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CYCLES IN A COMPLETE GRAPH ORIENTED IN EQUILIBRIUM

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Throughout this paper we shall call a complete graph with m vertices, oriented in equilibrium, a $\varrho(m)$-graph. (According to [1] a graph is oriented in equilibrium if for each of its vertices the following holds: the number of edges outgoing from the vertex v is equal to the number of edges incoming at the vertex v.) If we use the terminology introduced by Berge in [2], a $\varrho(m)$-graph is a complete antisymmetric graph wherein each vertex has an equal inward demi-degree and outward demidegree. Since according to definition a $\varrho(m)$-graph is complete and oriented in equilibrium, it must be a regular graph of an even degree and thus we have $m \equiv 1 \pmod 2$.

Remark 1. It would seem that with n given, all $\varrho(2n + 1)$-graphs are isomorphic. This is the case only with $n = 1$ and $n = 2$. Fig. 1 represents

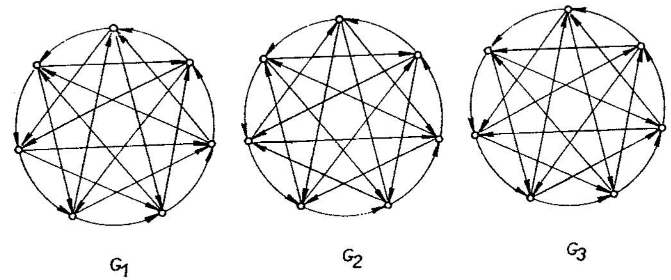

Fig. 1.

three different kinds of $\varrho(7)$-graphs. We can easily prove that any $\varrho(7)$-graph is isomorphic with exactly one of these three graphs. The answer to the following problem is not known to the author of the present paper: How many different mutually non-isomorphic $\varrho(2n + 1)$-graphs do there exist for each given $n > 3$?

Let x be any vertex of a $\varrho(2n + 1)$-graph G. We shall use the symbol $P(x)$ (or $Q(x)$) for denoting the sets of those vertices from G from which in the graph G the edge is incoming at the vertex x (or outgoing from it, respectively); by $|P(x)|$ or $|Q(x)|$ resp. we shall denote the number of its elements. It follows directly from the definition of a $\varrho(2n+1)$-graph and the sets $P(x)$, $Q(x)$ that for any vertex x we have: $|P(x)| = |Q(x)| = n$.

Theorem 1. *Let G be any $\varrho(2n+1)$-graph and h any of its edges. In the graph there exists at least one 3-cycle containing the edge h.*

Proof. Let the edge h in G be oriented from its vertex u into its vertex v. Let W be the set of all vertices of G not belonging into $\{u, v\}$. We obviously have $P(u) < W$; $Q(v) < W$ and since $|W| = 2n - 1$, $|P(u)| = n$, $|Q(v)| = n$, then necessarily $P(u) \cap Q(v) \neq 0$.

Then, however, there is at least one vertex $w \in W$ belonging both to $P(u)$ and $Q(v)$. The vertices u, v, w together with the edges joining these vertices form the 3-cycle of G containing h. This proves the theorem.

Theorem 2. *Let v be any vertex of a $\varrho(2n+1)$-graph G. The number of different 3-cycles of graph G, containing v, is exactly $\binom{n+1}{2}$.*

Proof. Let us denote by P (or Q resp.) the complete subgraph of the graph G containing all vertices and only vertices of the set $P(v)$ (or the set $Q(v)$, resp.) and all the edges joining these vertices. Let w be any vertex of the graph X (where $X \in \{G, P, Q\}$). Let us denote by $\sigma_X(\to w)$ the number of edges in X incoming at w and by $\sigma_X(w \to)$ the number of edges in X outgoing from w. Since $|P(v)| = |Q(v)| = n$, we have: the number of edges of both P and Q is $\binom{n}{2}$.

Whence it follows:

$$\sum_{x \in P} \sigma_P(x \to) = \sum_{x \in P} \sigma_P(\to x) = \sum_{x \in Q} \sigma_Q(x \to) = \sum_{x \in Q} \sigma_Q(\to x) = \binom{n}{2}.$$

Besides we have: $\sigma_G(x \to) = \sigma_G(\to x) = n$ for any vertex $x \in G$. Thus it follows that:

$$\sum_{x \in P} \sigma_G(\to x) = n^2$$

and since there is no edge oriented from the vertex v into a vertex of $P(v)$, we necessarily have: the number of edges of G oriented from some vertex of $Q(v)$ at a vertex of $P(v)$, is $n^2 - \binom{n}{2} = \binom{n+1}{2}$. Each of these edges and only such an edge together with v and the two edges incident at it form a 3-cycle containing v. This proves the theorem.

