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TWO OPERATIONS WITH FORMAL LANGUAGES AND THEIR INFLUENCE UPON STRUCTURAL UNAMBIGUITY

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1. INTRODUCTION

The formal languages here considered form a class $\mathscr{C}_0$ which contains the class of Chomsky's context-free grammars. Language ALGOL 60 (if considered without the limitations given in the non-formal parts of [1]) belongs to $\mathscr{C}_0$, too.

Recently the problem of semantics definition for languages from $\mathscr{C}_0$ has been raised (in connection with the unsatisfactory exactness of ALGOL 60 description). This problem was studied in Fabian's paper [4]. He investigated such semantics (a semantics is simply a transformation defined on the set of all terminal texts derivable in a given language), that the semantics value of a text t derivable from a non-terminal symbol A is determined, roughly speaking, by the way in which the text t is derivative from the symbol A and showed, that for such definition of semantics the weak structural unambiguity (see Def. 7.1, [4]) of a given language is very important. (Also some ambiguities of ALGOL 60 were a consequence of the fact that ALGOL 60 is not weakly structurally unambiguous.) But the concept of structural unambiguity (see Def. 7.1, [4]) is more convenient for the study. It has been proved (see[5]) that it is possible to transfer the investigation of weak structural unambiguity of a given language on the investigation of structural unambiguity of another language. Hence it is sufficient to study the structural unambiguity (s. u.) of formal languages.

In this paper the influence of language reduction (a non-terminal symbol is removed from the language by replacing, in all metatexts of the language (a metatext is simply such text by which a non-terminal symbol may be replaced, this symbol with its metatexts) and the language extension (a part of a metatext is replaced by new non-terminal symbol), on the structural unambignity is studied. (The operations of reduction and extension have been introduced in Čulík's paper [2].) It is proved that the extension and, under certain easily verified assumptions, even reduction have no influence upon structural unambiguity.

The operation of extension has been used in the proof of structural unambiguity of the language ALGOL MOD which is a slight modification of the language ALGOL 60 (see [6]).

The present paper uses notations and definitions of [4]. The reader should be familiar with section 1 to 7, [4].

2. REDUCTION OF LANGUAGES

A language $\mathscr{L}$ is said to be cyclic if there is a text t such that $\mathscr{L}: t \to t$. It has been proved (see [5]), that a language $\mathscr{L}$ is cyclic if and only if there is an $A \in \mathbf{d}\mathscr{L}$ such that $\mathscr{L}$: $[A] \to [A]$. Moreover, (see [5]) the structurally unambiguous language is not cyclic. Denote by $\mathscr{C}_0$ the class of all non-cyclic languages and by $\mathscr{C}_2$ the class of non-cyclic languages such that $\mathbf{d}\mathscr{L}$ and $\{\alpha;\ A \in \mathbf{d}\mathscr{L},\ \alpha \in \mathscr{L}A\}$ are finite sets.

2.1. Notations. If $\mathscr{L}$ is a language, $g \in \mathbf{g}\mathscr{L}$, then by $S_{\mathscr{L}}g$ $(\bar{S}_{\mathscr{L}}g)$ we shall denote the set of all structures $[\alpha, \tau]$ (such that $\alpha \neq [A]$) of g in $\mathscr{L}$. By $\mathbf{g}_u\mathscr{L}$ $(\mathbf{g}_a\mathscr{L})$ we shall denote the set of all structural unambiguous (structural ambiguous) grammatical elements of $\mathscr{L}$.

2.2. Definition. A metasymbol $A \in \mathbf{d}\mathscr{L}$ is called simple if there is only one α such that $\alpha \in \mathscr{L}A$. A metasymbol A is called reductible if $A \notin \mathbf{symb}\ \mathscr{L}A$, $\mathbf{symb}\ \mathscr{L}A \neq A$ and $A \in \mathbf{symb} \bigcup \{\mathscr{L}B;\ B \in \mathbf{d}\mathscr{L}\}$.

Let A be a reductible metasymbol, $\alpha \in \mathscr{L}A$, $\alpha \neq A$. Denote ψ the transformation defined on $\boldsymbol{\sigma}\mathscr{L}$ in the following manner:

(1) If A is a simple metasymbol, then

(1a) $\psi t = \Pi\xi$, where ξ is the decomposition defined on $\mathbf{d}t$ such that, for each $i \in \mathbf{d}\xi$, $\xi i = [ti]$ $(=\alpha)$ if $ti \neq A$ $(=A)$.

(2) If A is not a simple metasymbol, then

[2a] $\psi t = \{\Pi\xi;\ \xi$ is a decomposition defined on $\mathbf{d}t$ such that, for each $i \in \mathbf{d}\xi$, either $\xi i = [ti]$ or $\xi i = \alpha$ and $ti = A\}$.

