

REFLECTORS AND COREFLECTORS ON DIAGRAMS

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I. INTRODUCTION

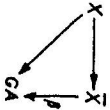
In the fall of 1957 the writer began a Ph.D. dissertation under the direction of E. B. Leach investigating what Kan [1] was to call direct and inverse limits and what Freyd [2] was to call left and right roots (or reflections and coreflections). The work was essentially complete by the time Kan's article [1] on adjoint functors appeared in the Transactions during the following year. Due to circumstances beyond the control of the writer there was a delay in the publication of his results and in the meantime some of the results such as the factorization of left roots into differences of products were published independently by other writers [2]. However, since the results in which the dissertation culminates have not to the writer's knowledge yet appeared it seemed to him worthwhile to write them up for publication, adapting for this purpose the elegant language invented by Kan.

The main tool of this paper is the concept suggested by E. B. Leach of a relative reflection: an object \bar{X} in a category \mathcal{C} is a *relative reflection of an object* X in \mathcal{C} with respect to a functor $G: \mathcal{A} \rightarrow \mathcal{C}$ provided there is a morphism $X \rightarrow \bar{X}$ in \mathcal{C} satisfying the universal mapping property with respect to GA for all objects A in \mathcal{A} . We define a category \mathcal{D} of diagrams over a category \mathcal{A} , in which the diagrams are not necessarily of the same form, and imbed \mathcal{A} as a subcategory of \mathcal{D} by means of a functor $J: \mathcal{A} \rightarrow \mathcal{D}$. If a diagram D has a *subdiagram functor* $D': \mathcal{A} \rightarrow \mathcal{D}$ (see below) and if $L: \mathcal{D} \rightarrow \mathcal{A}$ is a reflector [2] then LD' is a relative reflection of D . Since a reflection of relative reflection of D is a reflection of D (and dually for coreflections) a procedure is obtained for the iteration of reflections and coreflections which leads naturally to the investigation of the associativity, "commutativity", and distributivity of reflectors and coreflectors. Categories $\mathcal{D}(M)$ of diagrams of the form $D: \mathcal{A} \rightarrow \mathcal{A}$ are defined in which for each morphism α in \mathcal{A} , $D\alpha$ is constrained

to lie in a class $M(\alpha)$ of morphisms in \mathcal{A} . Reflectors and coreflectors on these categories are studied. Examples are sums, products, quotients, subobjects, etc. Given functors $\Phi: \mathcal{O}_I(M) \rightarrow \mathcal{A}$ and $\Psi: \mathcal{O}_J(M') \rightarrow \mathcal{A}$ and a diagram D in $\mathcal{O}_{I \times J}(M \times M')$ we define subdiagram functors $\bar{D}_1: \mathcal{O}_I \rightarrow \mathcal{O}_J(M')$ and $\bar{D}_2: \mathcal{O}_J \rightarrow \mathcal{O}_I(M)$ and define the compositions $\Phi \Psi: \mathcal{O}_{I \times J}(M \times M') \rightarrow \mathcal{A}$ and $\Psi \Phi: \mathcal{O}_{I \times J}(M \times M') \rightarrow \mathcal{A}$ by setting $(\Psi \Phi)D = \Psi(\Phi \bar{D}_2)$ and $(\Phi \Psi)D = \Phi(\Psi \bar{D}_1)$. If $\Psi(\Phi \bar{D}_2)$ is isomorphic to $\Phi(\Psi \bar{D}_1)$ then Φ and Ψ are said to commute even though Φ and Ψ have different domains. *Invariance* of reflectors and coreflectors under one another is defined and it is shown that one reflector $\Phi: \mathcal{O}_I(M) \rightarrow \mathcal{A}$ commutes with another $\Psi: \mathcal{O}_J(M') \rightarrow \mathcal{A}$ provided each is invariant under the other. There is a dual result on coreflectors. If $\Phi: \mathcal{O}_I(M) \rightarrow \mathcal{A}$ is a coreflector and $\Psi: \mathcal{O}_J(M') \rightarrow \mathcal{A}$ is a reflector such that each is invariant under the other then Φ does not generally commute with Ψ but there exists a natural transformation $\Psi \Phi \rightarrow \Phi \Psi$. The latter specializes to the celebrated minimax theorem and may be further specialized to the one-sided distributive law $(x \cdot y) + (x \cdot z) < x \cdot (y + z)$ of lattice theory [3].

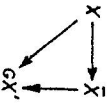
II. RELATIVE REFLECTIONS

Definition. Let $G: \mathcal{A} \rightarrow \mathcal{B}$ be a functor and let $X \rightarrow \bar{X}$ be a morphism in \mathcal{A} . Suppose that for any morphism $X \rightarrow GA$ in which A is an object in \mathcal{A} there exists a unique morphism $\bar{X} \xrightarrow{\beta} GA$ such that



commutes. Then $X \rightarrow \bar{X}$ is a relative reflection and \bar{X} is a relative reflection of X (both with respect to G). When there exists an object X' and a unique morphism $X' \xrightarrow{\alpha} A$ in \mathcal{A} such that $\beta = G\alpha$ then the relative reflection is absolute: in this case \bar{X} is called a reflection of X and $X \rightarrow GX'$ is called a reflection (both with respect to G).

