ABSTRACT FORMULATION OF SOME THEOREMS OF MEASURE THEORY

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It is well known, that some theorems of measure theory and its applications can be formulated and proved by means of some properties of the system of all sets of measure zero only(1). In this paper three theorems will be proved by means of sets of measure „less than ε". Instead $\mu(E) < 1/n$ we shall write $E \in \mathcal{N}_n$, where $\mathcal{N}_n$ signifies some system of sets. In the abstract form (i. e. without measure) we shall prove Egoroff's theorem, Luzin's theorem and the statement that every Baire measure is regular.

Throughout the article we shall suppose that some σ-ring $\mathcal{S}$ of subsets of an abstract space X is given. Sets belonging to $\mathcal{S}$ will be called measurable (as usually in measure theory). Also some other notions, as measurable function or monotone system will be understood in the usual sense, laid down in book [1].

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In this part Egoroff's theorem will be proved. Let E be an abstract set, $\mathcal{S}$ a σ-algebra of subsets of E, $\{\mathcal{N}_n\}$ a sequence of subsystems of the system $\mathcal{S}$. We shall use some of the following properties of the sequence $\{\mathcal{N}_n\}$:

(i) $\emptyset \in \mathcal{N}_n$ for all n.

(ii) For any positive integer n there is a sequence $\{k_i\}$ of positive integers such, that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{N}_n$, if $E_i \in \mathcal{N}_{k_i}$ $(i = 1, 2, \ldots)$.

(iii) If $\{E_i\}$ is a sequence of sets in $\mathcal{S}$, $E_{i+1} \subset E_i$ $(i = 1, 2, \ldots)$, $\bigcap_{i=1}^{\infty} E_i = \emptyset$, then for any n there is an m with $E_m \in \mathcal{N}_n$.

The property (ii) substitutes the σ-subadditivity, the property (iii) the continuity of a measure. If $(X, \mathcal{S}, \mu)$ is a measure space, $E \in \mathcal{S}$, $\mu(E) < \infty$, $\mathcal{N}_n = \{F \in \mathcal{S} : F \subset E,\ \mu(F) < 1/n\}$, then the sequence of the systems $\{\mathcal{N}_n\}$ satisfies the suppositions (i)—(iii).

(1) See [2], [3].

Theorem 1. *Let $\{\mathcal{N}_n\}$ be a sequence of subsystems of the system $\mathcal{S}$ fulfilling the conditions (ii) and (iii). If $\{f_k\}$ is a sequence of finite measurable functions which converges on E to a finite function f, then for any n there is a set $F \in \mathcal{N}_n$ such that $\{f_k\}$ converges uniformly on $E - F$.*

Proof. Put

$$E_{m,p} = \left\{ x : |f_k(x) - f(x)| < \frac{1}{m} \text{ for any } k \geqq p \right\}.$$

Clearly

$$\bigcup_{p=1}^{\infty} E_{m,p} = E,\ E_{m,p} \subset E_{m,p+1}\ (p = 1, 2, \ldots),$$

hence

$$(1) \qquad \bigcap_{p=1}^{\infty} (E - E_{m,p}) = \emptyset,\ E - E_{m,p} \supset E - E_{m,p+1}\ (p = 1, 2, \ldots).$$

Let n be any positive integer, $\{k_i\}$ the sequence of positive integers from the condition (ii). As it follows by (iii) and (1), to any m there is a positive integer $p(m)$ such that $E - E_{m,p(m)} \in \mathcal{N}_m$.

Hence

$$(2) \qquad E - E_{k_i,p(k_i)} \in \mathcal{N}_{k_i}.$$

If we put $F = E - \bigcap_{i=1}^{\infty} E_{k_i,p(k_i)}$, then $\{f_k\}$ converges uniformly on $E - F$ to f and by (2) and (ii) it is

$$F = E - \bigcap_{i=1}^{\infty} E_{k_i,p(k_i)} = \bigcup_{i=1}^{\infty} (E - E_{k_i,p(k_i)}) \in \mathcal{N}_n.$$

Corollary (Egoroff's theorem). *If $\{f_k\}$ is a sequence of finite measurable functions converging on a measurable set E of finite measure to finite function f, then to any $\varepsilon > 0$ there is a measurable set F such, that $\mu(F) < \varepsilon$ and $\{f_k\}$ converges to f uniformly on $E - F$.*

Note. Theorem 1 can be formulated more generally for the convergence almost everywhere. The system of zero sets can be changed by the system $\mathcal{N} = \bigcap_{n=1}^{\infty} \mathcal{N}_n$. But we should have to demand some other postulate regarding $\mathcal{N}$.

