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GENERALISATION OF A NUMBER-THEORETICAL RESULT

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I

Let $k \geqq 3$ be a natural number and let M be a set of natural numbers. We say that M is a k-thin set if from the condition

$$a_1, a_2, \ldots, a_{k-1} \in M$$

it follows that

$$a_1 + a_2 + \ldots + a_{k-1} \notin M.$$

With other words: the set M is k-thin if in its numbers the equation

$$a_1 + a_2 + \ldots + a_{k-1} = a_k$$

is not solvable (the numbers a_i can be equal).

Let us denote by $f(k, p)$ the greatest natural number for which there exist p disjoint k-thin sets $S_1, S_2, \ldots, S_p$ such that

$$\{1, 2, \ldots, f(k, p)\} = \bigcup_{i=1}^{p} S_i .$$

The existence of $f(k, p)$ for arbitrary k and p follows from Theorems 3 and 4 of article [3].

The case $k = 3$ was treated by I. Schur in article [4]. He proved namely the inequalities

$$f(3, p + 1) \geqq 3 \,.\, f(3, p) + 1, \tag{1}$$

$$f(3, p) \geqq \frac{3^p - 1}{2}. \tag{2}$$

In our article we shall generalize the inequalities (1) and (2) for the case of an arbitrary $k \geqq 3$ and show their application to the theory of graphs.

II

Theorem 1. *Let $k \geqq 3$ and p be natural numbers. We have*

$$f(k, p+1) \geqq k \, . \, f(k, p) + (k-2). \tag{3}$$

Note 1. For an arbitrary $k \geqq 3$ we have $f(k, 1) = k - 2$ and therefore because of (3)

$$f(k, p) \geqq k \, . \, f(k, p-1) + (k-2) \geqq k^2 \, . \, f(k, p-2) + k(k-2) + (k-2) \geqq$$
$$\geqq \ldots \geqq k^{p-1} \, . \, f(k, 1) + k^{p-2}(k-2) + \ldots + k(k-2) + (k-2) =$$
$$= (k-2)(k^{p-1} + k^{p-2} + \ldots + k + 1) = \frac{k-2}{k-1}(k^p - 1),$$

i. e.

$$f(k, p) \geqq \frac{k-2}{k-1}(k^p - 1)$$

and this is a generalisation of the relation (2) for any $k \geqq 3$.

Note 2. If in (3) we put $k = 3$, we get the relation (1).

Proof of the Theorem 1. From the definition of $f(k, p)$ it follows that there exist p disjoint k-thin sets $S_1, S_2, \ldots, S_p$ such that

$$\{1, 2, \ldots, f(k, p)\} = \bigcup_{i=1}^{p} S_i.$$

Let us put

$$S_{p+1} = \{f(k, p) + 1, f(k, p) + 2, \ldots, (k-1) f(k, p) + (k-2)\}.$$

From the inequality

$$(k-1)\,[f(k, p) + 1] > (k-1) f(k, p) + (k-2)$$

it follows that S_{p+1} is a k-thin set. Now, to accomplish the proof, it is sufficient to show that the numbers

$$[(k-1) f(k, p) + (k-1)], [(k-1) f(k, p) + k], \ldots, [kf(k, p) + (k-2)] \tag{4}$$

(the number of which is $f(k, p)$) can be divided into $S_1, S_2, \ldots, S_p$ so that after adding some numbers from (4) to S_i we get again a k-thin set A_i.

Let us denote $d = (k-1)f(k, p) + (k-2)$. Every number a from (4) can be written in the form $a = c(a) + d$, where $c(a)$ is a natural number fulfilling the condition

$$0 < c(a) \leqq f(k, p).$$

Now let us add each number a from (4) to the same set to which the number $c(a)$ belongs. The sets arisen in this way denote by $A_1, A_2, \ldots, A_p$. We shall prove that every A_i ($i = 1, 2, \ldots, p$) is a k-thin set.

Let be $a_1, a_2, \ldots, a_{k-1} \in A_i$. We shall distinguish three cases.

