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A GENESIS FOR COMBINATORIAL IDENTITIES

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The object of this article is to prove the following

Theorem. *Let*

$$a_i,\ i = 0, 1, 2, \dots, n,$$

and

$$x_j,\ j = 1, 2, 3, \dots, m$$

be the given complex numbers with the condition that the numbers a_i are distinct.

If we denote

$$S(m, n) = \sum_{i=0}^{n} \frac{(a_i - x_1)(a_i - x_2)\dots(a_i - x_m)}{(a_i - a_0)\dots(a_i - a_{i-1})(a_i - a_{i+1})\dots(a_i - a_n)},$$

then

$$S(n + 1, n) = \sum_{i=0}^{n} a_i - \sum_{j=1}^{n+1} x_j, \tag{1}$$

$$S(n, n) = 1, \tag{2}$$

$$S(m, n) = 0, \quad m < n. \tag{3}$$

The next sections contain two proofs of this Theorem. The first proof uses the mathematical induction, the second proof is based on the calculus of residues. By the method used in this proof we can evaluate also the sums $S(n + 2, n)$, $S(n + 3, n)$, ...

Finally, using the above Theorem, we derive in the last section some combinatorial identities.

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a) First, we can show that for $S(m, n)$ the following recurrence holds

$$S(m, n) = S(m - 1, n - 1) + (a_n - x_m)S(m - 1, n), \tag{4}$$

$$m > 1,\ n > 1.$$

For this purpose we write successively

$$S(m, n) = \sum_{i=0}^{n-1} \frac{(a_i - x_1)(a_i - x_2) \dots (a_i - x_{m-1})(a_i - a_n + a_n - x_m)}{(a_i - a_0) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)} +$$

$$+ \frac{(a_n - x_1)(a_n - x_2) \dots (a_n - x_m)}{(a_n - a_0)(a_n - a_1) \dots (a_n - a_{n-1})} =$$

$$= \sum_{i=0}^{n-1} \frac{(a_i - x_1)(a_i - x_2) \dots (a_i - x_{m-1})}{(a_i - a_0) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_{n-1})} +$$

$$+ (a_n - x_m) \sum_{i=0}^{n-1} \frac{(a_i - x_1)(a_i - x_2) \dots (a_i - x_{m-1})}{(a_i - a_0) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)} +$$

$$+ (a_n - x_m) \frac{(a_n - x_1)(a_n - x_2) \dots (a_n - x_{m-1})}{(a_n - a_0)(a_n - a_1) \dots (a_n - a_{n-1})} = S(m-1, n-1) +$$

$$+ (a_n - x_m) \sum_{i=0}^{n} \frac{(a_i - x_1)(a_i - x_2) \dots (a_i - x_{m-1})}{(a_i - a_0) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)} =$$

$$= S(m-1, n-1) + (a_n - x_m) S(m-1, n)$$

so that the required relation is proved.

For the following we need also the expressions $S(1, n)$, $n > 1$.

Denoting

$$A(n) = \sum_{i=0}^{n} \frac{a_i}{(a_i - a_0) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)},$$

$$B(n) = \sum_{i=0}^{n} \frac{1}{(a_i - a_0) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)},$$

we have

(5) $$S(1, n) = \sum_{i=0}^{n} \frac{a_i - x_1}{(a_i - a_0) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)} =$$

$$= A(n) - x_1 B(n)$$

and

$$S(1, n+1) = \sum_{i=0}^{n+1} \frac{a_i - x_1}{(a_i - a_0) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)(a_i - a_{n+1})} =$$

$$= \sum_{i=0}^{n} \frac{a_i - a_{n+1} + a_{n+1} - x_1}{(a_i - a_0) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)(a_i - a_{n+1})} +$$

$$+ (a_{n+1} - x_1) \frac{1}{(a_{n+1} - a_0)(a_{n+1} - a_1) \dots (a_{n+1} - a_n)} =$$

$$= B(n) + (a_{n+1} - x_1) \sum_{i=0}^{n+1} \frac{1}{(a_i - a_0) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_{n+1})} =$$