The subsequent corollary follows directly from Theorem 2:

Corollary 1. *In any $\varrho(2n+1)$-graph the number of different 3-cycles is* $\frac{1}{6}(2n+1)(n+1)n$.

Remark 2. We obtain the result $\frac{1}{4}(2n+1)(n+1)n$ so that the number of the 3-cycles containing the chosen vertex, i.e. the number $\binom{n+1}{2}$ is multiplied by the number of vertices and divided by three. Berge in [2], p. 145, Theorem 3 gives a more general formula for computing the number of 3-cycles no orientation in equilibrium is required. In the special case of the $\varrho(2n+1)$-graph its formula acquires the form given in Corollary 1.

Remark 3. While the number of 3-cycles in an $\varrho(2n+1)$-graph is not dependent — with n given — on the choice of the $\varrho(2n+1)$-graph, this does not hold for 4-cycles. Thus in the graphs G_1, G_2, G_3 given in Fig. 1 the number of 4-cycles is 25, 28, 21, though each of these three graphs is a $\varrho(7)$-graph.

Let C be any cycle of the $\varrho(2n+1)$-graph G. By the symbol $S(C)$ denote the set of vertices defined as follows: the vertex $x \in G$ belongs to $S(C)$ if anf only if it does not belong to C and when in the graph G there exist two such edges that one of them is oriented from a vertex of C into x and the other from x into a vertex of C. By the symbol $P(C)$ (or $Q(C)$, resp.), denote the set of the vertices from G that do not belong to C and have the property: any edge from G joining a vertex from $P(C)$ (or a vertex from $Q(C)$, resp.) with the vertex of C is incoming at (or outgoing from) the vertex of C.

Lemma 1. *Let C be any r-cycle of a $\varrho(2n+1)$-graph G where $r < 2n+1$ and let w be any vertex from $S(C)$. In the graph G there is at least one $(r+1)$-cycle C' containing both the vertex w and all vertices from C.*

Proof. According to the definition of $S(C)$ there is in G an edge (denote it by h) oriented from a vertex v_1 of C into w. Denote the other vertices of C by $v_2, v_3, \ldots, v_r$ in the order in which we pass through them by proceeding along the cycle C in the direction of the orientation of its edges, starting from v_1. From the definition of $S(C)$ it also follows that among the vertices $v_2, v_3, \ldots, v_r$ there exists such a vertex that the edge joining it with w is outgoing from w. Let v_s be the one from among such vertices that has with the given notation the smallest index. Then we necessarily have: there exists an edge of G

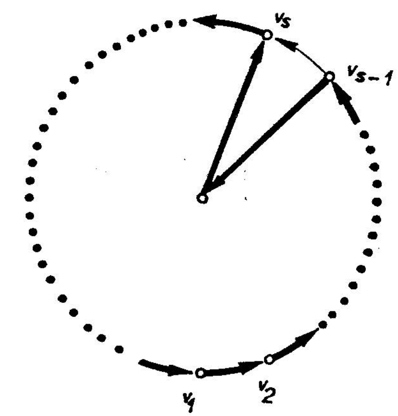

Fig. 2.

oriented from v_{s-1} into w and an edge g of G oriented from w into v_s. If in C we replace the edge oriented from v_{s-1} into v_s by the edges f, g and by the vertex w, we get a $(r+1)$-cycle C' of G having the required properties (see Fig. 2 — the edges from C' are accentuated).

Definition. *We shall say that the cycle C' from Lemma 1 arose by a λ-extension of the cycle C through the vertex w.*

Lemma 2. *Let C be any r-cycle of a $\varrho(2n+1)$-graph where $r < 2n$ and let v_r be any vertex from C; let w be any vertex from the set $P(C) \cup Q(C)$. In G there is at least one $(r+2)$-cycle C'' containing w and all vertices from C and in G there exists a $(r+1)$-cycle C^* containing w and all vertices from C except the vertex v_r.*

Proof. Denote the vertices of the cycle C — others than the vertex v_r — by the symbols v_i, where $i \in \{1, 2, \ldots, r-1\}$ so that we proceed along the cycle C in the direction of the orientation of its edges through its vertices in the following order: $v_1, v_2, \ldots, v_{r-1}, v_r$. Let h_i be the edge from G joining the vertices w and v_i. According to Theorem 1 there is in G at least one 3-cycle containing the edge h_i. Let x_i be the third vertex of such a cycle, hence let x_i be the vertex for which the following holds: $w \neq x_i \neq v_i$.