Relative reflections with respect to a functor $G: \mathcal{A} \rightarrow \mathcal{B}$ form a subcategory of \mathcal{A} . A reflection of a relative reflection of an object X is a reflection of X . Furthermore if $X \rightarrow \bar{X}$ is a relative reflection and $X \rightarrow GX'$ is a reflection then there is a unique morphism $\bar{X} \rightarrow GX'$ such that (1) the diagram



commutes and (2) the morphism $\bar{X} \rightarrow GX'$ is a reflection.

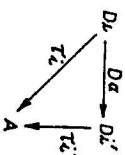
Recall that a diagram D over a category \mathcal{A} is a functor $D: \mathcal{I} \rightarrow \mathcal{A}$ in which \mathcal{I} is a small category [1]. If $D: \mathcal{I} \rightarrow \mathcal{A}$ and $D': \mathcal{I}' \rightarrow \mathcal{A}$ are diagrams, then a mapping $\tau: D \rightarrow D'$ consists of a functor $\tau_1: \mathcal{I} \rightarrow \mathcal{I}'$ onto \mathcal{I}' , together with a natural transformation $\tau_2: D \rightarrow D'\tau_1$. We will write $\tau = (\tau_1, \tau_2)$. (If τ_1 is obvious we will write only τ_2 instead of (τ_1, τ_2) .) A morphism equal to τ_2 for some i in \mathcal{I} is called a component of the mapping.

Definition. If $\sigma: F \rightarrow F'$ is a natural transformation between functors $F, F': \mathcal{X} \rightarrow \mathcal{B}$ and if $G: \mathcal{C} \rightarrow \mathcal{X}$ is a functor then $\sigma G: FG \rightarrow F'G$ is the natural transformation defined by $(\sigma G)c = \sigma(Gc)$ for each object c in \mathcal{C} .

Definition. If $\tau': D' \rightarrow D''$ is another mapping then $\tau: D \rightarrow D''$ is defined by $(\tau' \tau)_1 = \tau'_1 \tau_1$ and $(\tau' \tau)_2 = (\tau'_2 \tau_1) \tau_2$. Consequently we obtain the category \mathcal{D} of diagrams over \mathcal{A} .

There is an obvious imbedding functor $J: \mathcal{A} \rightarrow \mathcal{D}$ and under this imbedding we may regard \mathcal{A} as a subcategory of \mathcal{D} . It follows that a reflector [2] $F: \mathcal{A} \rightarrow \mathcal{A}$ is a (direct) limit functor. By a reflection of a diagram D over \mathcal{A} we mean a reflection of D with respect to J . We will usually suppress mention of J and identify objects A and morphisms α in \mathcal{A} with their corresponding diagrams JA and $J\alpha$.

A mapping $\tau: D \rightarrow A$ in which A is an object of \mathcal{A} consists of a family of morphisms $\tau_i: D_i \rightarrow A$ indexed by objects in \mathcal{I} such that for each morphism $\alpha: i \rightarrow i'$ in \mathcal{I} the diagram

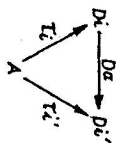


commutes.

A category \mathcal{A} has an opposite category \mathcal{A}^{op} in which the objects and morphisms of \mathcal{A} are the objects and morphisms of \mathcal{A}^{op} but $\text{hom}_{\mathcal{A}}(A, B) = \text{hom}_{\mathcal{A}^{op}}(B, A)$ [4]. Moreover the product $\alpha \circ \beta$ of morphisms α, β in \mathcal{A}^{op} is defined by $\alpha \circ \beta = \beta\alpha$ whenever $\beta\alpha$ is a product in \mathcal{A} .

If $D: \mathcal{I} \rightarrow \mathcal{A}$ and $D': \mathcal{I}' \rightarrow \mathcal{A}$ are diagrams then a mapping $\tau: D \rightarrow D'$ is a functor $\tau_1: \mathcal{I} \rightarrow \mathcal{I}'$ together with a natural transformation $\tau_2: D\tau_1 \rightarrow D'$. In effect a mapping has the same definition as a mapping except that domain and range are interchanged and components of τ from \mathcal{A} are replaced by morphisms from \mathcal{A}^{op} . The converse category \mathcal{D}^* of diagrams is the category whose objects are diagrams and whose morphisms are mappings. Coreflections are defined using mappings.

A comapping $\tau: A \rightarrow D$ in which A is an object in \mathcal{A} and $D: \mathcal{J} \rightarrow \mathcal{A}$ is a diagram consists of a family of morphisms $\tau_i: A \rightarrow D_i$ indexed by objects in \mathcal{J} such that for each morphism $\alpha: i \rightarrow i'$ in \mathcal{J} the diagram



commutes.

Definition. Let $P: \mathcal{J} \rightarrow \mathcal{K}$ be a functor. A functor induced category $P^{-1}(\mathcal{K})$ is defined as follows: the objects are diagrams $E_K: \mathcal{J} \rightarrow \mathcal{K}$ for each object K in \mathcal{K} and there corresponds to each morphism $\beta: K \rightarrow K'$ in \mathcal{K} a mapping $(F_\beta, \sigma_\beta): E_K \rightarrow E_{K'}$. The functor induced category satisfies the properties:

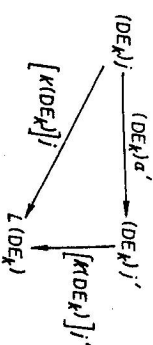
- (1) A morphism α in \mathcal{J} is an image under E_K if and only if $P\alpha = e_K$.
- (2) Each morphism in \mathcal{J} is the product of factors α such that α is either an image under one of the E_K or a component of a mapping (F_β, σ_β) such that $P\alpha = \beta$.
- (3) E_K is an imbedding, i. e. is one-to-one into.
- (4) (F_β, σ_β) is an identity if and only if β is an identity.
- (5) (F_β, σ_β) $(F_{\beta\alpha}, \sigma_{\beta\alpha}) = (F_{\beta\alpha}, \sigma_{\beta\alpha})$.

