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MODIFICATION ON SEMI-COMPLETENESS FOR INTEGER SEQUENCES

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Let there be given a natural number k and two sequences $P = \{p_i\}_{i=1}^{\infty}$, $M = \{m_i\}_{i=1}^{\infty}$ of natural numbers. A sequence $A = \{a_i\}_{i=1}^{\infty}$ of non-negative integers is called a (P, k, M)-representation of the integer c, if

(A) $$0 \leqq a_i \leqq p_i \quad \text{for} \quad i = 1, 2, \ldots;$$

(B) $$c = \sum_{i=1}^{\infty} a_i m_i;$$

(C) $$\sum_{i=1}^{\infty} a_i \leqq k.$$

A sequence M is called a (P, k)-sequence, if (D) M is a (strictly) increasing sequence; (E) any non-negative integer has at least one (P, k, M)-representation; (F) every member of M has only one (P, k, M)-representation.

The aim of the present paper is to construct all (P, k)-sequences.

Remark. (P, k, M)-representation of a given number can be interpreted as the weighing of a solid by a set of weights having $m_1, m_2, m_3, \ldots$ units so that each m_i can be used at most p_i times and the total number of used weights cannot exceed k. The (P, k)-sequence can be interpreted as such an „economical" system of weights by means of which it is possible to determine the gravity of an arbitrary solid (whose gravity can be represented by an integral number of units) with the following property: the solids with the gravity equal to the gravity of one of the weights of the set can be (under the conditions given above) weighed only by the same weight from the set. In all weighing of this type the weights of the set are put only on one side of the scales.

Theorem 1 of [5] implies the following

Lemma 1. *Let $\{q_i\}_{i=1}^{\infty}$ be a sequence of non-negative integers and let $\{f_i\}_{i=1}^{\infty}$ be the sequence of naturals such that $f_1 = 1$, $f_{j+1} = 1 + \sum_{i=1}^{j} q_i f_i$ $(j = 1, 2, \ldots)$.*

Then for all natural number we have: if an integer a fulfils the inequality $0 \leqq a \leqq \sum_{i=1}^{n} q_i f_i$, then it can be written in the form $a = \sum_{i=1}^{n} a_i f_i$, where a_i are integers fulfilling the inequalities $0 \leqq a_i \leqq q_i$ $(i = 1, 2, \ldots, n)$.

Theorem. *To every P and k there exist one and only one (P, k)-sequence, namely $M = \{m_i\}_{i=1}^{\infty}$, where*

$$m_i = \begin{cases} \prod_{j=1}^{i-1} (p_j + 1), \text{ if } i \leqq r + 1; \\ \prod_{j=1}^{r} (p_j + 1) + (i - r - 1)d, \text{ if } i \geqq r + 1, \end{cases} \tag{1}$$

and

$$r = \begin{cases} \max \{x \mid \sum_{j=1}^{x} p_j \leqq k\}, \text{ if } p_1 \leqq k; \\ 0, \text{ if } p_1 > k; \end{cases}$$

$$d = -1 + (k + 1 - \sum_{j=1}^{r} p_j) \prod_{j=1}^{r} (p_j + 1).\text{(1)}$$

Proof. I. Define the sequence $M = \{m_i\}_{i=1}^{\infty}$ by the equation (1). We shall prove that M has properties (D), (E) and (F) and it is uniquely determined.

Obviously M is an increasing sequence, i.e. (D) is valid. It can be easily shown that for $k = 1$ the sequence $M = \{m_i\}_{i=1}^{\infty}$ defined by (1) has the form $\{i\}_{i=1}^{\infty}$ and it is the only $(P, 1)$-sequence. Therefore we can suppose in the following that $k \geqq 2$. By induction we can easily establish that for $i = 1, 2, \ldots, r + 1$

$$m_i = 1 + \sum_{j=1}^{i-1} p_j m_j, \tag{2}$$

so that

$$m_{r+1} = \prod_{j=1}^{r} (p_j + 1) = 1 + \sum_{j=1}^{r} p_j m_j,$$

from which it follows by elementary consideration that

$$d = \sum_{j=1}^{r} p_j m_j + (k - \sum_{j=1}^{r} p_j) m_{r+1}. \tag{3}$$

II. Put

$$q_i = \begin{cases} p_i, \text{ if } i \leqq r; \\ k - \sum_{j=1}^{r} p_j, \text{ if } i = r + 1; \\ 1, \text{ if } i \geqq r + 2. \end{cases}$$

(1) The sum (product) with a lesser upper index than the lower one is defined as 0 (1, respectively).

