ON CYCLIC DECOMPOSITIONS OF THE COMPLETE GRAPH INTO $(4m + 2)$-GONS

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The construction of a cyclic decomposition of the complete graph into p-gons, where $p \equiv 0 \pmod 4$, was given in paper [1]; the case $p \equiv 1 \pmod 2$ was investigated in [2]. This article gives the solution of the problem of a cyclic decomposition of the complete graph in the remaining case $p \equiv 2 \pmod 4$.

Let k be natural, and let p of the form $p = 4m + 2$ be given, where m is natural. Denote $n = 2kp + 1$. In agreement to [2] the $(k \times p)$-matrix $\mathbf{A} = \|a_{ij}\|$ will be called a matrix of type (1), if $\{a_{11}, \ldots, a_{kp}\} = \{1, 2, \ldots, kp\}$ holds.

Theorem 1. *For arbitrary k and p of the form $p = 4m + 2$ there exists a $(k \times p)$-matrix $\mathbf{A} = \|a_{ij}\|$ of the type (1) and constants $\varepsilon_{ij} = 1$ or -1 such that*

$$\sum_{j=1}^{p} a_{ij}\varepsilon_{ij} \equiv 0 \pmod n$$

holds for all $i = 1, \ldots, k$.

Proof. The matrix $\mathbf{A} = \|a_{ij}\|$ and the constants ε_{ij} satisfying the conditions of the theorem can be determined as follows:

$$a_{ij} = \begin{cases} (i-1)p + j & 1 \leqq j \leqq p-2 \\ (k-i+1)p - 1 & j = p-1 \\ (k-i+1)p & j = p, \end{cases}$$

where $\varepsilon_{i,4}$ equals -1 and all remaining ε_{ij} equal $+1$ if $m = 1$, $\varepsilon_{i,4}$; $\varepsilon_{i,6}$, $\varepsilon_{i,7}$; $\varepsilon_{i,10}$, $\varepsilon_{i,11}$, $\ldots$, $\varepsilon_{i,p-4}$, $\varepsilon_{i,p-3}$ equal -1 and all remaining ε_{ij} equal $+1$ if $m \geqq 2$.

One can see easily that the conditions of the theorem are satisfied. Obviously each of the numbers $1, 2, \ldots, kp$ appears in the matrix $\mathbf{A}$ exactly once. The i-th row of the matrix $\mathbf{A}$ is of the form:

$(i-1)p + 1$, $(i-1)p + 2$, $\ldots$, $ip - 4$, $ip - 3$, $ip - 2$, $(k-i+1)p - 1$, $(k-i+1)p$. We obtain

$$\sum_{j=1}^{p} a_{ij}\varepsilon_{ij} = [(i-1)p+1] + [(i-1)p+2] + [(i-1)p+3] -$$
$$-[(i-1)p+4] + \{(i-1)p+5] - [(i-1)p+6] -$$
$$-[(i-1)p+7] + [(i-1)p+8]\} + \{(i-1)p+9) -$$
$$-[(i-1)p+10] - [(i-1)p+11] + [(i-1)p+12]\} + \dots$$
$$\dots + \{(ip-5) - (ip-4) - (ip-3) + (ip-2)\} +$$
$$+ [(k-i+1)p-1] + (k-i+1)p = 2(i-1)p+2+$$
$$+ (k-i+1)p - 1 + (k-i+1)p = 2kp+1.$$

Let there be given a complete graph $\langle n\rangle$ with n vertices $v_1, \dots, v_n$, where n is of the form $n = 2kp + 1$, p is of the form $p = 4m + 2$, k is natural.

The length of an edge v_iv_j in the graph $\langle n\rangle$ is defined as a minimum of the numbers $|i-j|$, $n - |i-j|$. By the turning of an edge v_iv_j in the graph $\langle n\rangle$ we mean the adding of a 1 to the indices, whereby we get the edge $v_{i+1}v_{j+1}$ from the edge v_iv_j (the indices are taken modulo n). By the turning of a polygon in the graph $\langle n\rangle$ we mean a simultaneous turning of all edges of the polygon.

A decomposition $\mathscr{R} = \{K_1, \dots, K_r\}$ of the complete graph into r polygons $K_1, \dots, K_r$ is called cyclic if the following holds: If $\mathscr{R}$ contains a polygon K, then $\mathscr{R}$ contains also the polygon K' obtained from K by turning.