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For the sake of simplicity the theorem on regularity will be proved with some special assumptions. We shall suppose that $X = \langle 0,1\rangle$ and $\mathcal{S}$ is the system of all Borel subsets of X. The following condition will be supposed on $\mathcal{N}_n$:

(iv) If $E \in \mathcal{N}_n$, $F \subset E$, $F \in \mathcal{S}$, then $F \in \mathcal{N}_n$.

Theorem 2. *Let $\{\mathcal{N}_n\}$ be a sequence of subsystems of the system $\mathcal{S}$ fulfilling the conditions* (i)—(iv). *Then to any $E \in \mathcal{S}$ and any n there are a closed set C and an open set U for which $C \subset E \subset U$, $U - E \in \mathcal{N}_n$ and $E - C \in \mathcal{N}_n$.*

Proof. Let $\mathcal{P}$ be the system of all regular sets i. e. sets E with the following property: For any n there are a closed set C and an open set U such, that

$$C \subset E \subset U \text{ and } E - C \in \mathcal{N}_n,\ U - C \in \mathcal{N}_n.$$

First we prove that $\mathcal{P}$ is a ring. The properties (i) and (ii) imply the following property: For any n there are positive integers m, k such that $M \in \mathcal{N}_m$ $K \in \mathcal{N}_k \Rightarrow M \cup K \in \mathcal{N}_n$.

Hence, let $E, F \in \mathcal{P}$ be any sets, n a positive integer. Let m, k be numbers taken from the property above. By the assumption there are open sets U, V and closed sets C, D such that

$$U \supset E \supset C,\ U - E \in \mathcal{N}_k,\ E - C \in \mathcal{N}_k,$$
$$V \supset F \supset D,\ V - F \in \mathcal{N}_m,\ F - D \in \mathcal{N}_m.$$

By these relations it follows

$$U \cup V \supset E \cup F \supset C \cup D,\ U \cup V - E \cup F \subset (U - E) \cup (V - F) \in \mathcal{N}_n,$$
$$E \cup F - C \cup D \subset (E - C) \cup (F - D) \in \mathcal{N}_n.$$

By (iv) $E \cup F \in \mathcal{P}$. Similarly the relation $E - F \in \mathcal{P}$ follows from (iv) and by the following relations

$$U - D \supset E - F \supset C - V,\ (U - D) - (E - F) \subset (U - E) \cup (F - D) \in \mathcal{N}_n,$$
$$(E - F) - (C - V) \subset (E - C) \cup (V - F) \in \mathcal{N}_n.$$

The proof of the theorem will be complete, if we show that $\mathcal{P}$ is a monotone system. For, since every closed set is G_σ, the system $\mathcal{P}$ includes by (iii) all closed sets. Since every monotone ring is a σ-ring, the inclusion $\mathcal{P} \supset \mathcal{S}$ is true and, hence the statement of the theorem also.

Hence let $\{E_i\}$ and $\{F_i\}$ be sequences of sets from $\mathcal{P}$, $E_i \subset E_{i+1}$ $(i = 1, 2, \ldots)$, $F_i \supset F_{i+1}$ $(i = 1, 2, \ldots)$. Put $E = \bigcup_{i=1}^{\infty} E_i$, $F = \bigcap_{i=1}^{\infty} F_i$. Let n be any positive integer.

Let us construct a sequence $\{k_i\}$ according to (ii). To any i there are an open set U_i and a closed set D_i such that

$$U_i \supset E_i,\ F_i \supset D_i,\ U_i - E_i \in \mathcal{N}_{k_i},\ F_i - D_i \in \mathcal{N}_{k_i}.$$

If we put $U = \bigcup_{i=1}^{\infty} U_i$, $D = \bigcap_{i=1}^{\infty} D_i$, then

(3) $$U \supset E,\ U \text{ open},\ U - E \subset \bigcup_{i=1}^{\infty} (U_i - E_i) \in \mathcal{N}_n,$$

(4) $$F \supset D,\ D \text{ closed},\ F - D \subset \bigcup_{i=1}^{\infty} (F_i - D_i) \in \mathcal{N}_n.$$

Let us take k, m such that $M \in \mathcal{N}_m$, $K \in \mathcal{N}_k$ implies $M \cup K \in \mathcal{N}_n$. Then by (iii) there are i_0 resp. j_0 such that

$$E - E_{i_0} \in \mathcal{N}_m,\ F_{j_0} - F \in \mathcal{N}_m.$$

Let us construct an open set V and a closed set C such that

$$E_{i_0} \supset C,\ V \supset F_{j_0},\ E_{i_0} - C \in \mathcal{N}_k,\ V - F_{j_0} \in \mathcal{N}_k.$$