Moreover, denote $\mathscr{L}^\alpha_A$ the transformation defined as follows:

$$\mathbf{d}\mathscr{L}^\alpha_A = \begin{cases} \mathbf{d}\mathscr{L} - \{A\} & \text{if } A \text{ is a simple metasymbol} \\ \mathbf{d}\mathscr{L} & \text{otherwise,} \end{cases}$$

and

$$\mathscr{L}^\alpha_A B = \begin{cases} \bigcup\ \{\psi\beta;\ \beta \in \mathscr{L}B\} & \text{if } B \neq A \\ \mathscr{L}A - \{\alpha\} & \text{if } B = A \in \mathbf{d}\mathscr{L}^\alpha_A. \end{cases}$$

The language $\mathscr{L}^\alpha_A$ will be called (A, α) — reduction of $\mathscr{L}$.

2.3. Theorem. *Let A be a reductible metasymbol of a language $\mathscr{L} \in \mathscr{C}_0$ and let $A \neq \alpha \in \mathscr{L}A$. Then $\mathscr{L}^{\alpha}_{A} \in \mathscr{C}_0$ and if*

(1) for each $B \in \mathbf{d}\mathscr{L}$, and $\alpha_1, \alpha_2 \in \mathscr{L}B$ the inequality $\alpha_1 \neq \alpha_2$ implies $\psi\alpha_1 \cap \psi\alpha_2 = \Lambda$,

then $\mathscr{L}^{\alpha}_{A}$ is s. u. if and only if so is $\mathscr{L}$. If (1) does not hold then $\mathscr{L}$ is not s. u.

(In the case A is a simple metasymbol we received the language $\mathscr{L}^{\alpha}_{A}$ from $\mathscr{L}$ by omitting the metasymbol A from $\mathbf{d}\mathscr{L}$ and by replacing, in all metatexts of $\mathscr{L}$, the symbol A with α. If A is not a simple metasymbol then the matter is a little more complicated. In that case we received the language $\mathscr{L}^{\alpha}_{A}$ from $\mathscr{L}$ in such a way that each metatext β is replaced with new metatexts which are obtained from β by replacing some symbols A in β with α. In this case we received 2^n new metatexts from every β where n is the number of all A in β. Moreover, α is omitting from the metatexts of the symbol A in $\mathscr{L}^{\alpha}_{A}$.)

Proof. Denote briefly $\mathscr{L}_0 = \mathscr{L}^{\alpha}_{A}$. In order to prove $\mathscr{L}_0$ is a language, it suffices to show according to the definition of $\mathscr{L}_0$ and Def. 5.1, [4], that $[B] \notin \mathscr{L}_0 B$ if $B \in \mathbf{d}\mathscr{L}_0$. But it follows straightforward from the definition of $\mathscr{L}_0$ and from non-cyclicity of $\mathscr{L}$.

Next, it is obvious that $\mathscr{L}\colon [B] \to t$ if $\mathscr{L}_0 : [B] \Rightarrow t$. Hence,

(2) $\mathscr{L}\colon [B] \to t$ if $\mathscr{L}_0\colon [B] \to t$

and $\mathscr{L}_0$ is the non-cyclic language, i. e. $\mathscr{L}_0 \in \mathscr{C}_0$.

Now suppose that (1) does not hold. Then there are $B \in \mathbf{d}\mathscr{L}$, $\alpha_1, \alpha_2 \in \mathscr{L}B$ such that $\alpha_1 \neq \alpha_2$ and $\psi\alpha_1 \cap \psi\alpha_2 \neq \Lambda$. Let $\alpha_0 \in \psi\alpha_1 \cap \psi\alpha_2$. Recalling the definition of ψ we have $\mathscr{L}\colon \alpha_1 \stackrel{\to}{=} \alpha_0$, $\mathscr{L}\colon \alpha_2 = \alpha_0$, and therefore, since $\alpha_1 \neq \alpha_2$, $[B, \alpha_0] \in \mathbf{g}_a\mathscr{L}$ and the second assertion of Theorem is proved. In what follows we shall suppose that (1) holds.

In the following we shall say that a text t does not contain the symbol A if $A \notin \mathbf{symb}\ \{t\}$. We proceed to prove some auxiliary results.

(3) If $\mathbf{g}_a\mathscr{L} \neq \Lambda$, there is a $[B, t] \in \mathbf{g}_a\mathscr{L}$ such that t does not contain A.