If such a category $P^{-1}(\mathcal{K})$ exists then the mapping $P^{-1}: \mathcal{K} \rightarrow P^{-1}(\mathcal{K})$, defined by $P^{-1}K = E_K$ for each object K in \mathcal{K} and $P^{-1}\beta = (F_\beta, \sigma_\beta)$ for each morphism β in \mathcal{K} , is by (4) and (5) a functor. The functor P induces a factorization of \mathcal{J} into subcategories and morphisms between the subcategories. Thus $P^{-1}: \mathcal{K} \rightarrow P^{-1}(\mathcal{K})$ may be called a factorization of \mathcal{J} .

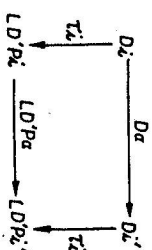
Definition. Let $D: \mathcal{J} \rightarrow \mathcal{A}$ be a diagram and let $\bar{D}: P^{-1}(\mathcal{K}) \rightarrow \mathcal{A}$ be a functor such that $D(E_K) = DE_K$ and $D(F_\beta, \sigma_\beta) = (F_\beta, D\sigma_\beta)$. Thus \bar{D} is a restriction of D to the subcategories of \mathcal{J} in $P^{-1}(\mathcal{K})$ and to the mappings between the subcategories. Let $D' = DP^{-1}: \mathcal{K} \rightarrow \mathcal{A}$ then $D'K = \bar{D}P^{-1}K = DE_K: \mathcal{J} \rightarrow \mathcal{A}$ is a diagram for each object K in \mathcal{K} . D' is called a subdiagram functor of D and the functor P is a projection functor of D .

Theorem 1. Suppose $L: \mathcal{A} \rightarrow \mathcal{A}'$ is a reflector and $\kappa: E_{\mathcal{A}} \rightarrow L$ is the natural transformation induced by L , [1] (called a front adjunction by Mac Lane [4]). Let $D: \mathcal{J} \rightarrow \mathcal{A}$ be a diagram and let $D': \mathcal{K} \rightarrow \mathcal{A}$ be a subdiagram functor of D . If $P: \mathcal{J} \rightarrow \mathcal{K}$ is a projection functor of D and $\tau: D \rightarrow LD'P$ is the transformation such that $\tau E_K = \kappa(D'k)$ for each object k in \mathcal{K} then $(P, \tau): D \rightarrow LD'P$ is a relative reflection. (By the object $L(D'E_k)$ of \mathcal{A}' , the range-object of $\kappa(D'k)$, is here meant the diagram $JL(D'E_k)$ where $J: \mathcal{A} \rightarrow \mathcal{A}'$ is the chosen imbedding of \mathcal{A} in \mathcal{A}' .)
Proof. In order to establish that the transformation $\tau: D \rightarrow LD'P$ is natural let $\alpha: i \rightarrow i'$ be a morphism in \mathcal{J} . There are two cases to consider.

Case 1. There exists a morphism $\alpha': j \rightarrow j'$ in \mathcal{J}_k such that $E_k\alpha' = \alpha$ for some object k in \mathcal{K} . It follows that $(D'k)\alpha' = (DE_k)\alpha = D\alpha$ and since the natural transformation $\kappa(DE_k): DE_k \rightarrow L(DE_k)$ may be regarded as a mapping, the diagram

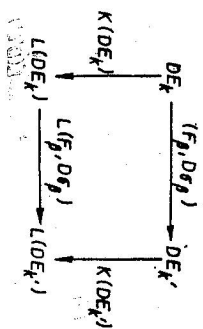


commutes. But $L(DE_k) = L(D'Pj) = L(D'Pj')$ and $P\alpha = \alpha'$ and consequently the diagram

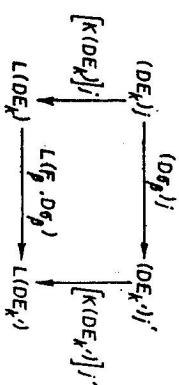


commutes.

Case 2. $P\alpha = \beta$ and α is a component of (F_β, σ_β) , i. e. there exists j such that $\sigma_j = \alpha$. Let k and k' be objects in \mathcal{K} such that $E_k j = i$ and $E_{k'} j' = i'$. The diagram



commutes and consequently for each j in \mathcal{J}_k the diagram



commutes. But

$$\begin{aligned} L(DE_k) &= L(D'k) = LD'P_i \\ L(DE_{k'}) &= L(D'k') = LD'P_i', \\ LD'P_\alpha &= LD'\beta = L(F_\beta, D\sigma_\beta), \end{aligned}$$

and therefore the diagram

$$\begin{array}{ccc} D_i & \xrightarrow{D\alpha} & D_i' \\ \tau_i \downarrow & & \downarrow \tau_i' \\ LD'P_i & \xrightarrow{LD'P_\alpha} & LD'P_i' \end{array}$$

commutes.