From the preceeding relations it follows

(5) $$E \supset C,\ E - C \subset (E - E_{i_0}) \cup (E_{i_0} - C) \in \mathcal{N}_n,$$

(6) $$V \supset F,\ V - F \subset (V - F_{j_0}) \cup (F_{j_0} - F) \in \mathcal{N}_n.$$

From the preceeding relations it follows $E, F \in \mathcal{P}$, hence $\mathcal{P}$ is a monotone system.

Corollary. *Every finite measure μ defined on the system $\mathcal{S}$ of all Borel subsets of $\langle 0, 1\rangle$ is regular.*

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Luzin's theorem will also be proved with special assumptions, so that we may use the results of the preceding two parts. Hence $X = \langle 0, 1\rangle$, $\mathcal{S}$ is the system of all Borel subsets of X. We shall need the following property of the sequence $\{\mathcal{N}_n\}$:

(v) System $\bigcap_{n=1}^{\infty} \mathcal{N}_n$ is hereditary, i.e. if $E \in \bigcap_{n=1}^{\infty} \mathcal{N}_n$, $F \subset E$, then $F \in \bigcap_{n=1}^{\infty} \mathcal{N}_n$.

This condition is satisfied by any complete measure.

Theorem 3. *Let $\{\mathcal{N}_n\}$ be a sequence of subsystems of the system $\mathcal{S}$, satisfying the conditions* (i)—(iv). *If f is a finite measurable function on $M \in \mathcal{S}$, then for any n there is $F \in \mathcal{N}_n$ such that f is continuous on $M - F$.*

Proof. Let f be a simple function, i.e. $f = \sum_{i=1}^{m} c_i \chi_{E_i}$, E_i pairwise disjoint, $\bigcup_{i=1}^{m} E_i = M$. Let $\{k_i\}$ be a sequence according to the condition (ii). By theorem 2 there are closed sets $F_i \subset E_i$ such that $E_i - F_i \in \mathcal{N}_{k_i}$. Put $F = \bigcup_{i=1}^{m} F_i$. Hence $M - F \subset \bigcup_{i=1}^{m} (E_i - F_i) \in \mathcal{N}_n$. Besides f is continuous on F.

Let f be now any finite measurable function. Take a sequence of simple functions $\{f_i\}$ such that $\{f_i\}$ converges to f on M. Construct a sequence $\{k_i\}$ by (ii). We have proved the existence of closed sets F_i, where f_i is continuous on $F_i \subset M$ and

$$M - F_i \in \mathcal{N}_{k_{i+1}} \ (i = 1, 2, \ldots). \tag{7}$$

From Theorem 1 the existence of a set K follows such that

$$M - K \in \mathcal{N}_{k_1} \tag{8}$$

and $\{f_i\}$ converges uniformly on K. Hence the function f, as a limit of a uniformly convergent sequence of continuous functions on $F = K \cap \bigcap F_i$, is continuous on this set. On the other side from (7) and (8) it follows

$$M - F = M - K \cap \bigcap_{i=1}^{\infty} F_i = (M - K) \cup \bigcup_{i=1}^{\infty} (M - F_i) \in \mathcal{N}_n.$$

Theorem 4. *Let $\{\mathcal{N}_n\}$ be a sequence of systems of subsets of X satisfying the conditions* (iv) *and* (v). *Let f be almost continuous on a measurable set $M \subset X$, i.e. for any n there is $F \in \mathcal{N}_n$ such that f is continuous on $M - F$. Then f is measurable on M.*

Proof. By the assumption there is for any n an F_n such that $F_n \in \mathcal{N}_n$ and f is continuous on $M - F_n$. From this it follows that f is measurable on $M - F_n$ and hence also on $\bigcup_{n=1}^{\infty} M - F_n = M - \bigcap_{n=1}^{\infty} F_n$. By (iv) it is $\bigcap_{n=1}^{\infty} F_n \in \mathcal{N}_n$ $(n = 1, 2, \ldots)$, hence by (v) f is measurable on M.

Corollary (Luzin's theorem). *A finite real function f is measurable on a set $M \in \mathcal{S}$ if and only if for any $\varepsilon > 0$ there is a set $F \in \mathcal{S}$ such that $\mu(F) < \varepsilon$ and f is continuous on $M - F$.*

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