Proof. Let $g = [B, t] \in \mathbf{g}_a\mathscr{L}$. If t does not contain A, then (3) holds trivially. Now suppose that t contains A. Let us define the transformation ξ on $\mathbf{d}t$ as follows: $\xi i = \alpha$ if $ti = A$ and $\xi i = [ti]$ if $ti \neq A$. Put $u = \Pi\xi$. Then $\mathscr{L}\colon [B] \to u$ and u does not contain A. Denote $g_0 = [B, u]$. We shall prove that $g_0 \in \mathbf{g}_a\mathscr{L}$. Let $[\alpha_1, \tau_1]$ and $[\alpha_2, \tau_2]$ be two different structures in $S_{\mathscr{L}}g$. Fixed an i. If $\alpha_i \neq [B]$, then $[\alpha_i, \tau_i \otimes \xi] \in \bar{S}_{\mathscr{L}}g_0$ and if $\alpha_i = [B]$, then $[t, \xi] \in \bar{S}_{\mathscr{L}}g_0$. From Lemma 4.11, [4] we conclude $[\alpha_1, \tau_1 \otimes \xi] \neq [\alpha_2, \tau_2 \otimes \xi]$ if $\alpha_1 \neq [B] \neq \neq \alpha_2$. If $\alpha_1 = [B] \neq \alpha_2$, we have $[t, \xi] \neq [\alpha_2, \tau_2 \otimes \xi]$ because the equality implies $\mathscr{L}\colon t = \alpha_2 \to t$ which contradicts the non-cyclicity of $\mathscr{L}$. Similarly can be proved $g_0 \in \mathbf{g}_a\mathscr{L}$ if $\alpha_1 \neq [B] = \alpha_2$. This completes the prove of (3).

(4) If $g = [B, t] \in \mathbf{g}\mathscr{L}$ and t does not contain A, then either $\mathscr{L}\colon [B] = [A] \Rightarrow \Rightarrow \alpha \stackrel{\to}{=} t$, $\mathscr{L}_0\colon \alpha \stackrel{\to}{=} t$(and $\mathscr{L}_0\colon \alpha \to t$ if $\mathscr{L}\colon \alpha \to t$) or $g \in \mathbf{g}\mathscr{L}_0$.

Proof. Denote M the set of all $g \in \mathbf{g}\mathscr{L}$ such that (4) holds. If $\mathscr{L}\colon [B] \Rightarrow t$,

then, according to the definition of $\mathscr{L}_0$, $[B, t] \in M$. Now suppose that $[B, t]$ has a M-regular structure $[\beta, \tau]$ (see Def. 6.6, [4]) in $\mathscr{L}$. In order to prove (4), it suffices, by Theorem 6.7, [4], to show $[B, t] \in M$. By the preceding it suffices to investigate the case $t \notin \mathscr{L}B$ and hence, $[\beta, \tau] \in \bar{S}_{\mathscr{L}}g$. If $B = A$ and $\alpha = \beta$, then $\beta i \neq A$ and because either $\beta i = \tau i$ or $[\beta i, \tau i] \in M$, we get $\mathscr{L}_0\colon [\beta i] \stackrel{\to}{=} \tau i$. Thus $\mathscr{L}_0\colon \alpha = \beta \stackrel{\to}{=} t$ (and $\mathscr{L}_0\colon \alpha \to t$ if $\mathscr{L}\colon \alpha \to t$), (1) holds and $g \in M$. If it is not the case $B = A$ and $\alpha = \beta$, then we get $[B, t] \in M$ as follows: define ξ on $\mathbf{d}\beta$ by putting $\xi i = [\beta i]$ if $\mathscr{L}_0\colon [\beta i] \stackrel{\to}{=} \tau i$ and $\xi i = \alpha$ otherwise. According to M-regularity of $[\beta, \tau]$, we obtain in this second case $\mathscr{L}_0\colon \alpha \stackrel{\to}{=} \tau i$ and hence $\mathscr{L}_0\colon \Pi\xi \to t$. Recalling the definition of ξ we have $\Pi\xi \in \psi\beta$ and hence $\Pi\xi \in \in \mathscr{L}_0 B$. (If $A = B$, then β does not contain A, $\psi\beta = \{\beta\}$ and $\alpha \neq \beta = \Pi\xi \in \in \mathscr{L}_0 B$). Therefore, $\mathscr{L}_0\colon [B] \Rightarrow \Pi\xi \stackrel{\to}{=} t$, $[B, t] \in M$ and the proof of (4) is finished.

Now we introduce the following notation: If $[B, t] \in \mathbf{g}\mathscr{L}$, t does not contain A and $[\beta, \tau] \in \bar{S}_{\mathscr{L}}[B, t]$, then by $\bar{\beta}$ and $\bar{\tau}$ we shall denote the text $\Pi\xi^{\tau}_{\beta}$ and the decomposition $\Pi\zeta^{\tau}_{\beta}$, respectively, where ξ^{τ}_{β} and ζ^{τ}_{β} are transformations defined on $\mathbf{d}\beta$ as follows: If $\mathscr{L}\colon [\beta i] = [A] \Rightarrow \alpha \stackrel{\to}{=} \tau i$, then $\xi^{\tau}_{\beta} i = \alpha$ and $\zeta^{\tau}_{\beta} i$ is an α-decomposition of τi in $\mathscr{L}_0$; otherwise $\xi^{\tau}_{\beta} i = [\beta i]$ and $\zeta^{\tau}_{\beta} i = \tau i$. From this definition and from (4) we conclude:

(5) If $[B, t] \in \mathbf{g}\mathscr{L}$, t does not contain A and $[\beta, \tau] \in \bar{S}_{\mathscr{L}}[B, t]$, then $\bar{\beta} \in \psi\beta$, $\mathscr{L}_0\colon \bar{\beta} \stackrel{\to}{=} t$ and $\bar{\tau}$ is a $\bar{\beta}$-decomposition of t in $\mathscr{L}_0$.

