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PROBLEMS ON PERIODICITY — FUNCTIONS AND SEMIGROUPS

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In Memoriam — George Earle Schweigert (1907—1965)

It is assumed throughout that $f\colon X \to X$ is a continuous function and that X is a non-void *compact* Hausdorff space. The hypothesis of compactness is by no means always necessary and is made to obtain a uniform hypothesis. (Cf., e.g., Whyburn [14]).

Also, S will be assumed to be a *compact* semigroup — that is, a non-void compact Hausdorff space provided with a continuous associative multiplication, denoted by juxtaposition. (In the case of the discrete topology see Clifford—Preston [1] and Ljapin [8], and in the general case see Paalman—de Miranda [10]).

For notions concerning periodicity we refer to Gottschalk—Hedlund [3] and Whyburn [14], though the language in these monographs is not uniform.

An element $a \in X$ may be termed *periodic* under f (and f periodic at a) iff $f^m(a) = a$ for some positive integer m. Also, f is *pointwise periodic* iff f is periodic at each $a \in X$.

Proposition 1 — F. *In order that f be pointwise periodic it is necessary and sufficient that $f(A) \subset A$ imply $f(A) = A$ for all $A \subset X$.* (Whyburn [14]).

An element a of S is termed *periodic* (this differs from the language of Š. Schwarz [11]) iff $a^{m+1} = a$ for some positive integer m. Also, S is *pointwise periodic* iff each element of S is periodic.

Proposition 1 — S. *In order that S be pointwise periodic it is necessary and sufficient that $A^2 \subset A$ imply $A^2 = A$ for all $A \subset S$.*

It should be pointed out that

$$AB = \{ab \mid a \in A \quad \text{and} \quad b \in B\}$$

and that

$$A^2 = AA,$$

so that A^2 is *not* the set $\{a^2 \mid a \in A\}$.

Proposition 2 — F. *If X is totally disconnected, if f is pointwise periodic and if $a \in X$ satisfies $f^m(a) = a$ then, inside any open set containing a, there is a compact open set V about a such that $f^m(V) = V$.*

This result has its origin in a theorem of Hall—Schweigert [5] and has been extended in Hofmann—Wright [6], as well as in [13].

Problem 2 — S. Is there an analog of Proposition 2 — F for semigroups?

It cannot be insisted that the analog, if there is one, should take that form in Proposition 1 — S. Thus it might be desirable to consider a formulation involving $A^{(m)} = \{a^m \mid a \in A\}$.

It is readily seen that f is a homeomorphism if it is pointwise periodic.

Proposition 3 — F. *If f is pointwise periodic and if X is a manifold then f is periodic.* (Montgomery [9] and the references in Whyburn [14, p. 265]).

Problem 3 — S. Is there an analog of Proposition 3 — F for semigroups?

There is, in a sense, a partial analog due to Anne Lester Hudson [7].

Hudson's Theorem. *If S is topologically an n-cell and if S is pointwise periodic then $p(x) \leq 2$ for each $x \in X$, where $p(x)$ is the least positive integer such that $x^{p(x)+1} = x$.*

Of course one may introduce a periodic multiplication by the device of defining, say, $xy = x$ for all x and y. But it is known that if S is a connected manifold which is pointwise periodic (or even recurrent, defined later) then S cannot have a zero element. It seems plausible that if S is a pointwise periodic connected manifold then S is a group or the multiplication is trivial, as above.

Proposition 4 — F. *In order that f be pointwise periodic it is necessary and sufficient that f be open onto and that, for each $x \in X$, the set*

$$\{y \mid f^p(x) = f^q(y) \quad \text{for some integers} \quad p, q > 1\}$$

be closed (Whyburn [14]).

Problem 4 — S. Is there an analog of Proposition 4 — F for semigroups?

The function f is termed *recurrent* if for each x and each open set U about x there is a positive integer m such that $f^m(x) \in U$. And the semigroup S is termed *recurrent* if for each x and each open set U about x there is an integer $m \geq 2$ such that $x^m \in U$. For the next proposition see Whyburn [14].

Proposition 5 — F. *In order that f be recurrent it is necessary and sufficient that $f(A) \subset A$ imply $f(A) = A$ for each closed $A \subset X$.*

The set $G \subset S$ is a subgroup of S iff $xG = G = Gx$ for each $x \in G$. Each subgroup of S is contained in a maximal such and no two of these intersect. Also, S is a *Clifford semigroup* iff S is the union of its subgroups.

Proposition 5 — S. *These are equivalent:* (i) S *is recurrent* (ii) $A^2 \subset A$ *implies* $A^2 = A$ *for each closed* $A \subset S$ (iii) S *is Cliffordian.*

It may be shown (Hall—Kelley [4] and Gottschalk—Hedlund [3]) that there exists a non-void closed set $A \subset X$ minimal relative to satisfying $f(A) = A$ and that f is recurrent on any such A. The corresponding proposition concerning semigroups is uninteresting if one takes for the analog of $f(A)$ the set A^2.

The next result is a very special instance of a theorem of Gottschalk's [2], see also Whyburn [14].

Proposition 6 — F. *Suppose that X is the union of two proper closed connected sets A and B which intersect in a point p and each of which intersects its image under f. If f is a recurrent homeomorphism then $f(p) = p$.*

Problem 6 — S. Is there an analog of Proposition 6 — F for semigroups?

Other examples may be given of parallel propositions for functions and for semigroups, and other problems may be stated. It is clear that the principal problem is this — Is there a mathematical system whose propositions properly subsume these analagous results?

This note closes with an historical commentary. George Schweigert obtained the doctorate at John Hopkins with G.T. Whyburn and later was instructor at the University of Virginia, to which institution Whyburn had gone. Among Schweigert's colleagues and friends were D. W. Hall, J. L. Kelley and myself and the three of us wrote our dissertations with Whyburn. W. H. Gottschalk and Schweigert were later colleagues at the University of Pennsylvania. Deane Montgomery, as a visiting member of the faculty at Virginia, was influential in some of the work done by Schweigert. Although W. L. Ayres is not mentioned in this bibliography (but see Whyburn [14]) his work — and he himself — made a strong impression on Schweigert.

Although uniformly agreeable and pleasant, Schweigert was rarely vivacious and was introverted rather than extroverted. His lectures were complete and clear, though he had the rather unusual manner of standing quite close to the chalkboard and holding the chalk as though he were holding a pencil, resting the heel of his hand upon the board. In letters, and sometimes in conversation, he could become quite enthusiastic and excited, though these occasions were rare, so far as I know. As may be seen from several of my papers, his remarks and results were inspiring, and had he been outgoing it is reasonable to expect that his impact upon mathematics and mathematicians would have been considerable.

He had a rather keen interest in abstraction and generalization, quite disparate from his published work and, at the same time, a good grip on reality. Starting from the trivial observation that a homeomorphism of the interval upon itself must have at least two fixed points if it has a fixed endpoint, he produced the result which was reformulated in [12] as — If X is a locally connected continuum and if f is a homeomorphism of X onto itself which leaves an endpoint fixed, then there is at least one other fixed point. This has evoked, in one sense or another, a series of some ten papers by at least four mathematicians, a couple being now in press. It is a reflection upon humanity in general, and not upon individuals in particular, that later papers carry no reference to him.

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