According to the assumption w belongs to $P(C) \cup Q(C)$. All edges $h_1, h_2, \ldots, h_r$ therefore are incoming at the vertex w or they are outgoing from the vertex w. Hence for all $i \in \{1, 2, \ldots, r\}$ we have: x_i does not belong to C. If w belongs to $P(C)$ then the sequence $w, v_r, v_1, \ldots, v_{r-1}, x_{r-1}$ gives the order in which we pass through the vertices of a $(r+2)$-cycle C'' if we proceed along it in the direction of the orientation of its edges. The sequence $w, v_1, \ldots, v_{r-1}, x_{r-1}$ determines in the given way a $(r+1)$-cycle C^*. The cycles C'', C^* obviously have the required properties. If w belongs to $Q(C)$ then the required cycle C'' is given by the sequence $w, x_1, v_1, \ldots, v_r$ and the cycle C^* by the sequence $w, x_1, v_1, \ldots, v_r$ (see Fig. 3). Hence the cycles C'' and C^* with the required properties exist, Q.E.D.

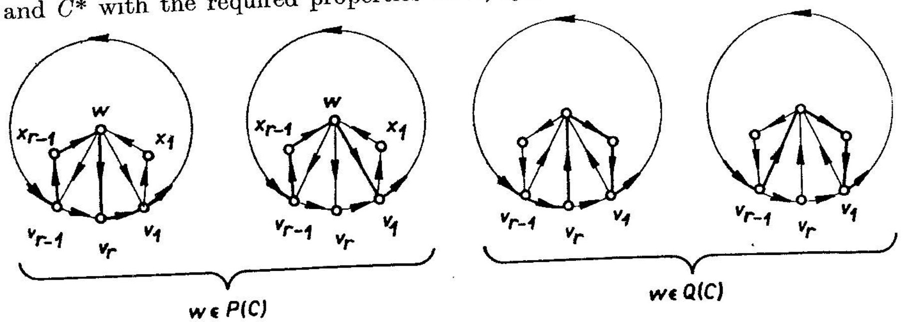

Fig. 3.

Definition. *We say that the cycle C'' from Lemma 2 arose from the cycle C by a μ-extension through the vertex w, and we say that the cycle C^* from the same lemma arose from C through a ν-extension through the vertex w with a simultaneous replacement of the vertex v_r.*

Theorem 3. *Let x, y be any two vertices of a $\varrho(2n+1)$-graph G and let k be any number from the set $\{3, 4, \ldots, 2n+1\}$. In G there is at least one k-cycle containing both vertices x and y.*

Proof. According to Theorem 1 there is in G a 3-cycle containing an edge joining the vertices x, y. Hence for $k = 3$ the theorem holds. Let us prove the following: If the theorem holds for $k = r$ (where r is a natural number, $3 \leqq r \leqq 2n$), then it holds also for $k = r + 1$. Suppose that in G there is an r-cycle C containing the vertices x, y. If $S(C)$ is a non-empty set, then, according to Lemma 1 we shall obtain by a λ-extension of the cycle C through any its vertex an $(r+1)$-cycle containing the vertices x, y. Let $S(C) = \emptyset$ and w be any vertex of the set $P(C) \cup Q(C)$. Since $r > 2$, we have in C a vertex (denote it by v_r) for which $x \neq v_r \neq y$. According to Lemma 2 we get by a ν-extension of the cycle C through the vertex w with a replacement of the vertex v_r an $(r+1)$-cycle C^* containing the vertices x, y. Hence if the theorem holds for $k = r$, it holds also for $k = r + 1 \leqq 2n + 1$. Thus the theorem holds for $k = 3$, hence it also holds for all $k \in \{3, 4, \ldots, 2n+1\}$.