Now we can start the own proof of Theorem. First we prove that $\mathbf{g}_a\mathscr{L} \neq \Lambda$ implies $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$. Let $\mathbf{g}_a\mathscr{L} \neq \Lambda$.

By (3) there is a $g = [B, t] \in \mathbf{g}_a\mathscr{L}$ such that t does not contain A. Let $[\alpha_1, \tau_1]$ and $[\alpha_2, \tau_2]$ be two different structures in $S_{\mathscr{L}}\,[B, t]$. Let us distinguish two cases.

1. $A \neq B$. If $t \in \mathscr{L}B$ and $[\beta, \tau] \in \bar{S}_{\mathscr{L}}\ [B, t]$, then, by non-cyclicity of $\mathscr{L}$ and by (1), $\beta \neq t \neq \bar{\beta}$. From this and from (5) we conclude $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$ if $\{[\alpha_1, \tau_1], [\alpha_2, \tau_2]\} \notin \bar{S}_{\mathscr{L}}\ [B, t]$. Now let $[\alpha_1, \tau_1]$, $[\alpha_2, \tau_2] \in \bar{S}_{\mathscr{L}}\ [B, t]$. Straightforward from (5) we have $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$ if $\bar{\alpha}_1 \neq \bar{\alpha}_2$. At last we have to investigate the case $\bar{\alpha}_1 = \bar{\alpha}_2$. By (1) $\alpha_1 = \alpha_2$ and hence $\tau_1 \neq \tau_2$. Next we prove $\bar{\tau}_1 \neq \bar{\tau}_2$ and the inequality $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$ will be proved for the case $A \neq B$.

Denote $x_i = \iota\tau_i$, $\bar{x}_i = \iota\bar{\tau}_i$ for $i = 1, 2$. Since $\tau_1 \neq \tau_2$ there is the smallest j_0 such that $x_1 j_0 \neq x_2 j_0$. Obviously $j_0 > 1$. Put $\nu_i = \sum_{j=1}^{j_0-1} \lambda(\xi^{\tau_i}_{\alpha_i} j) + 1$. Because $\bar{\alpha}_1 = \bar{\alpha}_2$ we have $\nu_1 = \nu_2$ and it is the case $\bar{x}_1\,\nu_1 = x_1\,j_0 \neq x_2\,j_0 = \bar{x}_2\nu_2$. Thus, $\bar{\tau}_1 \neq \bar{\tau}_2$.

2. $A = B$. We first set down some additional notation. By the assumptions of Theorem there are $C \in \mathbf{d}\mathscr{L}$ and $\gamma \in \mathscr{L}C$ such that γ contains A. Define the decomposition $\bar{\xi}$ on $\mathbf{d}\gamma$ as follows: $\bar{\xi} i = t$ if $\gamma i = A$ and $\bar{\xi} i = [\gamma i]$ otherwise. Put $u = \Pi\bar{\xi}$. As a consequence of the definition of $\bar{\xi}$ we have that $\mathscr{L}_0\colon [A] \to t$

implies $[C, u] \in \mathbf{g}_a\mathscr{L}_0$ and $[\gamma, \xi] \in \bar{S}_{\mathscr{L}_0} [C, u]$. The case $\mathscr{L}_0$: $[A, t]$ is, for instance, if $\mathscr{L}$: $[A] \Rightarrow \alpha_0 \to t$ and $\alpha_0 \neq \alpha$. For each $[\alpha, \tau] \in \bar{S}_{\mathscr{L}} [A, t]$ we define α' and τ' as follows: $\alpha' = \Pi\xi'$, $\tau' = \Pi\zeta'$ where ξ' and ζ' are defined on $\mathbf{d}\gamma$ in the following manner: if $\gamma i = A$ then $\xi' i = \alpha$, $\zeta' i = \tau$; otherwise $\xi' i = [\gamma i]$, $\xi i = [[\gamma i]]$. Put $u = \Pi\Pi\zeta'$. As α does not contain A we have, by the previous definition and by (4) $\mathscr{L}_0$: $[C] \Rightarrow \alpha' \to u$, $[\alpha', \tau'] \in \bar{S}_{\mathscr{L}_0} [C, u]$. Moreover, $\tau_1' \neq \tau_2'$ if $\tau_1 \neq \tau_2$.