Since all morphisms α in \mathcal{S} are products of morphisms of the types in cases 1 and 2 it follows that τ is natural.

Now let $\pi: D \rightarrow A$ be an arbitrary mapping into \mathcal{A} . For each object k in \mathcal{K} there is a unique morphism ωk such that

$$\begin{array}{ccc} D'k & \xrightarrow{K(D'k)} & L(D'k) \\ & \searrow \pi E_k & \downarrow \omega k \\ & & A \end{array}$$

commutes.

The transformation $\omega: LD' \rightarrow A$ is a mapping because if $\beta: k \rightarrow k'$ is a morphism in \mathcal{K} then $LD'\beta$ is the unique morphism such that

$$\begin{array}{ccccc} D'k & & D'\beta & & D'k' \\ \downarrow K(D'k) & \xrightarrow{\pi E_k} & & \xrightarrow{\pi E_{k'}} & \downarrow K(D'k') \\ & \searrow \omega k & A & \swarrow \omega k' & \\ & & L(D'\beta) & & \\ \downarrow L(D'k) & & & & \downarrow L(D'k') \end{array}$$

commutes. Since $\kappa(D'k) = \tau E_k$ it follows from diagram (1) that $\omega: LD' \rightarrow A$ is the unique mapping such that

$$\begin{array}{ccc} D & \xrightarrow{\tau} & LD' \\ & \searrow \kappa & \downarrow \omega \\ & & A \end{array}$$

commutes.

Theorem 1 can be readily applied to proving known theorems such as the associativity of sums and products and that reflections of diagrams may be factored into a summation followed by a "quotient" and that coreflections may be factored into a multiplication followed by a "difference". (This may be seen from the following.)

Definition. $\hat{\mathcal{S}}$ is the largest discrete subcategory of the category \mathcal{A} . (A discrete category [2] is a category whose only morphisms are identities.)

Definition. If $D: \mathcal{S} \rightarrow \mathcal{A}$ is a diagram and $E: \hat{\mathcal{S}} \rightarrow \mathcal{S}$ is an injection then $\hat{D}: \hat{\mathcal{S}} \rightarrow \mathcal{A}$ is the family defined by $\hat{D} = DE$. (A family $(D_i)_{i \in I}$ is a diagram $D: \mathcal{S} \rightarrow \mathcal{A}$ in which \mathcal{S} is a discrete category). Thus \hat{D} is the largest family in D .

We finally show that every diagram $D: \mathcal{S} \rightarrow \mathcal{A}$ has a subdiagram functor $D': \mathcal{K} \rightarrow \mathcal{D}$ of the form $\Delta: \hat{D} \rightarrow D$ in which Δ is a family of mappings. Such a functor may be called an *object-map factorization*.

Let $E: \hat{\mathcal{S}} \rightarrow \mathcal{S}$ be an injection and for each morphism $\alpha: i \rightarrow i'$ in $\hat{\mathcal{S}}$ define $(F_\alpha, \sigma_\alpha): E \rightarrow E$ as a mapping having α as a component and whose other components are identity morphisms. Let \mathcal{K} be the category whose only object is E and whose morphisms are mappings $(F_\alpha, \sigma_\alpha): E \rightarrow E$. Define $P_i = E$ and $P_\alpha = (F_\alpha, \sigma_\alpha)$ for each object i and each morphism α in \mathcal{K} . Define $D': \mathcal{K} \rightarrow \mathcal{D}$ by setting $D'E = \hat{D}$ and $D'P_\alpha = (F_\alpha, D\sigma_\alpha): \hat{D} \rightarrow D$. Then D is of the form $\Delta: \hat{D} \rightarrow D$ in which Δ is a family of mappings $D'P_\alpha$ each of which consists of D_α together with identity morphisms.

III. "COMMUTATIVITY" OF REFLECTORS AND COREFLECTORS

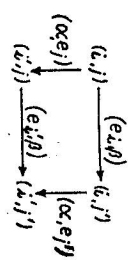
In the category of R -modules in which R is a commutative ring with identity the direct sum functor Φ has as its domain families of modules and the quotient functor Ψ has as its domain pairs of modules of which one is a submodule of the other. It is known that "the quotient of the sums is the sum of the quotients" so that in a sense Φ "commutes" with Ψ although Φ and Ψ have different domains. In this section we characterize such functors in the cases they may be regarded as reflectors and coreflectors and define the composition of such functors relative to which they commute.

Given a diagram $D: \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{A}$ there correspond subdiagram functors of the forms $D_1: \mathcal{S} \rightarrow \mathcal{D}$ and $D_2: \mathcal{S} \rightarrow \mathcal{D}$. A theorem on the commutativity of reflectors and coreflectors will be proved by applying Theorem 1 to these functors.