The following corollary is a direct consequence of Theorem 2:

Corollary 2. *Each $\varrho(2n+1)$-graph with any natural n contains a Hamiltonian cycle.*

Lemma 3. *Let r, n, s be natural numbers, where $2 < s < r < 2n$ and let $v_1, v_2, \ldots, v_s$ be mutually different vertices of a $\varrho(2n+1)$-graph G. If there is in G a r-cycle containing all vertices of the set $V = \{v_1, v_2, \ldots, v_s\}$ then for each $k = r+1, r+2, \ldots, 2n+1$, there is in G also a k-cycle containing all vertices from V.*

Proof. Let there be in graph G a p-cycle C_0 containing all vertices of the set V. The cycle C_0 may be successively extended by λ-extensions and ν-extensions through suitably chosen vertices into the cycles $C_1, C_2, \ldots, C_{2n+1-p}$, where C_i is the $(p+i)$-cycle containing all vertices from V. This can be done so that in case of $S(C_i) = 0$ at the ν-extension of cycle C_i into cycle C_{i+1} through a certain vertex with the replacement of the vertex v_r from C_i, we must chose for v_r where $(r = p + i)$ always such a vertex from C_i that does not belong to V. Since such a cycle always exists with $r + i > s$, the lemma evidently holds.

Remark 4. In Fig. 4 we have a $\varrho(9)$-graph with the following property: In the graph there does not exist a 4-cycle containing the vertices u, v, w though there is in the same graph a 3-cycle with such vertices. Whence it follows that the condition $s < r$ must not be omitted from Lemma 3.

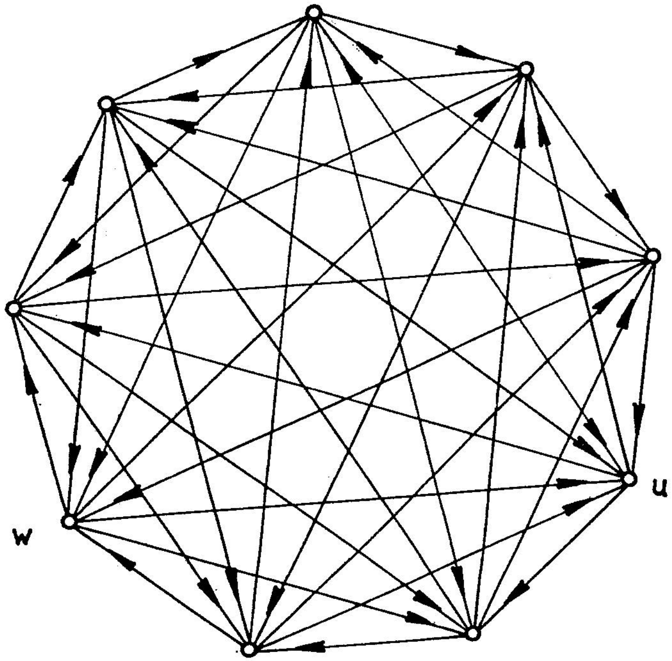

Fig. 4.

Lemma 4. *Let n, p be natural numbers and let C be the $2p$-cycle of the $\varrho(2n+1)$-graph C containing all vertices of a set V, then for any $k = 2p+1, 2p+2, \ldots, 2n+1$ there is in G a k-cycle containing all vertices of the set V.*

Proof. The cycle C contains according to the assumption an even number of vertices, therefore necessarily $S(C) \neq \emptyset$ (in the reverse case we would have $|P(C)| = |Q(C)| = \frac{1}{2}(2n+1-2p)$, which is impossible as $|P(C)|$ must be an integer). But then it is possible to extend the cycle C by a λ-extension through a vertex from $S(C)$ into a $(2p+1)$-cycle containing all vertices from V. If we put $r = 2p+1$, $s = |V|$, then $s < r$ and the validity of Lemma 5 follows from Lemma 3.

Remark 5. The difference between Lemma 3 and Lemma 4 is that in the case of an even s we may have $r = s$, hence in the case of an even $|V|$, V may be the set of all vertices of the cycle C.

Lemma 5. *Let C be any $(2p+1)$-cycle of a $\varrho(2n+1)$-graph G $(p < n)$ and let V be the set of all vertices of the cycle C. Let k be any number from the set*

$\{2p+3, 2p+4, \ldots, 2n+1\}$, then there exists in graph G such a k-cycle that contains all vertices from V.