Now we can begin the investigation of the case $A = B$. First suppose $[\alpha_1, \tau_1]$, $[\alpha_2, \tau_2] \in \bar{S}_{\mathscr{L}}[A, t]$. Then α_i does not contain A and therefore, by (4), $[\alpha_i, \tau_i] \in \bar{S}_{\mathscr{L}_0} [A, t]$ if $\alpha_i \neq \alpha$. That is $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$ if $\alpha_1 \neq \alpha \neq \alpha_2$. If $\alpha_1 = \alpha \neq \alpha_2$, then $[\gamma, \xi]$ and $[\alpha', \tau_1']$ are two different structures in $S_{\mathscr{L}_0} [C\ u]$ and again $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$. Similarly for the case $\alpha_1 \neq \alpha = \alpha_2$. If $\alpha_1 = \alpha = \alpha_2$, then $\tau_1 \neq \tau_2$ and $[\alpha', \tau_1']$, $[\alpha', \tau_2']$ are again two different structures in $S_{\mathscr{L}_0} [C, u]$. Thus, $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$. Finally suppose $\alpha_1 = [A] \neq \alpha_2$. As $\mathscr{L}$ is not cyclic, then either $\alpha_2 \neq \alpha$ or $\alpha \neq t$. If $t \neq \alpha \neq \alpha_2$ then obviously $[[A]. [t]] \in S_{\mathscr{L}_0} [A, t]$ and similarly as above we can prove $[\alpha_2, \tau_2] \in S_{\mathscr{L}_0} [A, t]$; that is $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$. If $t = \alpha \neq \alpha_2$, then $[[C], [u]]$ and $[\gamma, \xi]$ are two different structures in $S_{\mathscr{L}_0} [C, u]$. If $t \neq \alpha = \alpha_2$, then two different structures from $S_{\mathscr{L}_0} [C, u]$ are $[\alpha', \tau_2']$ and $[\gamma, \xi]$. Similarly for the case $\alpha_1 \neq [A] = \alpha_2$. This completes the proof that $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$ if $\mathbf{g}_a\mathscr{L} \neq \Lambda$.

In the following part of this proof the converse implication, i. e. $\mathbf{g}_a\mathscr{L} \neq \Lambda$ if $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$, will be proved. Let $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$.

If $B \in \mathbf{d}\mathscr{L}_0$, $\beta \in \mathscr{L}_0 B$ then by $\bar{\beta}$ we shall denote an element in $\mathscr{L}B$ such that $\beta \in \psi\bar{\beta}$; by ξ_β an $\bar{\beta}$-decomposition of β in $\mathscr{L}$ such that for each $i \in \mathbf{d}\bar{\beta}$ either $[\bar{\beta} i] = \xi_\beta i$ or $\bar{\beta} i = A$, $\xi_\beta i = \alpha$. Since $A \notin \mathbf{symb}\ \{\mathscr{L}A\}$ and (1) holds, $\bar{\beta}$ and ξ_β are determined uniquely and $\mathscr{L}$: $\bar{\beta} \mathrel{\underline{\Rightarrow}} \beta$. From this and from (2) we conclude

(6) $[\bar{\beta}, \xi_\beta \otimes \tau] \in \bar{S}_{\mathscr{L}} g$ if $[\beta, \tau] \in \bar{S}_{\mathscr{L}_0} g$

Now let $g = [B, t] \in \mathbf{g}_a\mathscr{L}_0$ and let $[\alpha_1, \tau_1]$, $[\alpha_2, \tau_2]$ be two different structures in $S_{\mathscr{L}_0} g$.

First investigate the case $\alpha_1 = [B] \neq \alpha_2$. If $t = \bar{t}$ then $[[B], [t]] \in S_{\mathscr{L}} g$ and, choosing suitable $\bar{\tau}_2$, also $[\bar{\alpha}_2, \bar{\tau}_2] \in S_{\mathscr{L}} g$ and hence $\mathbf{g}_a\mathscr{L} \neq \Lambda$. Next we shall investigate the case $t \neq \bar{t}$. Then $[\bar{t}, \xi_t]$ and $[\bar{\alpha}_2, \xi_{\alpha_2} \otimes \tau_2]$ are, by (2), from $S_{\mathscr{L}} g$. They are different, and hence $\mathbf{g}_a\mathscr{L} \neq \Lambda$ if either $\bar{t} \neq \bar{\alpha}_2$ or $\xi_t \neq \neq \xi_{\alpha_2} \otimes \tau_2$. Now consider the case $\bar{t} = \bar{\alpha}_2$ and $\xi_t = \xi_{\alpha_2} \otimes \tau_2$. Since $\alpha_2 \neq t$ (by non-cyclicity of $\mathscr{L}_0$), $\xi_t \neq \xi_{\alpha_2}$ and therefore there is the smallest integer i such that $\xi_t i \neq \xi_{\alpha_2} i$. This means that either $\xi_t i = [A]$ and $\xi_{\alpha_2} i = \alpha$ or $\xi_{\alpha_2} i = [A]$ $\xi_t i = [\alpha]$. Since $\xi_t = \xi_{\alpha_2} \otimes \tau_2$, we have $\mathscr{L}_0$: $\alpha \to [A]$ in the former case and $\mathscr{L}_0$: $[A] \to \alpha$ in the latter one. The relation $\mathscr{L}_0$: $\alpha \to [A]$ implies, by (2), $\mathscr{L}$: $[A] \Rightarrow \alpha \to [A]$ which contradicts the non-cyclicity of $\mathscr{L}$. Since $\alpha \notin \mathscr{L}_0 A$, there is. in the case $\mathscr{L}_0$: $[A] \to \alpha$. an $\alpha_1 \in \mathscr{L}_0 A$ such that $\mathscr{L}_0$: $\alpha_1 \to \alpha$. Thus $\mathscr{L}$: $[A] \Rightarrow \alpha_1 \to \alpha$ and $[A, \alpha] \in \mathbf{g}_a\mathscr{L}$.

