N = 5$, we find out that to be able to do it, we must know the roots of the polynomial of the sixth degree. The increase of the polynomial degree (property e)) is too quick and, as we see, it puts before us basic difficulties at the investigation of factors affecting the cancellation of transcendental terms in the functions $\beta_1^N(r)$.

The expression of the $z_N(r)$-th polynomial in the form (73) satisfies the recurrent equation (71), as it can be easily verified, however, it cannot be used for the above reasons for the polynomials with high degrees. In order to be able to extend the mentioned class of rational functions $\beta_1^N(r)$, in spite of this, we employ in a suitable way equation (71) which is a consequence of the fundamental properties of the functions $\beta_1^N(r)$. If we know the polynomial $z_{N-1}(r)$, equation (71) enables us to calculate $z_N(r)$ and with the help of (69) also the function $\beta_1^N(r)$.

Let us assume thus the polynomials $z_{N-1}(r)$ and $z_N(r)$ in the following form

$$z_{N-1}(r) = \sum_{j=0}^{N(N-1)/2} A_j r^j, \qquad z_N(r) = \sum_{j=0}^{\infty} B_j r^j, \tag{74}$$

where A_j are known coefficients and B_j are the ones to be determined. Inserting (73) into (71) we obtain a system of algebraic equations for the unknown coefficients B_j

$$\sum_{\nu=0}^{l}\sum_{\substack{\mu=l+2\\ \nu+\mu=l+2}}^{2}[(\mu-1)\mu(A_\nu B_\mu + A_\mu B_\nu) - 2(\nu+1)(\mu-1)A_{\nu+1}B_{\mu-1}] = 0. \tag{75}$$

$$l = 0, 1, 2, 3, \ldots$$

For the calculation of the coefficients B_j concerning the given N it is sufficient to consider the first $1 + N(N+1)/2$ equations from the infinite number of the equations of (75). It may be shown namely, as it might have been even expected, that the constants B_j in (74) are identically equal to zero at powers higher than $N(N+1)/2$. From (75) it follows that in every polynomial one must arbitrarily choose two constants: in z_2 B_3 and B_0, in z_3 B_6 and B_1, in z_4 B_{10} and B_3, in z_5 B_{15} and B_6 etc. If we choose these constants so that the first pair is put equal to 1 and the second pair $3c_2$, $5c_3$, $7c_4$, $9c_5$, respectively, etc., we get the polynomials of our particular cases. Since system (75) helps to construct the polynomials $z_N(r)$, we can regard it as a convenient means to gain additional information on system (1). We can thus extend by it the class of the functions $\beta_1^N(r)$.

From the applications of (75) let us quote at least the result obtained for the polynomial $z_5(r)$:

$$z_5(r) = r^{15} + 3 \,.\, 5 \,.\, 7c_2r^{12} + 4 \,.\, 5 \,.\, 7c_3r^{10} + 3^2 \,.\, 5^2 \,.\, 7c_2^2r^9 + 7 \,.\, 9c_4r^8 -$$
$$- 3^2 \,.\, 5^2 \,.\, 7c_2c_3r^7 + 9c_5r^6 + 3 \,.\, 7^2(C - 2 \,.\, 3^2c_2c_4)r^5 +$$
$$+ 3^3 \,.\, 5^3 \,.\, 7^2c_2^2c_3r^4 + (-3^5 \,.\, 5^3 \,.\, 7^2c_2^4 - 3 \,.\, 5 \,.\, 7^2c_3c_4 + 3 \,.\, 5 \,.\, 9c_2c_5)r^3 -$$
$$- 3^2 \,.\, 5 \,.\, 7^2Cc_2r^2 + \left(-\frac{3^2 \,.\, 7^2}{5}c_4^2 + 5 \,.\, 9c_3c_5\right)r +$$
$$+ \frac{3^4 \,.\, 7^2}{5} \,.\, \frac{c_2^2c_4^2}{c_3} - \frac{7^2C^2}{5c_3} - 3^2 \,.\, 5 \,.\, 9c_2^2c_5, \qquad (76)$$

where
$$C = (3^2c_2c_4 - 5^2c_3^2).$$

CONCLUSION

As the cases A, B, C and D in Section 3 show, the solutions of system (1) are rational functions. It was our task to find from the functions $\beta_j(r)$ the function $\beta_1(r)$ connected with the potential of long-range forces for the arbitrary number of the CDD poles. However, if we want to investigate the proper long-range asymptotics of the potential it is suffecient to solve the system (1) approximately (see [2]). We can convince ourselves that the solutions obtained in this paper coincide in the limit $r \to \infty$ with the solutions from [2]. For a sufficiently large r we can also obtain the solutions of more general equations occuring in papers [3] and [4].

The exact solutions have some interesting properties (see Section 4). The rationality of the solutions is evident in the logarithmical derivative of the polynomials $z_N(r)$. The solutions of individual cases are mutually connected, which is obvious from the iterative connection of the polynomials of particular cases according to (72). Between the properties of the polynomials one property is particularly remarkable (the property expressed by (73)): the product of a given polynomial and the integral of the squared ratio of the foregoing and given polynomial is again a polynomial. The determination of a criterion explaining this property is associated with the difficulty which is the rapid rise

of the polynomial degree (Section 4, property e)). Therefore we can only assume that the following requirement plays here a role: all integration constants — with the exception of two — must coincide with the constants of the polynomials taking part in the formation of the mentioned polynomial. This dependence of the constants of structural polynomials on the constants of the considered polynomial affects certainly the roots of the polynomials and trough them also the cancellation of the transcendental functions.

In spite of the difficulty we have with the integration according to (72) for the polynomials with high degrees, we are able, due to equation (71), to extend the system of our polynomials and with regard to (69) thus also the solutions $\beta_1(r)$ for the arbitrary finite number of the equations in system (1), namely on the basis of an algebraic approach (Section 4). In the system of the algebraic equations (75) for the coefficients of the sought polynomial $z_N(r)$ it is always necessary to choose the coefficients of the highest power of r, i. e. $N(N + 1)/2$ and of the power $(N - 1)(N - 2)/2$. To obtain the system of our polynomials the first coefficient has to be chosen 1, the second one $(2N - 1)c_N$.

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REFERENCES

[1] Castillejo L., Dalitz R., Dyson F., Phys. Rev. *101* (1956), 453.
[2] Weiss J., Czech. J. Phys. (to be published).
[3] Petráš M., Mat.-fyz. časopis *12* (1962), 136.
[4] Weiss J., Mat.-fyz. časopis *13* (1963), 58.

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Katedra fyziky
Strojníckej fakulty SVŠT,
Bratislava