Let $E_i: \mathcal{S} \rightarrow \mathcal{S} \times \mathcal{S}$ be the imbedding functor defined by $E_i i = (i, j)$, $E_i \alpha = (\alpha, e_j)$ and let $E^i: \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$ be the imbedding functor defined by $E^i i = (i, j)$, $E^i (e_i, \beta) = \beta$ for objects i in \mathcal{S} , j in \mathcal{S} and morphisms α in \mathcal{S} , β in \mathcal{S} . Since the E^i are all of the same form, a mapping or a comapping $E^i \rightarrow$

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E' may be regarded as a natural transformation. For each morphism $\alpha: i \rightarrow i'$ in \mathcal{J} let $\alpha^*: E^i \rightarrow E^{i'}$ be the transformation defined by $(\alpha^x)j = (\alpha, e_j)$ for each object j in \mathcal{J} . To show that α^* is natural let $\beta: j \rightarrow j'$ be a morphism in \mathcal{J} . Then $E^i\beta = (e_i, \beta)$ and $E^{i'}\beta = (e_{i'}, \beta)$ and the diagram



commutes. The natural transformation $\alpha^*: E^i \rightarrow E^{i'}$ induces a natural transformation $D^{\alpha^*}: DE^i \rightarrow DE^{i'}$. Define a functor $D_1: \mathcal{J} \rightarrow \mathcal{D}$ by setting $D_1 i = DE^i$ and $D_1 \alpha = D^{\alpha^*}$ for each object i and each morphism α in \mathcal{J} .

A similar procedure generates a functor $D_2: \mathcal{J} \rightarrow \mathcal{D}$. For each morphism $\beta: j \rightarrow j'$ in \mathcal{J} and for each object i in \mathcal{J} we set $\sigma \beta i = (e_i, \beta)$ and then define $D_2 \beta = D \sigma \beta$ and set $D_2 j = DE_j$ for each object j in \mathcal{J} . If $\alpha: i \rightarrow i'$ is a morphism in \mathcal{J} and $\beta: j \rightarrow j'$ is a morphism in \mathcal{J} then

$$\begin{aligned} (\alpha, \beta) &= (\alpha, e_j) \circ (e_i, \beta) \\ &= E_j \alpha \circ \sigma \beta i \\ &= \alpha^x j' \circ E^i \beta. \end{aligned}$$

It follows that each morphism (α, β) in $\mathcal{J} \times \mathcal{J}$ is the product of two factors one of which is an image under E_j and the other is a component of one the mappings $E_j \rightarrow E_{j'}$, and furthermore one of them is an image under E^i and the other is a component of a mapping $E^i \rightarrow E^{i'}$. Consequently $D_1: \mathcal{J} \rightarrow \mathcal{D}$ and $D_2: \mathcal{J} \rightarrow \mathcal{D}$ are subdiagram functors.

Corresponding to the subdiagram functors are the projection functors $P_1: \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$ and $P_2: \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$. It follows from Theorem 1 and its dual that if $L: \mathcal{D} \rightarrow \mathcal{A}$ is a reflector or a coreflector then $L(LD_1)$ is isomorphic to $L(LD_2)$. We now apply this result to the study of the commutativity of reflectors and coreflectors.

Definition. Let \mathcal{J} be a small category, let \mathcal{A} be a category and let \mathcal{D} be the category of diagrams of the form $D: \mathcal{J} \rightarrow \mathcal{A}$. For every morphism α in \mathcal{J} let $M(\alpha)$ be a class of morphisms in \mathcal{A} such that if α is an identity then $M(\alpha)$ is the class of identity morphisms in \mathcal{A} . Define $\mathcal{D}_1(M)$ as the full subcategory of \mathcal{D} such that $D\alpha$ belongs to $M(\alpha)$ for every morphism α in \mathcal{J} .

Examples. 1. If \mathcal{J} is a discrete category then $\mathcal{D}_1(M) = \mathcal{D}$ and a reflector $\mathcal{D}_1(M) \rightarrow \mathcal{A}$ is a sum functor and a coreflector $\mathcal{D}_1(M) \rightarrow \mathcal{A}$ is a product functor. 2. Let \mathcal{A} be the category of modules over a commutative ring R with identity and let \mathcal{J} be a category consisting of two objects i and i' (with corresponding identities) and two morphisms $\alpha: i \rightarrow i'$ and $\alpha': i' \rightarrow i$. Let $M(\alpha)$ consist of

Джонсон А. А., *Рефлекторы и коррефлекторы на диаграммах*, *Мат.-физ. вестн.* 16 (1966), 309—319. (Англ.)

Определяются понятия относительной рефлексии и поддиаграммных функторов. Показывается, что если $D': \mathcal{J} \rightarrow \mathcal{D}$ — поддиаграммный функтор и $L: \mathcal{D} \rightarrow \mathcal{A}$ — рефлектор, то LD' есть относительно рефлексивная D . Приводится общее исследование коммутативности функторов на диаграммах подобно тому как, например, при коммутативности прямых сумм с частными в теории модулей над кольцом.

Ричан Б., *Замечания о эргодичности*, *Мат.-физ. вестн.* 16 (1966), 320—323. (Англ.)

В статье исследуется критерий эргодичности, доказанный в одной статье Р. В. Прейсэндорфера и Б. В. Рооса. Автор доказывает, что в этом критерии можно заменить условие сохранения меры более слабым условием нескжимаемости.

Юнович Э., Лешо И., *Замечание о покрытии плоскости неконгруэнтными кругами*, *Мат.-физ. вестн.* 16 (1966), 324—328. (Немецк.)

В статье дается для $0,088 \leq q \leq 0,169$ верхнее ограничение минимальной плотности покрытия эвклидовой плоскости кругами, радиусы которых выты из интервала $\langle q, 1 \rangle$.