Define the sequence $\{f_i\}_{i=1}^{\infty}$ recurrently by the relations $f_1 = 1$, $f_i = 1 + \sum_{j=1}^{i-1} q_j f_j$ $(i = 2, 3, \ldots)$. According to (2) and by the definition of q_i for $i = 1, 2, \ldots, r + 1$ we have $f_i = m_i$. Further, put $n = r + 1$. By applying (3) from Lemma 1 it follows that every non-negative integer $a \leqq d$ can be represented in the form

$$a = a_1 m_1 + a_2 m_2 + \ldots + a_{r+1} m_{r+1},$$

where $a_1 \leqq p_1$, $a_2 \leqq p_2, \ldots, a_r \leqq p_r$, $a_{r+1} \leqq k - \sum_{i=1}^{r} p_i$. Obviously

$$\sum_{i=1}^{r+1} a_i \leqq k,$$

and the equality can occur only if $a_1 = p_1$, $a_2 = p_2, \ldots, a_r = p_r$, $a_{r+1} = k - \sum_{i=1}^{r} p_i$; but in such case by (3) we get a (P, k, M)-representation of d. Therefore numbers smaller than d have a $(P, k - 1, M)$-representation.

III. From Lemma 1 (for $n = r + 1$) it follows that every non-negative integer c not greater than

$$\sum_{i=1}^{r} p_i m_i + (k + 1 - \sum_{i=1}^{r} p_i) m_{r+1} = m_{r+2}$$

can be written in the form

$$c = \sum_{i=1}^{r+1} a_i m_i,$$

where $a_1 \leqq p_1$, $a_2 \leqq p_2, \ldots, a_r \leqq p_r$,

$$a_{r+1} \leqq k + 1 - \sum_{i=1}^{r} p_i.$$

Obviously $a_1 + a_2 + \ldots + a_r + a_{r+1} \leqq k + 1$. The equality can occur only in the case of $(P, k + 1, M)$-representation of m_{r+2}; thus any $c < m_{r+2}$ has a (P, k, M)-representation. We shall prove that any integer $c \geqq m_{r+2}$ has a (P, k, M)-representation as well. Let $m_t \leqq c < m_{t+1}$, where $t \geqq r + 2$. Evidently $c - m_t < d < m_t$, so that the number $c - m_t$ has according to II a $(P, k - 1, M)$-representation $B = \{b_i\}_{i=1}^{\infty}$, whence it follows that

$$c - m_t = \sum_{i=1}^{\infty} b_i m_i,$$

where $b_t = 0$ since $c - m_t < m_t$. Therefore the sequence $C = \{c_i\}_{i=1}^{\infty}$, where

$$c_i = \begin{cases} b_i \text{ for } i \neq t, \\ 1 \text{ for } i = t, \end{cases}$$

is a (P, k, M)-representation of c. Thus we have proved that every non-negative integer has a (P, k, M)-representation, i.e. the sequence fulfils the condition (E).

IV. Let m_j be any member of the sequence M. If $j \leqq r + 1$, the uniqueness of its (P, k, M)-representation follows from Theorem 2 of [5]. Therefore suppose that $j \geqq r + 2$ and let m_j have a (P, k, M)-representation $\{e_i\}_{i=1}^{\infty}$, i.e.

$$m_j = \sum_{i=1}^{\infty} e_i m_i,$$

$$0 \leqq e_i \leqq p_i \qquad (i = 1, 2, 3, \ldots),$$

$$\sum_{i=1}^{\infty} e_i \leqq k.$$

Denote

$$s = k + 1 - \sum_{i=1}^{r} p_i,$$

so that

$$d = -1 + s \prod_{i=1}^{r} (p_i + 1).$$

Suppose that $j \geqq r + 1$. Then the following equalities and congruences modulo d hold:

$$sm_j = s \prod_{i=1}^{r} (p_i + 1) + (j - r - 1)sd \equiv s \prod_{i=1}^{r} (p_i + 1) = d + 1 \equiv 1.$$