1. $a_m \leqq f(k, p)$ for every $m = 1, 2, \ldots, k-1$. In this case we have: $a_1 + a_2 + \ldots + a_{k-1} < d$. From the construction of the set A_i it follows that $a_1 + a_2 + \ldots + a_{k-1} \notin A_i$.

2. Let exactly one of the numbers a_m be greater than d and the other less or equal to $f(k, p)$. We can assume that just a_1 is greater than d and so $a_2, a_3, \ldots, a_{k-1} \in S_i$. Since $a_1 > d > f(k, p)$, a_1 is one of the numbers (4); hence $c(a_1) = a_1 - d \leqq f(k, p)$ and from the construction of the set A_i it follows, that $a_1 - d \in S_i$. The set S_i is k-thin, hence we have

$$(a_1 - d) + a_2 + \ldots + a_{k-1} \notin S_i.$$

We shall show that $a = a_1 + a_2 + \ldots + a_{k-1} \notin A_i$. We shall prove indirectly. Assume that a belongs to A_i. Since $a > d$, we can write $a = d + c(a)$; obviously

$$c(a) = (a_1 - d) + a_2 + \ldots + a_{k-1}.$$

From the construction of the set A_i it follows that $c(a)$ belongs to S_i. This is a contradiction.

3. Let at least two of the numbers a_m be greater than d. Then we have

$$a_1 + a_2 + \ldots + a_{k-1} > 2d > kf(k, p) + (k-2)$$

(since $k \geqq 3$) and therefore $a_1 + a_2 + \ldots + a_{k-1} \notin A_i$.

The proof of the Theorem is completed, because the above considerations are correct for arbitrary $i = 1, 2, \ldots, p$.

Note 3. The Theorem gives in fact also a method of the direct splitting of the numbers

$$1, 2, \ldots, \frac{k-2}{k-1}(k^p - 1)$$

into p k-thin sets. We shall illustrate this method on the case $k = 5$, $p = 3$. Because of note 1 we have

$$f(5, 3) \geqq \tfrac{3}{4}(5^3 - 1) = 93.$$

The division of the numbers $1, 2, \ldots, 93$ into three 5-thin sets is the following:

$A_1 =$	{1, 2, 3,		16, 17, 18,		76, 77, 78,		91, 92, 93}
$A_2 =$		{4, 5,…, 15,				79, 80,…, 90}	
$A_3 =$				{19, 20,…, 75}			

III

We shall apply the above results to the solving of a known problem of the theory of graphs. All considerations of part III are direct generalisation of those of [1].

Let $g(k, p)$ denote the greatest natural number such that all edges of a complete graph of $g(k, p)$ vertices can be coloured by p colours so that there does not arise a complete subgraph of k vertices, all edges of which are coloured by the same colour.

The existence of $g(k, p)$ for any natural k and p follows from the article [2].

Theorem 2. *Let $k \geqq 3$ and p be natural numbers. We have:*

$$g(k, p) \geqq f(k, p) + 1. \tag{5}$$

Proof. Let $A_1, A_2, \ldots, A_p$ be such k-thin sets that each of the numbers $1, 2, \ldots, f(k, p)$ belongs exactly to one of them (existence of such sets follows from the definition of the number $f(k, p)$). Let G be a complete graph with $f(k, p) + 1$ vertices. Let us denote them by $P_0, P_1, \ldots, P_{f(k,p)}$. Colour all edges of graph G by the colours $C_1, C_2, \ldots, C_p$ in the following way: colour the edge interconnecting vertices P_i and P_j by the colour C_m if and only if $|i - j| \in A_m$. Let us suppose that all edges of a complete subgraph with k vertices $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ are in this colouring coloured by the same colour C_{m_0}. We can suppose that

$$i_1 > i_2 > \ldots > i_k,$$

i. e.

$$(i_1 - i_2), (i_2 - i_3), \ldots, (i_{k-1} - i_k), (i_1 - i_k) \in A_{m_0}.$$

Since A_{m_0} is a k-thin set, this is a contradiction, because

$$(i_1 - i_2) + (i_2 - i_3) + \ldots + (i_{k-1} - i_k) = (i_1 - i_k).$$

The proof of Theorem 2 is complete.

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