Якубин Я., *Дедекндовы сечения в прямом произведении частично упорядоченных групп*, *Мат.-физ. вестн.* 16 (1966), 329—336. (Немецк.)

Пусть $E(G)$ — дедекндово расширение направленной вполне незамкнутой группы G . Доказывается теорема: Если G — полное прямое произведение частично упорядоченных групп G_i ($i \in I$), то $E(G)$ является полным прямым произведением структурно упорядоченных групп $E(G_i)$ ($i \in I$).

Вага И., *Системы Рикати со совпадающими или различными фундаментальными линиями на линейчатой поверхности*, *Мат.-физ. вестн.* 16 (1966), 337—348. (Немецк.)

Исследуются свойства квадратичных и аксиальных систем Рикати, которые имеют заданные совпадающие или различные фундаментальные линии, на линейчатой неэвклидовой поверхности в проективном пространстве R_3 .

Роса А., *О циклических разложениях полного графа на $(4n+2)$ -углышки*, *Мат.-физ. вестн.* 16 (1966), 349—352. (Англ.)

В статье дается построение циклического разложения полного графа с $2kr + 1$ вершинами на r -угольники для случая $r \equiv 2 \pmod{4}$.

Копит А., *О δ -преобразовании антисимметрических графов*, *Мат.-физ. вестн.* 16 (1966), 353—356. (Словацк.; реф. англ.)

Приводится необходимое и достаточное условие для того, чтобы заданный антисимметрический граф без петель можно было преобразовать с помощью δ -преобразований в произвольный граф с тем же множеством вершин и с теми же внешними и внутренними подстепенями отдельных вершин; под δ -преобразованием понимается обращение направления ребер некоторого цикла длины 3.

SYNOPSIS

Знам Ш., *Обобщение одного результата из теории чисел*, Mat.-fyz. časop. 16 (1966), 357—361. (Англ.)

Статья занимается обобщением результатов И. Штра о разложении натуральных чисел в множества, не содержащие сумм своих элементов, а также применение этих результатов при решении известной задачи из теории графов.

Сиглеш Э., Тучек Й., *Эффективность ядерных эмulsion при изучении мембранураст*, Mat.-fyz. časop. 16 (1966), 362—365. (Русск.)

В статье исследуются ионизационные обмены тренов частиц в ядерных фотографических эмульсиях НИКФИ ВР-2, охлажденных до температур жидкого азота и гелия.

Паулин М.-Тотова В., *Диффузия цинка в примесных кристаллах NaCl*, Mat.-fyz. časop. 16 (1966), 366—376. (Англ.)

Статья содержит доказательства концентрации независимости коэффициента диффузии цинка в примесных кристаллах NaCl, а также анализ диффузии без градиента вакансий. Приводится сравнение изотерм для NaCl + CaCl₂ и NaCl + CdCl₂.

Johnson A. A., *Reflectors and coreflectors on diagrams*, Mat.-fyz. časop. 16 (1966), 309—319. (English.)

The concepts of relative reflection and subdiagram functors are defined and it is shown that if $D' : \mathcal{K} \rightarrow \mathcal{D}$ is a subdiagram functor and $L : \mathcal{D} \rightarrow \mathcal{A}$ is a reflector then LD' is a relative reflection of D . A general investigation is made of the "commutativity" of functors on diagrams such as occurs, for example, in the "commutativity" of direct sums with quotients in theory of modules over a ring.

Riečan V., *Note on ergodicity*, Mat.-fyz. časop. 16 (1966), 320—323. (English.)

The article deals with a criterion of ergodicity demonstrated in an article of R. W. Prinsendorfer and V. W. Roos. The author proves that in the criterion the assumption of measure preservation can be replaced by the weaker assumption of incompressibility.

Jucović E., Leško J., *A note on covering of the plane by non-congruent circles*, Mat.-fyz. časop. 16 (1966), 324—328. (German.)

In this paper the upper bound of the minimal covering density of the Euclidean plane with circles of which radii are from the interval $\langle q, 1 \rangle$, $0,088 \leq q \leq 0,169$, is given.

Jakubík J., *The Dedekind cuts in the direct product of partially ordered groups*, Mat.-fyz. časop. 16 (1966), 329—336. (German.)

Let G be a directed group which is completely integrally closed and let $E(G)$ be the Dedekind extension of G . Main result: If G is the complete direct product of partially ordered groups G_i ($i \in I$), then $E(G)$ is the complete direct product of the lattice-ordered groups $E(G_i)$.

Vala J., *Riccati's families with coincident or imaginary fundamental curves on the ruled surface*, Mat.-fyz. časop. 16 (1966), 337—348. (German.)

This paper deals with the properties of the quadratic and the axial Riccati's families with given coincident or imaginary fundamental curves on the nondevelopable ruled surface in a projective space P_3 .

Rosa A., *On cyclic decompositions of the complete graph into $(4m+2)$ -gons*, Mat.-fyz. časop. 16 (1966), 349—352. (English.)

In the paper a cyclic decomposition of the complete graph with $2kp+1$ vertices into p -gons in the case $p \equiv 2 \pmod{4}$ is constructed.

Kotzig A., *On δ -transformations of antisymmetric graphs*, Mat.-fyz. časop. 16 (1966), 353—356. (Slovak, English summary.)