Thus we have proved that for $j \geqq r + 1$

(4) $$sm_j \equiv 1 \pmod{d}.$$

V. From (4) it follows (the congruences being considered modulo d):

$$\sum_{i=1}^{r} sm_i e_i + \sum_{i=r+1}^{\infty} e_i \equiv \sum_{i=1}^{r} sm_i e_i + \sum_{i=r+1}^{\infty} sm_i e_i = s \sum_{i=1}^{\infty} m_i e_i = sm_j \equiv 1 \pmod{d}.$$

VI. By a simple way we obtain:

$$\sum_{i=1}^{r} sm_i e_i + \sum_{i=r+1}^{\infty} e_i = \sum_{i=1}^{r} e_i(sm_i - 1) + \sum_{i=1}^{r} e_i + \sum_{i=r+1}^{\infty} e_i \leqq \sum_{i=1}^{r} p_i(sm_i - 1) + \sum_{i=1}^{\infty} e_i =$$

$$= \sum_{i=1}^{r} p_i(sm_i - 1) + k = \sum_{i=1}^{r} p_i(sm_i - 1) + \sum_{i=1}^{r} p_i + (k - \sum_{i=1}^{r} p_i) = \sum_{i=1}^{r} p_i m_i + k -$$

$$- \sum_{i=1}^{r} p_i = s \left(\prod_{i=1}^{r} (p_i + 1) - 1 \right) + s - 1 = s \prod_{i=1}^{r} (p_i + 1) - 1 = d.$$

VII. From the definition of s it follows that $s > 0$. From V and VI we have:

$$\sum_{i=1}^{r} sm_i e_i + \sum_{i=r+1}^{\infty} e_i = 1.$$

Therefore

$$\sum_{i=1}^{\infty} e_i = \sum_{i=1}^{r} e_i + \sum_{i=r+1}^{\infty} e_i \leqq \sum_{i=1}^{r} sm_i e_i + \sum_{i=r+1}^{\infty} e_i = 1,$$

so that

$$\sum_{i=1}^{\infty} e_i = 1,$$

which is possible only if

$$e_i = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases}$$

whence it follows that there exist only one (P, k, M)-representation of m_j. Thus the validity of (F) for M has been proved.

VIII. Using induction we shall prove that if $N = \{n_i\}_{i=1}^{\infty}$ is a (P, k)-sequence, then $m_i = n_i$ $(i = 1, 2, 3, \ldots)$. Evidently $m_1 = n_1 = 1$. Suppose that $m_i = n_i$ for $i = 1, 2, \ldots, j$. We shall show that $m_{j+1} = n_{j+1}$. Let the sequence $\{b_i\}_{i=1}^{\infty}$ be a (P, k, N)-representation of m_{j+1}. If $n_{j+1} > m_{j+1}$, then we should have

$$m_{j+1} = \sum_{i=1}^{\infty} b_i n_i = \sum_{i=1}^{j} b_i n_i = \sum_{i=1}^{j} b_i m_i.$$

In that case m_{j+1} would have two different (P, k, M)-representations, which is in contradiction to (F). Therefore $n_{j+1} \leqq m_{j+1}$. Analogously it can be proved that $m_{j+1} \leqq n_{j+1}$, whence $m_{j+1} = n_{j+1}$.

Remarks. 1. If M is a (P, k)-sequence, then not only every member of M, but also every non-negative integer $c < m_{r+2}$ has the only (P, k, M)-representation. In the special case $p_1 = p_2 = \ldots = p$ this representation corresponds to the representation of c in the $(p + 1)$-adic number system.

2. (P, k)-sequences are a special case of semi-complete sequences of [1], [3], [4]. If we solve our problem without restriction relative to k (condition (C)) — which can be interpreted as weighing without limitation of the total number of weigths — we obtain another semi-complete sequence $M = \{1, p_1 + 1, (p_1 + 1)(p_2 + 1), (p_1 + 1)(p_2 + 1)(p_3 + 1), \ldots\}$. Sequences with similar properties were treated in [1]—[8]; representations of arbitrary integers (both positive and negative) are investigated in [5], representations of positive real numbers in [6] and [9].

3. Another modification of our problem can be obtained by omitting the limitation concerning P (condition (A)); this means when interpreted that only the total number of the used weights is limited but there are no restrictions concerning single weights m_i of the set. The solution of this problem is the sequence $M = \{1, k + 1, 2k + 1, 3k + 1, \ldots\}$. If zero is added to the set of all members of this sequence, we get the reccurent basis of the k-th

order belonging to the empty set [11, pp. 54—55]. An extensive survey of properties of the bases for natural numbers is in [10] and [11].

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