Similarly we can prove that $\mathbf{g}_a\mathscr{L} \neq \Lambda$ if $\alpha_1 \neq [B] = \alpha_2$. Finally consider the case $\alpha_1 \neq [B] \neq \alpha_2$. If either $\bar{\alpha}_1 \neq \bar{\alpha}_2$ or $\xi\alpha_1 \times \tau_1 \neq \xi_{\alpha_1} \otimes \tau_2$ then it is easy to see that $[B, t] \in \mathbf{g}_a\mathscr{L}$. Now let $\bar{\alpha}_1 = \bar{\alpha}_2$ and $\xi_{\alpha_1} \otimes \tau_1 = \xi_{\alpha_2} \otimes \tau_2$. Denote $\zeta = \xi_{\alpha_1} \otimes \tau_1 = \xi_{\alpha_2} \otimes \tau_2$. We shall distinguish two cases:

1. $\alpha_1 \neq \alpha_2$. Then $\xi_{\alpha_1} \neq \xi_{\alpha_2}$. Hence. there is an i such that $\xi_{\alpha_1} i \neq \xi_{\alpha_2} i$. Now there are two possibilities: either $\xi_{\alpha_1} i = [A]$ and $\xi_{\alpha_2} i = \alpha$ or $\xi_{\alpha_2} i = \alpha$ and $\xi_{\alpha_2} i = [A]$. Consider the first possibility. Then

(7) $\mathscr{L}_0$: $\alpha \mathrel{\underline{\Rightarrow}} \zeta i$ and $\mathscr{L}_0$: $[A] \mathrel{\underline{\Rightarrow}} \zeta i$.

If $[A] = \zeta i$, then (7) implies $\mathscr{L}_0$: $\alpha \mathrel{\underline{\Rightarrow}} [A]$ and hence $\mathscr{L}$: $[A] \Rightarrow \alpha \to [A]$, which contradicts the non-cyclicity of $\mathscr{L}$. Hence $\mathscr{L}_0$: $[A] \to \zeta i$. But it means that there is an $\alpha_1 \in \mathscr{L}_0 A$ such that $\mathscr{L}_0$: $[A] \Rightarrow \alpha_1 \mathrel{\underline{\Rightarrow}} \zeta i$. Obviously $\alpha_1 \neq \alpha$ and, moreover, $\mathscr{L}$: $[A] \Rightarrow \alpha_1 \mathrel{\underline{\Rightarrow}} \zeta i$. By (7) we also have $\mathscr{L}$: $\alpha \mathrel{\underline{\Rightarrow}} \zeta i$ and hence $[A, \zeta i] \in \mathbf{g}_a\mathscr{L}$. Similarly we can prove that $\mathbf{g}_a\mathscr{L} \neq \Lambda$ if $\xi_{\alpha_1} i = \alpha$, $\xi_{\alpha_2} i = [A]$.

2. $\alpha_1 = \alpha_2$. Then $\tau_1 \neq \tau_2$. Denote $x = \iota\xi_{\alpha_1} = \iota\xi_{\alpha_2}$, $x_1 = \iota\tau_1$, $x_2 = \iota\tau_2$. Since $\xi_{\alpha_1} \otimes \tau_1 = \xi_{\alpha_2} \otimes \tau_2$ we have $x_1 x i = x_2 x i$ for each $i \in \mathbf{d}x$. Because of $\tau_1 \neq \tau_2$ it is also $x_1 \neq x_2$. Hence, there is an $i \in \mathbf{d}x$ such that $x_1 x i = x_2 x i$, $x_1 x(i + 1) = x_2 x(i + 1)$ and a j such that $xi < j < x(i + 1)$, $x_1 j \neq x_2 j$. But it means that $\tau_1^{(xi, x(i+1)-1)}$ and $\tau_2^{(xi, x(i+1)-1)}$ are two different α-decomposition of ζi in $\mathscr{L}_0$ and hence in $\mathscr{L}$, too. Thus $[A, \zeta i] \in \mathbf{g}_a\mathscr{L}$. This completes the proof of Theorem.