The paper deduces the necessary and sufficient condition for the possibility of changing a given antisymmetric graph without loops into any graph with the same set of vertices and same outward and inward demi-degrees of the vertices by a δ -transformations, i. e. by the successive turning of orientation of edges of some cycle of the length 3.

Znám Š., *Generalisation of a number-theoretical problem*, Mat.-fyz. časop. 16 (1966), 357—361. (English.)

The article contains the generalisation of Schur's results regarding the splitting of natural numbers in sum-free sets and application of results by solving of a known problem on colouring of edges of complete graphs.

Sileš E., Tuček J., *Efficiency of nuclear emulsions under low temperatures*. Mat.-fyz. časop. 16 (1966), 362—365. (Russian.)

In this paper we have examined the changes of the ionization of particles tracks in nuclear photographic emulsions NIKFI BR-2, cooled on the temperature of liquid nitrogen and helium.

Pauliny-Tóthová V., *Diffusion of zinc in doped NaCl crystals*. Mat.-fyz. časop. 16 (1966), 366—376. (English.)

In this paper the concentration independence of the diffusion coefficient of zinc in doped NaCl crystals has been proved and diffusion without vacancy gradient has been analyzed. The comparison of the diffusion isotherms for NaCl + CaCl₂ and NaCl + CdCl₂ has been done.

monomorphisms in \mathcal{A} and let $M(\alpha')$ consist of trivial homomorphisms so that $D\alpha': D_i \rightarrow D'_i$ maps D_i onto the zero element of D'_i . Then a reflector $\Phi: \mathcal{D}_I(M) \rightarrow \mathcal{A}$ is a quotient functor and $\Phi D = D'_i/D_i$ for every diagram D in $\mathcal{D}_I(M)$.

Definition. Let $\Phi: \mathcal{D}_I(M) \rightarrow \mathcal{A}$ and $\Psi: \mathcal{D}_I(M') \rightarrow \mathcal{A}$ be reflectors or coreflectors.

Then Ψ is invariant under Φ (or Φ -invariant) provided for every morphism τ in $\mathcal{D}_I(M)$ the morphism $\Phi\tau$ is in class $M'(\alpha')$ of Ψ whenever every component of τ is in $M'(\alpha')$.

Examples. 3. Sum and product functors are invariant under any reflector of corefactor $\Phi: \mathcal{D}_I(M) \rightarrow \mathcal{A}$ since as shown in example 1 the classes $M'(\alpha')$ of sum and product functors contain only identities.

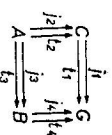
4. A quotient functor as in example 2 is direct sum invariant. For if $\tau: D \rightarrow D'$ is a morphism in \mathcal{D}_I and τ_i is a monomorphism for each object i in \mathcal{J} then $\sum_{i \in I} \tau_i$ is a monomorphism and if τ_i is trivial for each object i in \mathcal{J} then $\sum_{i \in I} \tau_i$ is trivial.

Definition. Let $\mathcal{D}_{I \times J}(M \times M')$ be the full subcategory of $\mathcal{D}_{I \times J}$ such that $D(\alpha, e) \in M(\alpha)$, $D(e, \beta) \in M'(\beta)$ whenever α (or β) is a morphism in \mathcal{J} (or \mathcal{J}') and e (or e') is an identity in \mathcal{J} (respectively). Let $D: \mathcal{J} \times \mathcal{J}' \rightarrow \mathcal{A}$ be a diagram in $\mathcal{D}_{I \times J}(M \times M')$ and let $D_1: \mathcal{J} \rightarrow \mathcal{A}$ and $D_2: \mathcal{J}' \rightarrow \mathcal{A}$ be the subdiagram functors of D as defined above. Let $\bar{D}_1: \mathcal{J} \rightarrow \mathcal{D}_I(M)$ and $\bar{D}_2: \mathcal{J}' \rightarrow \mathcal{D}_I(M')$ be restrictions of D_1 and D_2 and suppose $\Phi: \mathcal{D}_I(M) \rightarrow \mathcal{A}$ and $\Psi: \mathcal{D}_I(M') \rightarrow \mathcal{A}$ are functors. Define the composition $\Phi\Psi: \mathcal{D}_{I \times J}(M \times M') \rightarrow \mathcal{A}$ and the composition $\Psi\Phi: \mathcal{D}_{I \times J}(M \times M') \rightarrow \mathcal{A}$ by setting $(\Phi\Psi) D = \Phi(\Psi\bar{D}_1)$ and $(\Psi\Phi) D = \Psi(\Phi\bar{D}_2)$.

Theorem 2. If Φ and Ψ are reflectors then under the composition just defined Φ and Ψ commute provided each is invariant under the other.

Proof. Since Φ is Ψ -invariant it follows that $\Psi\bar{D}_1: \mathcal{J} \rightarrow \mathcal{A}$ is a diagram in $\mathcal{D}_I(M)$ and since Ψ is Φ -invariant it follows that $\Phi D_2: \mathcal{J}' \rightarrow \mathcal{A}$ is a diagram in $\mathcal{D}_I(M')$. Furthermore $\Psi(\Phi\bar{D}_2)$ is isomorphic to $\Phi(\Psi\bar{D}_1)$ since Φ and Ψ are reflectors.

