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# ADDITIONAL NOTE TO OUR PAPER „A GENESIS FOR COMBINATORIAL IDENTITIES"

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In the paper [1] we have described a certain method by means of which we can derive some combinatorial formulas. In this note we introduce another similar method.

Theorem. *Let $n$ be a natural number, $x$ an arbitrary complex number and $a_1, a_2, \ldots, a_n, a_{n+1}$ the given distinct complex numbers, with the condition $a_k = a_{k-n-1}$ for $k > n + 1$. Then the following relation holds*

$$\sum_{i=1}^{n+1} \frac{(x + a_i)(x + a_{i+1}) \ldots (x + a_{i+n-1})}{(a_i - a_{i-1})(a_{i+1} - a_{i-1}) \ldots (a_{i+n-1} - a_{i-1})} = 1 . \tag{1}$$

Proof. (1) is an algebraic equation of degree $n$ in $x$. But it has $(n + 1)$ roots

$$-a_1, -a_2, \ldots, -a_n, -a_{n+1}. \tag{2}$$

Therefore it is an identity.

In fact the factor $(x + a_k)$, $k = 1, 2, \ldots, n$, $(n + 1)$ occurs in all members on the left side of this equation except in member with $i = k + 1$. Thus for $x = -a_k$ only the member

$$\frac{(-a_k + a_{k+1})(-a_k + a_{k+2}) \ldots (-a_k + a_{k+n})}{(a_{k+1} - a_k)(a_{k+2} - a_k) \ldots (a_{k+n} - a_k)} = 1 \tag{3}$$

is different from zero.

Example. Let $a_i = i$. In this case equation (1) gives

$$\frac{(x + 1)(x + 2) \ldots (x + n)}{(-n)[-(n - 1)] \ldots (-2)(-1)} + \frac{(x + 2)(x + 3) \ldots (x + n + 1)}{1 . 2 . \ldots . n} +$$
$$+ \frac{(x + 3)(x + 4) \ldots (x + n + 1)}{1 . 2 . \ldots . (n - 1)} \cdot \frac{x + 1}{-1} + \frac{(x + 4)(x + 5) \ldots (x + n + 1)}{1 . 2 . \ldots . (n - 2)} \cdot$$
$$\cdot \frac{(x + 1)(x + 2)}{(-1)(-2)} + \ldots + \frac{x + n + 1}{1} \cdot$$
$$\cdot \frac{(x + 1)(x + 2) \ldots (x + n - 1)}{[-(n - 1)][-(n - 2)] \ldots (-2)(-1)} = 1$$

or

$$\sum_{k=0}^{n} (-1)^k \binom{x + n + 1}{n - k} \binom{x + k}{k} = 1. \tag{4}$$

In virtue of identity

$$\binom{x + n + 1}{n - k} \binom{x + k}{k} = \binom{x + n + 1}{n + 1} \binom{n}{k} \frac{n + 1}{x + k + 1} \tag{5}$$

we have therefrom

$$\sum_{k=0}^{n} (-1)^k \frac{1}{x + k + 1} \binom{n}{k} = \left[\binom{x + n + 1}{n + 1} (n + 1)\right]^{-1}. \tag{6}$$

This is a generalisation of the well-known relation

$$\sum_{k=0}^{n} (-1)^k \frac{1}{n + k + 1} \binom{n}{k} = \frac{(n!)^2}{(2n + 1)!} . \tag{7}$$

See [2].

Remark. Let us only remark that the identity (4) can be obtained in another with the aid of Cauchy's identity $\sum_{k=0}^{n} \binom{x}{k} \binom{y}{1 - k} = \binom{x + y}{n}$

$$1 = \binom{n}{n} = \binom{x + n + 1 - (x + 1)}{n} = \sum_{k=0}^{n} \binom{x + n + 1}{n - k} \binom{-x - 1}{k} = \tag{8}$$
$$= \sum_{k=0}^{n} (-1)^k \binom{x + n + 1}{n - k} \binom{x + k}{k}.$$

## REFERENCES


[1] Bartoš P., Kaucký J., *A genesis for combinatorial identities*, Mat.-fyz. časop. *16* (1966), 31–40.

[2] Turán P., *Problème* 51, Mat. lapok *7* (1956), 141.



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