As a consequence of the preceding Theorem we have:

2.4. Theorem. *Let $\mathscr{L} \in \mathscr{C}_0$ and A be a reductible metasymbol of $\mathscr{L}$, $A \notin \mathscr{L}A$. Denote for every $B \in \mathbf{d}\,\mathscr{L}$, $\beta \in \mathscr{L}\,B$, $\psi\,B = \{\Pi\xi;\ \xi$ is a decomposition defined on $\mathbf{d}\beta$ such that for each $i \in \mathbf{d}\beta$ either $\xi\,i = [\beta i] + [A]$ or $\xi i \in \mathscr{L}\,A$ and $\beta_i = A$. Denote $\mathscr{L}^A$ the language defined as follows:*

$\mathbf{d}\mathscr{L}^A = \mathbf{d}\mathscr{L} - \{A\}$, $\quad \mathscr{L}^A B = \{\psi\beta;\ \beta \in \mathscr{L}\,B\}$

If

(1) *there are $B \in \mathbf{d}\mathscr{L}$ and $\beta_1, \beta_2 \in \mathscr{L}B$ such that $\beta_1 \neq \beta_2$ and $\psi\beta_1 \cap \psi\beta_2 \neq \Lambda$ then $\mathscr{L}$ is s. a. If (1) does not hold then $\mathscr{L}$ is s. u . if and only if so is $\mathscr{L}^A$.*

2.5. Remark. According to previous theorem in studying of the structural unambiguity of languages from $\mathscr{C}_2$ it suffices to consider only languages $\mathscr{L}$ such that

(1) for each $A \in \mathbf{d}\mathscr{L}$ either $\mathscr{L}A = \{A\}$ or $A \in \mathbf{symb}\ \mathscr{L}A$ or $A \notin \mathbf{symb} \cup \cup\ \{\mathscr{L}B;\ B \in \mathbf{d}\mathscr{L}\}$.

Indeed, if $\mathscr{L}_0 \in \mathscr{C}_2$, then we can construct a finite sequence $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n$ of languages such that the language $\mathscr{L}_i$ is an (A_i, α_i)-reduction of $\mathscr{L}_{i-1}$ where A_i is a reductible metasymbol of $\mathscr{L}_{i-1}$, $\Lambda \neq \alpha_i \in \mathscr{L}_{i-1}A_i$ $(i = 1, 2, \ldots, n)$, and for the language $\mathscr{L}_n$ the condition (1) is already satisfied. If at least for one of the languages $\mathscr{L}_i$, $i = 0, 1, \ldots, n - 1$, condition (2.3.1) is not satisfied, then, by Theorem 2.3, $\mathscr{L}_0$ is not s. u. If for all languages $\mathscr{L}_i, i = 0, 1, \ldots, n-1$, the condition (2.3.1) holds, then, again by Theorem 2.3, $\mathscr{L}_n$ is s. u. if and only if so is $\mathscr{L}_0$.

This results with results of paper [5] show that in studying the weak structural unambiguity of regular languages from $\mathscr{C}_2$ (i. e. languages such that $t_l(\mathscr{L}, A) \neq \Lambda$ for $A \in \mathbf{d}\mathscr{L}$), it suffices to consider only languages $\mathscr{L}$ such that

(2) $A \in \mathbf{symb}\ \mathscr{L}A$ for every $A \in \mathbf{d}\mathscr{L}$ such that $A \in \mathbf{symb} \bigcup \{\mathscr{L}B; B \in \mathbf{d}\mathscr{L}\}$.

Indeed, suppose that we want to investigate the weak structural unambiguity of a $\mathscr{L} \in \mathscr{C}_2$. If $\mathscr{L}$ is not Λ-s. u. (see Def. 5.5, [5]), then by Lemma 5.6, [5] is not weakly structurally unambiguous, too. If $\mathscr{L}$ is Λ-s. u., then, by Theorem 5.12, [5], $\mathscr{L}$ is weakly s. u. if and only if the language $\mathscr{L}_0$, defined as in Def. 5.8, [5], is s. u. But for $\mathscr{L}_0$ it already holds $\Lambda \notin \bigcup \{\mathscr{L}_0 A; A \in \mathbf{d}\mathscr{L}_0\}$. As it was shown in the first part of this remark, the investigation of the structural unambiguity of the language $\mathscr{L}_0$ can be transferred, with suitable reductions, upon the investigation of the structural unambiguity of a language $\mathscr{L}_n$ which satisfies condition (1) and, since $\Lambda \notin \bigcup \{\mathscr{L}_0 A; A \in \mathbf{d}\mathscr{L}_0\}$, condition (2), too.

3. EXTENSION OF LANGUAGES

3.1. Theorem. *Let $\mathscr{L}$ be a language from $\mathscr{C}_0$, let $A \in \mathbf{d}\mathscr{L}$, $\alpha \in \mathscr{L}A$, $1 \leq i_1 \leq i_2 \leq \lambda\alpha$, $X \notin \mathbf{a}\mathscr{L}$. Define the transformation $\mathscr{L}_1$ as follows: $\mathbf{d}\mathscr{L}_1 = \mathbf{d}\mathscr{L} \cup \{X\}$, $\mathscr{L}_1 B = \mathscr{L}B$ if $B \notin \{A, X\}$; $\mathscr{L}_1 A = (\mathscr{L}A - \{\alpha\}) \cup \{\alpha^{(1, i_1 - 1)} \times [X] \times \alpha^{(i_2+1, \lambda\alpha)}\}$, $\mathscr{L}_1 X = \{\alpha^{(i_1, i_2)}\}$. Then $\mathscr{L}_1 \in \mathscr{C}_0$ (we shall say about a simple extension of $\mathscr{L}$ or about (A, α, i_1, i_2, X)-extension of $\mathscr{L}$), and $\mathscr{L}_1$ is s. u. if and only if so is $\mathscr{L}$.*