Examples. 5. In the category of R -modules direct sums commute with quotients. However, quotients are not self-invariant: otherwise we should be able to prove that if $A \subset B \subset C$ and if $A \subset C \subset G$ then $(G/C)/(B/A)$ is isomorphic to $(G/B)/(C/A)$ using the diagram



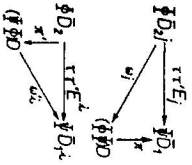
in which the j 's are monomorphisms and the t 's are trivial. In case $C = A$ the monomorphisms are preserved under quotients and it follows that $(G/A) / (B/A)$ is isomorphic to G/B .

By a trivial modification of the proof of Theorem 2 it follows that coreflectors $\Phi: \mathcal{D}_1(M) \rightarrow \mathcal{A}$ and $\Psi: \mathcal{D}_1(M') \rightarrow \mathcal{A}$ commute provided each is invariant under the other.

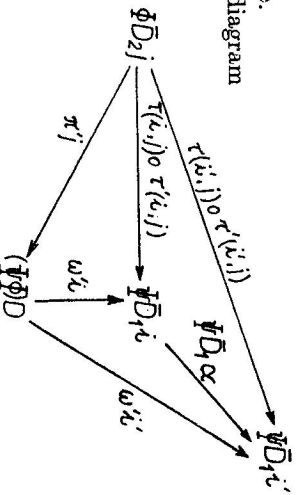
Definition. Morphisms between diagrams may be generalized as follows: Let \mathcal{K} be a small category and let $D: \mathcal{J} \rightarrow \mathcal{A}$ and $D': \mathcal{J} \rightarrow \mathcal{A}$ be diagrams such that there are projectors $P: \mathcal{K} \rightarrow \mathcal{J}$ and $P': \mathcal{K}' \rightarrow \mathcal{J}$ onto \mathcal{J} and onto \mathcal{J}' ; then a \mathcal{K} -morphism $D \rightarrow D'$ is a natural transformation $DP \rightarrow D'P'$. A mapping $D \rightarrow D'$ is an \mathcal{J} -morphism and a coreflection is a \mathcal{J} -morphism.

Let $\Phi: \mathcal{D}_1(M) \rightarrow \mathcal{A}$ be a coreflector and let $\Psi: \mathcal{D}_1(M') \rightarrow \mathcal{A}$ be a reflector and suppose that each is invariant under the other. Let $D: \mathcal{J} \times \mathcal{J}' \rightarrow \mathcal{A}$ be a diagram in $\mathcal{D}_1(M)$ and $\Phi \bar{D}_2: \mathcal{J} \rightarrow \mathcal{A}$ is a diagram in $\mathcal{D}_1(M')$.

There is a natural transformation $\tau: D \rightarrow \Psi \bar{D}_1 P_1$ that defines a relative reflection $D \rightarrow \Psi \bar{D}_1$ and there is a natural transformation $\tau': \Phi \bar{D}_2 P_2 \rightarrow D$ that defines a relative coreflection $\Phi \bar{D}_2 \rightarrow D$. Consequently the natural transformation $\tau \tau': \Phi \bar{D}_2 P_2 \rightarrow \Psi \bar{D}_1 P_1$ is an $\mathcal{J} \times \mathcal{J}'$ -morphism $\Phi \bar{D}_2 \rightarrow \Psi \bar{D}_1$. Hence for every object i in \mathcal{J} there is a mapping $(\tau \tau')^{E_i}: \Phi \bar{D}_2 \rightarrow \Psi \bar{D}_1 i$ and for every object j in \mathcal{J}' there is a mapping $(\tau \tau')^{E_j}: \Phi \bar{D}_2 j \rightarrow \Psi \bar{D}_1 j$. Let $\pi: \Phi \bar{D}_2 \rightarrow (\Psi \Phi) D$ be a reflection and let $\pi: (\Phi \Psi) D \rightarrow \Psi \bar{D}_1$ be a coreflection. Then obviously there exist unique morphisms ω_i 's and ω_j 's for each object i in \mathcal{J} and j in \mathcal{J}' that makes the diagrams

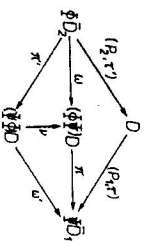


commutative.
Since the diagram



commutes it follows that $\omega_i: (\Psi \Phi) D \rightarrow \Psi \bar{D}_1$ is a comapping. Similarly $\omega_j: \Phi \bar{D}_2 \rightarrow (\Phi \Psi) D$ is a mapping.

We now show that there exists a unique morphism ν such that the diagram



commutes. The existence of such a morphism follows from the existence of a unique morphism ν such that $\nu \tau' = \omega$. This yields $\pi \nu \tau' = \pi \omega = \tau \tau'$ and therefore $\pi \nu = \omega'$, which establishes commutativity.

If the natural transformation $\Psi \Phi \rightarrow \Phi \Psi$ were an equivalence then reflection functors would commute naturally with coreflection functors provided that each is invariant under the other. However, there are counterexamples such as (disjoint) sums and products in the category of sets. Nonetheless, we obtain a generalization of the celebrated minimax inequality: namely that there is a natural transformation $\sum_{i \in I} \prod_{j \in J} \rightarrow \prod_{i \in I} \sum_{j \in J}$ from which we have as a special case the one-sided distributive law $(x \cdot y) + (x \cdot z) \rightarrow x \cdot (y + z)$. For the cases in which sum and product are lattice operations these reduce to the usual laws. [3]

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