Proof. If $S(C)$ is a non-empty set, then the cycle C may be extended by a λ-extension through a vertex of $S(C)$ into a $(2p+2)$-cycle C' which, apart from all vertices of the set V contains only one other vertex from $S(C)$. From the existence of the cycle C' there follows according to Lemma 3 the existence of a k-cycle containing all vertices of the set V also for all $k \in \{2p+3, 2p+4, \ldots, 2n+1\}$.

If $S(C) = \emptyset$ then there is in G at least one vertex w belonging to $P(C) \cap Q(C)$ and we get by a μ-extension of the cycle C through the vertex w according to Lemma 2 a $(2p+3)$-cycle C'' containing all vertices from V.

The validity of Lemma 5 then is evident from Lemma 3.

Lemma 6. *Let G be a $\varrho(2n+1)$-graph and let V be the set of certain of its r vertices, where $2 < r < 2n+1$. Let p be any natural number for which we have $1 < p < r$. If there is in G such a cycle C that contains apart from certain p vertices from V at least one vertex not belonging to V, then there is in G also a cycle $\bar{C}$ containing at least $p+1$ vertices from V and besides at least one vertex not belonging to V.*

Proof. Let C be a cycle containing p vertices from V and at most one vertex not belonging to V. We shall consider the following three possible cases:

1. $V \cap S(C) \neq \emptyset$.
2. $V \cap S(C) = \emptyset$, C containing only vertices from V.
3. $V \cap S(C) = \emptyset$, C containing one vertex — denote it by v_{p+1} — not belonging to V.

In the first case we get a λ-extension of the cycle C through any vertex from $V \cap S(C)$ a cycle with the required properties; in the second case we get such a cycle by a μ-extension of the cycle C through any vertex from the set $M = V \cap (P(C) \cap Q(C))$ and in the third case by a ν-extension of the cycle C through a vertex from M with the replacement of the vertex v_{p+1}. This proves the lemma.

Theorem 4. *Let G be any $\varrho(2n+1)$-graph and let V be the set of certain r vertices of G $(2 < r < 2n+1)$. If there is not in G an r-cycle containing all vertices from V, then there exists in G an $(r+1)$-cycle containing all vertices from V.*

Proof. Let there not be in G an r-cycle containing all vertices from V and let $x \neq y$ be any vertices from V. According to Theorem 1 there is in G a 3-cycle C containing the vertices x, y. Hence there is in G a cycle C which, with the exception of certain p vertices from $V (p \in \{2, 3\})$ contains at most one vertex not belonging to V. But then, according to Lemma 6, in case when $p < r$, there is in G a cycle $\bar{C}$ containing at least $p + 1$ vertices from V and at most one vertex not belonging to V. According to Lemma 6 the cycle C can be successively extended through the vertices from V so that the number of vertices of the cycle not belonging to V never exceeds one. After a finite number of steps we shall find such a cycle that contains all vertices from V and besides at most one vertex not belonging to V. Such cycle according to the assumption must be an $(r + 1)$-cycle. The Lemma follows.

The following corollary is a direct consequence of Lemma 4.

Corollary 3. *Let G be any $\varrho(2n + 1)$-graph and let V be the set of certain r vertices from G where $2 < r < 2n$. If there is not in G an $(r + 1)$-cycle containing all vertices from V then there is in G an r-cycle containing all vertices from V.*

Theorem 5. *Let n, r be natural numbers $2 < r < 2n$, $n > 1$ and let G be any $\varrho(2n + 1)$-graph. Let $R = \{r, r + 1, \ldots, 2n + 1\}$ and let V be any set of r vertices from G. In G there is a cycle containing all vertices from V either for all $k \in R$, all for all $k \in R$ with the exception of $k = r$, or for $k \in R$ with the exception of $k = r + 1$.*

Proof. If in G there are both an r-cycle and an $(r + 1)$-cycle containing all vertices from V, then there is, according to Lemma 3 in G a k-cycle containing all vertices from V for every $k \in R$.

If there is in G no $(r + 1)$-cycle containing all vertices from V then (see Corollary 3) there is in G an r-cycle containing all vertices from V and according to Lemmas 4 and 5 there exists such a k-cycle also for every $k > r + 1$, $k \leqq 2n + 1$.

Finally: If there is not in G an r-cycle containing all vertices from V, then, according to the theorem, there is in G an $(r + 1)$-cycle containing all vertices from V. According to Lemma 3 such a cycle exists for all $k \in R$ with one exception only: $k \neq r$. This proves the theorem.

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