Proof. Obviously $\mathscr{L}_1$ is a language and $\mathscr{L}$ is a $(X, \alpha^{(i_1, i_2)})$—reduction of $\mathscr{L}_1$. If $\mathscr{L}_1$ would be cyclic, there would be a $C \in \mathbf{d}\mathscr{L}_1$ such that $\mathscr{L}_1$: $[C] \to [C]$. By (2.3.4), we have (note that in proving (2.3.4) we have not used the assumption that the language $\mathscr{L}$ considered in Theorem 2.3 is not cyclic), that either $\mathscr{L}$: $[C] \to [C]$ or, if $C = X$, $\mathscr{L}: \alpha^{(i_1, i_2)} \to \alpha^{(i_1, i_2)}$, which contradicts the noncyclicity of $\mathscr{L}$. Thus, $\mathscr{L}_1 \in \mathscr{C}_0$. It is easy to see, from the definition of $\mathscr{L}_1$, that for $\mathscr{L}_1$, for X and for $\alpha^{(i_1, i_2)}$ condition (2.3.1) holds, and therefore, by Theorem 2.3, $\mathscr{L}_1$ is s. u. if and only if so is $\mathscr{L}$.

3.2. Corollary. *Let $\mathscr{L} \in \mathscr{C}_0$ and let $\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n$ be a sequence of transformations such that $\mathscr{L}_0 = \mathscr{L}$ and, for $i = 0, 1, \ldots, n-1$, $\mathscr{L}_{i+1}$ is a simple extension of $\mathscr{L}_i$. Then $\mathscr{L}_n \in \mathscr{C}_0$ ($\mathscr{L}_n$ is called extension of $\mathscr{L}$) and $\mathscr{L}_n$ is s. u. if and only if so is $\mathscr{L}$.*

3.3. Remark. In studying the structural unambiguity of languages from $\mathscr{C}_2$ it suffices to investigate the languages such that

(1) $\lambda\alpha \leq 2$ for each metatext α.

Indeed, let $\mathscr{L}$ be a language from $\mathscr{C}_2$. By suitable extension of $\mathscr{L}$ we can obtain a language $\mathscr{L}_0$ which satisfies condition (1) and, by Corollary 3.2, which is s. u. if and only if so is $\mathscr{L}$.

Moreover, by suitable extension of a language $\mathscr{L} \in \mathscr{C}_2$, we can obtain the language $\mathscr{L}$ satisfying not only condition (1) but also the following two conditions:

(2) If $B \in \mathbf{d}\mathscr{L}_1$, $\alpha_1, \alpha_2 \in \mathscr{L}_1 B$, $\alpha_1 \neq \alpha_2$, $\lambda\alpha_1 + \lambda\alpha_2 > 2$, then $\mathbf{symb}\ \{\alpha_1\} \cap \cap\ \mathbf{symb}\ \{\alpha_2\} = \Lambda$.

(3) If $B_1, B_2 \in \mathbf{d}\mathscr{L}_1$, $\alpha_1 \in \mathscr{L}_1 B_1$, $\alpha_2 \in \mathscr{L}_1 B_2$, $B_1 \neq B_2$, $\lambda\alpha_1 + \lambda\alpha_2 > 2$, then $\mathbf{symb}\ \{\alpha_1\} \cap \mathbf{symb}\ \{\alpha_2\} = \Lambda$.

Example. Let the language $\mathscr{L}$ be defined as follows: $\mathbf{d}\mathscr{L} = \{A, B, E\}$, $\mathscr{L}A = \{[B, C, D], [E, A]\}$, $\mathscr{L}B = \{[C, E]\}$, $\mathscr{L}E = \{[A]\}$. Let

$\mathscr{L}_1$ be an $(A, [B, C, D], 2, 3, F)$—extension of $\mathscr{L}$,
$\mathscr{L}_2$ be an $(A, [E, A], 1, 1, G)$ —extension of $\mathscr{L}_1$,
$\mathscr{L}_3$ be an $(A, [G, A], 2, 2, H)$ —extension of $\mathscr{L}_2$,
$\mathscr{L}_4$ be a $(B, [C, E], 1, 1, J)$ —extension of $\mathscr{L}_3$,
$\mathscr{L}_5$ be a $(B, [J, E], 2, 2, K)$ —extension of $\mathscr{L}_4$,
$\mathscr{L}_6$ be a $(F, [C, D], 1, 1, L)$ —extension of $\mathscr{L}_5$,

then $\mathscr{L}_6$ is the extension of $\mathscr{L}$ and $\mathscr{L}_6$ satisfies condition (1) to (3).

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