Theorem 2. *For an arbitrary natural k and for an arbitrary p of the form $p = 4m + 2$, where m is natural, there exists a cyclic decomposition of the graph $\langle 2kp + 1\rangle$ into p-gons.*

Proof. Let in the graph $\langle 2kp + 1\rangle$ be given k polygons, with p edges each:

$$K_j = \{v_{i_{j1}}v_{i_{j2}}, v_{i_{j2}}v_{i_{j3}}, \dots, v_{i_{jp}}v_{i_{j1}}\}; \{i_{j1}, \dots, i_{jp}\} \subseteq \{1, 2, \dots, 2kp+1\}, j = 1, 2, \dots, k.$$

If each of the possible lengths $1, 2, \dots, kp$ in the graph $\langle 2kp + 1\rangle$ is the length of exactly one of kp edges of the p-gons $K_1, \dots, K_k$, then call the system of p-gons $\mathscr{K} = \{K_1, \dots, K_k\}$ a basic system of p-gons in the graph $\langle 2kp + 1\rangle$. We obtain a cyclic decomposition of the graph $\langle 2kp + 1\rangle$ into p-gons if any of the p-gons of the basic system is turned successively $2kp$ times.

The basic system of p-gons in the graph $\langle 2kp + 1\rangle$ can be determined with the help of the matrix of the type (1) satisfying the condition of Theorem 1. Let $\mathbf{A} = \|a_{ij}\|$ be such a matrix and let $\mathbf{E} = \|\varepsilon_{ij}\|$ be the corresponding matrix of constants constructed to prove Theorem 1. Denote by $\mathbf{A}'$ and $\mathbf{E}'$ the matrix which arises from the matrix $\mathbf{A}$ and $\mathbf{E}$ if the elements of each row of the matrix $\mathbf{A}$ and $\mathbf{E}$ respectively are permuted:

a) for $m = 1$ under the identic permutation

b) for $m \geq 2$ under the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots & 2m+2 & 2m+3 & 2m+4 \\ 1 & 2 & 3 & 4 & 5 & 4m+1 & 4m-1 & 4m-3 & \dots & 9 & 7 & 4m \end{pmatrix}$$
$$\begin{pmatrix} 2m+5 & \dots & 4m & 4m+1 & 4m+2 \\ 4m-2 & \dots & 8 & 6 & 4m+2 \end{pmatrix}.$$

The matrix $\mathbf{A}'$ clearly also satisfies the conditions of Theorem 1 with the constants ε'_{ij}.

Choose an arbitrary vertex v_x ($x \in \{1, 2, \dots, 2kp + 1\}$). The p-gon K_i will be determined as follows:

$$K_i = \{v_xv_{x+c_{i1}}, v_{x+c_{i1}}v_{x+c_{i2}}, \dots, v_{x+c_{i,p-1}}, v_x\},$$

where

$$c_{ij} = \sum_{\nu=1}^{j} a'_{i\nu}\varepsilon'_{i\nu}, \; i = 1, \dots, k; \; j = 1, \dots, p.$$

It is easy to verify that no vertex appears in the sequence of the edges of K_i more than two times. Namely, it is easy to verify an equivalent statement that no pair of numbers a, b, for which $a \equiv b \pmod{2kp + 1}$, appears in the following sequence of p numbers:

$(i-1)p+1,\ 2(i-1)p+3,\ 3(i-1)p+6,\ 2(i-1)p+2,\ 3(i-1)p+7,\ p(2i+k-2)+6,\ p(i+k-2)+9,\ p(2i+k-2)+4,\ p(i+k-2)+11, \dots, p(i+k-2)+2m+3, p(2i+k-3)+2m+12, p(i+k-2)+2m+5,\ p(2i+k-2)+2m+3,\ p(i+k-2)+2m+7,\ p(2i+k-2)+2m+1,\ p(i+k-2)+2m+9,\ \dots,\ p(i+k-1)-1,\ p(2i+k-2)+7,\ p(i+k-1)+1,\ 2kp+1.$

This completes the proof of Theorem 2.