$$= B(n) + (a_{n+1} - x_1) B(n+1)$$

so that

(6) $$S(1, n+1) = B(n) + (a_{n+1} - x_1) B(n+1).$$

b) Now we prove that

(7) $$S(1, n) = 0, \ n > 1.$$

We proceed by induction. Because

$$S(1, 2) =$$

$$= \frac{a_0 - x_1}{(a_0 - a_1)(a_0 - a_2)} + \frac{a_1 - x_1}{(a_1 - a_0)(a_1 - a_2)} + \frac{a_2 - x_1}{(a_2 - a_0)(a_2 - a_1)} = 0,$$

the assertion is true for $n = 2$. We suppose further that (7) holds for some $n \geqq 2$. Because x_1 is an arbitrary number, this assumption says, with respect to (5), that

(8) $$B(n) = 0.$$

Let us now consider the equation in the variable x_1

(9) $$S(1, n+1) = 0.$$

From (6) with use of (8) we have

$$S(1, n+1) = (a_{n+1} - x_1) B(n+1)$$

so that this equation has the root $x_1 = a_{n+1}$. But $S(1, n+1)$ is a symmetric function in a_i so that (9) has also the roots a_i, $i = 0, 1, \dots, n$. The equation (9) of degree 1 in the variable x_1 has more than 1 roots, therefore it is an identity.

We have shown that if (7) holds for some $n \geqq 2$, then this relation holds also for $n + 1$, hence it holds generally.

c) To prove the correctness of (3), we proceed by induction with respect to m. Because (7) holds generally, the relation (3) holds for $m = 1$. We suppose now that (3) holds for a given $m = m' > 1$ and all $n > m'$ and we will prove that also

(10) $$S(m' + 1, n) = 0, \ n > m' + 1.$$

But in virtue of the recurrence (4) we have

$$S(m'+1, n) = S(m', n-1) + (a_n - x'_{m+1})S(m', n)$$

and both the sums on the right side of this equation are zeros in accordance with the assumption because from $n > m' + 1$ it follows that $n > m'$ and $n - 1 > m'$.

Thus the relation (10) holds, the induction is finished and the relation (3) established.

d) It remains to prove the equations (1) and (2). Because

$$S(1, 1) = \frac{a_0 - x_1}{a_0 - a_1} + \frac{a_1 - x_1}{a_1 - a_0} = 1,$$

the equation (2) is true for $n = 1$. Supposing that it holds for some n, we can show that also

(11) $$S(n+1, n+1) = 1.$$

But according to the recurrence (4) we have

$$S(n+1, n+1) = S(n, n) + (a_{n+1} - x_{n+1})\, S(n, n+1) = S(n, n) = 1$$

because in virtue of (3) $S(n, n+1) = 0$. Therefore (2) holds generally.

In the same way we prove the equation (1). For $n = 1$ the relation holds because

$$S(2, 1) = \frac{(a_0 - x_1)(a_0 - x_2)}{a_0 - a_1} + \frac{(a_1 - x_1)(a_1 - x_2)}{a_1 - a_0} = a_0 + a_1 - (x_1 + x_2).$$

Now we suppose that for some n

$$S(n+1, n) = \sum_{i=0}^{n} a_i - \sum_{j=1}^{n+1} x_j$$

and we will show that

$$S(n+2, n+1) = \sum_{i=0}^{n+1} a_i - \sum_{j=1}^{n+2} x_j.$$

But using again the recurrence (4) we obtain

$$S(n+2, n+1) = S(n+1, n) + (a_{n+1} - x_{n+2})\, S(n+1, n+1) =$$

$$= \sum_{i=0}^{n} a_i - \sum_{j=1}^{n+1} x_j + a_{n+1} - x_{n+2} = \sum_{i=0}^{n+1} a_i - \sum_{j=1}^{n+2} x_j$$

because in virtue of (2) $S(n+1, n+1) = 1$. (1) holds therefore generally.