Example 1. The cyclic decomposition of the complete graph $\langle 61\rangle$ into 10-gons will be obtained if each of the 10-gons K_1, K_2, K_3 is turned successivelly 60 times (the vertices are denoted by v_i, $i = 1, \dots, 61$):

$K_1 = \{v_1v_2, v_2v_4, v_4v_7, v_7v_3, v_3v_8, v_8v_{37}, v_{37}v_{30}, v_{30}v_{38}, v_{38}v_{32}, v_{32}v_1\}$
$K_2 = \{v_1v_{12}, v_{12}v_{24}, v_{24}v_{37}, v_{37}v_{23}, v_{23}v_{38}, v_{38}v_{57}, v_{57}v_{40}, v_{40}v_{58}, v_{58}v_{42}, v_{42}v_1\}$
$K_3 = \{v_1v_{22}, v_{22}v_{44}, v_{44}v_6, v_6v_{43}, v_{43}v_7, v_7v_{16}, v_{16}v_{50}, v_{50}v_{17}, v_{17}v_{52}, v_{52}v_1\}.$

By Theorem 2 with $p \equiv 2 \pmod 4$ there exists for an arbitrary $n \equiv 1 \pmod{2p}$ a cyclic decomposition of the graph $\langle n\rangle$ into p-gons. Obviously if $p \equiv 2 \pmod 4$ there exists no x, $x \not\equiv 1 \pmod{2p}$ so that for an arbitrary $n \equiv x \pmod{2p}$ there exists a cyclic decomposition of the graph $\langle n\rangle$ into p-gons. However, it is easy to verify that for some p, $p \equiv 2 \pmod 4$ there exist n and x, $x \neq 1$ so that $n \equiv x \pmod{2p}$ and there exists a cyclic decomposition of the graph $\langle n\rangle$ into p-gons. This fact is shown by the following example.

Table 1

K_1	K_2	K_3	K_4	K_5	K_6
$x\ \ , x+6$	$x\ \ , x+8$	$x\ \ , x+9$	$x\ \ , x+10$	$x\ \ , x+11$	$x\ \ , x+12$
$x+6\ , x+13$	$x+8\ , x+7$	$x+9\ , x+7$	$x+10, x+7$	$x+11, x+7$	$x+12, x+7$
$x+13, x+26$	$x+7\ , x+15$	$x+7\ , x+16$	$x+7\ , x+17$	$x+7\ , x+18$	$x+7\ , x+19$
$x+26, x+40$	$x+15, x+14$	$x+16, x+14$	$x+17, x+14$	$x+18, x+14$	$x+19, x+14$
$x+40, x+7$	$x+14, x+22$	$x+14, x+23$	$x+14, x+24$	$x+14, x+25$	$x+14, x+26$
$x+7\ , x+22$	$x+22, x+21$	$x+23, x+21$	$x+24, x+21$	$x+25, x+21$	$x+26, x+21$
$x+22, x+39$	$x+21, x+29$	$x+21, x+30$	$x+21, x+31$	$x+21, x+32$	$x+21, x+33$
$x+39, x+8$	$x+29, x+28$	$x+30, x+28$	$x+31, x+28$	$x+32, x+28$	$x+33, x+28$
$x+8\ , x+27$	$x+28, x+36$	$x+28, x+37$	$x+28, x+38$	$x+28, x+39$	$x+28, x+40$
$x+27, x+4$	$x+36, x+35$	$x+37, x+35$	$x+38, x+35$	$x+39, x+35$	$x+40, x+35$
$x+4\ , x+24$	$x+35, x+43$	$x+35, x+44$	$x+35, x+45$	$x+35, x+46$	$x+35, x+47$
$x+24, x+46$	$x+43, x+42$	$x+44, x+42$	$x+45, x+42$	$x+46, x+42$	$x+47, x+42$
$x+46, x+25$	$x+42, x+1$	$x+42, x+2$	$x+42, x+3$	$x+42, x+4$	$x+42, x+5$
$x+25, x$	$x+1\ , x$	$x+2\ , x$	$x+3\ , x$	$x+4\ , x$	$x+5\ , x$

Example 2. A cyclic decomposition of the graph $\langle 49 \rangle$ into 14-gons (in this case $49 \equiv 21 \pmod{28}$). This decomposition is given in Table 1. In this table the vertices v_i are denoted briefly as i; x denotes an arbitrary vertex v_x of the graph $\langle 49 \rangle$ (all numbers in Table 1 are taken modulo 49). The cyclic decomposition of the graph $\langle 49 \rangle$ into 14-gons will be obtained if the 14-gon K_1 is turned successively 48 times, and each of the 14-gons K_2, K_3, K_4, K_5, K_6 successively 6 times, which makes together 84 of 14-gons.

Now Theorem 2 can be combined with Theorem 1 of [1]:

Theorem 3. *For an arbitrary natural k and for an arbitrary even $p > 2$ there exists a cyclic decomposition of the graph $\langle 2kp + 1 \rangle$ into p-gons.*

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