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Let

$$f(x) = (x - x_1)(x - x_2) \ldots (x - x_m) = x^m - \sigma_1 x^{m-1} + \sigma_2 x^{m-2} - \ldots$$

and

$$g(x) = (x - a_0)(x - a_1) \ldots (x - a_n) = x^{n+1} - \tau_1 x^n + \tau_2 x^{n-1} - \ldots$$

be the given polynomials and let the numbers a_i be distinct.

We wish to find the values of the integrals in the equation

(13) $$\frac{1}{2\pi i} \int_c \frac{f(x)}{g(x)}\, dx = \frac{1}{2\pi i} \int_{c'} \frac{f\left(\frac{1}{y}\right)}{y^2 g\left(\frac{1}{y}\right)}\, dy,$$

where c is any circle with the center at $x = 0$ having the radius so large that the points a_i are all inside c and c' is a circle with the center in the origin.

The value of the left-hand expression is equal to the sum of residues of the function

$$F(x) = \frac{f(x)}{g(x)}$$

at the simple poles a_i. If we denote by A_i the residuum of $F(x)$ at a_i, then

$$A_i = \lim_{x \to a_i} (x - a_i) \frac{f(x)}{g(x)} = \frac{f(a_i)}{g'(a_i)} =$$

$$= \frac{(a_i - x_1)(a_i - x_2) \ldots (a_i - x_m)}{(a_i - a_0) \ldots (a_i - a_{i-1})(a_i - a_{i+1}) \ldots (a_i - a_n)}$$

so that the left-hand side expression of (13) equals $S(m, n)$.

To evaluate the expression on the right side of this equation, we denote

$$G(y) = \frac{f\left(\frac{1}{y}\right)}{y^2 g\left(\frac{1}{y}\right)}$$

and after some modifications we have

$$G(y) = y^{n-m-1}\{1 + (\tau_1 - \sigma_1)y + (\sigma_2 + \tau_1^2 - \tau_1\sigma_1 - \tau_2)\, y^2 + \ldots\}.$$

We are now in the condition to prove the required relations (1), (2), (3).

If for example $m < n$, then $n - m - 1 \geqq 0$ and $G(y)$ is an anylytic function inside c' and the mentioned integral is zero. We have the relation (3)

$$S(m, n) = 0, \; m < n .$$

For $m = n$ the origin is a simple pole for $G(y)$ with the residuum 1. Thus

$$S(n, n) = 1$$

and this is the equation (2).

Finally, for $m = n + 1$ the point $y = 0$ is a double pole for $G(y)$ with the respective residuum $\tau_1 - \sigma_1$ so that

$$S(n+1, n) = \tau_1 - \sigma_1 = \sum_{i=0}^{n} a_i - \sum_{j=1}^{n+1} x_j .$$

As we have already mentioned, we can evaluate by the same method the sums $S(m, n)$ for every $m > n$. In fact, if we have $m = n + 2$ for example, the origin is a pole of order 3 for $G(y)$ with the residuum

$$\sigma_2 + \tau_1(\tau_1 - \sigma_1) - \tau_2$$

so that

$$S(n+2, n) = \sigma_2 + \tau_1(\tau_1 - \sigma_1) - \tau_2$$

and so on.

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On the basis of the Theorem we can now derive any binomial formulas.

a) We put

$$a_i = i, \; x_j = -x,$$

x being an arbitrary complex number. Then

$$(a_i - x_1)(a_i - x_2) \dots (a_i - x_m) = (x + i)^m,$$
$$(a_i - a_0)(a_i - a_1) \dots (a_i - a_{i-1}) = i!$$
$$(a_i - a_{i+1})(a_i - a_{i+2}) \dots (a_i - a_n) = (-1)^{n-i}(n - i)!$$

so that

$$S(m, n) = \frac{(-1)^n}{n!} \sum_{i=0}^{n} (-1)^i (x + i)^m \binom{n}{i} .$$

Moreover

$$\sum_{i=0}^{n} a_i - \sum_{j=1}^{m} x_j = mx + \frac{n(n+1)}{2} .$$

The equations (1), (2), (3) give the following results:

$$\text{(15)} \qquad \sum_{i=0}^{n} (-1)^i (x + i)^{n+1} \binom{n}{i} = (-1)^n (n+1)! \left(x + \frac{n}{2}\right),$$

$$\sum_{i=0}^{n} (-1)^i (x + i)^n \binom{n}{i} = (-1)^n n!$$

$$\sum_{i=0}^{n} (-1)^i (x + i)^m \binom{n}{i} = 0, \; m < n\text{(1)}.$$

With $-x$ instead of x the last two equations (15) give

$$\text{(15')} \qquad \sum_{i=0}^{n} (-1)^i (x - i)^n \binom{n}{i} = n!,$$

$$\sum_{i=0}^{n} (-1)^i (x - i)^m \binom{n}{i} = 0, \; m < n .$$

(1) The first equation of (15) is a simple consequence of the second. In fact, if we denote

$$\bar{S}(m, n, x) = \sum_{i=0}^{n} (-1)^i (x + i)^m \binom{n}{i}$$

we have

$$\bar{S}(n+1, n, x) = \sum_{i=0}^{n} (-1)^i (x + i)^n (x + i) \binom{n}{i} = x \sum_{i=0}^{n} (-1)^i (x + i)^n \binom{n}{i} +$$
$$+ n \sum_{i=1}^{n} (-1)^i (x + i)^n \binom{n-1}{i-1} = (-1)^n x n! - n \sum_{i=0}^{n-1} (-1)^i (x + 1 + i)^n \binom{n-1}{i} =$$
$$= (-1)^n x n! - n\bar{S}(n, n-1, x+1)$$

so that for the sums $\bar{S}(m, n, x)$ the following recurrence holds

$$\bar{S}(n+1, n, x) = (-1)^n x n! - n\bar{S}(n, n-1, x+1).$$

Now putting $n - 1, n - 2, \dots, 2, 1, 0$ instead of n and $x + 1, x + 2, \dots, x + n$ instead of x, we obtain from here

$$\bar{S}(n, n-1, x+1) = (-1)^{n-1}(x+1)(n-1)! - (n-1)\bar{S}(n-1, n-2, x+2),$$
$$\bar{S}(n-1, n-2, x+2) = (-1)^{n-2}(x+2)(n-2)! - (n-2)\bar{S}(n-2, n-3, x+3),$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$\bar{S}(1, 0, x+n) = x + n .$$

Finally, multiplying these relations successively by

$$1, -n, n(n-1), \dots, (-1)^n n!$$

and adding the thus obtained results we have

$$\bar{S}(n+1, n, x) = (-1)^n x n! + (-1)^n (x+1) n! + \dots$$
$$\dots + (-1)^n (x+n) n! = (-1)^n n! \left\{(n+1)x + \frac{1}{2} n(n+1)\right\} =$$
$$= (-1)^n (n+1)! \left(x + \frac{n}{2}\right) .$$

The first equation of (15) is thus derived from the second.

For arbitrary real integer x these formulas are well-known. See for example [1] (p. 65, formula 4).

b) We put

$$a_i = i, \; x_j = -x - j,$$

x being an arbitrary complex number. Introducing the symbol

$$(y)_k = y(y-1)\dots(y-k+1)$$

we have

$$(a_i - x_1)(a_i - x_2)\dots(a_i - x_m) = (x + m + i)_m.$$

Further we have as in the preceeding case

$$(a_i - a_0)(a_i - a_1)\dots(a_i - a_{i-1}) = i!,$$

$$(a_i - a_{i+1})(a_i - a_{i+2})\dots(a_i - a_n) = (-1)^{n-i}(n-i)!$$

so that

$$S(m, n) = (-1)^n \frac{m!}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{x+m+i}{m} =$$

$$= \frac{m!}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{x+m+n-i}{m}.$$

Moreover

$$\sum_{i=0}^{n} a_i - \sum_{j=1}^{n+1} x_j = (n+1)(x+n+1).$$

The equations (1), (2) and (3) give the following results:

$$\frac{m!}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{x+m+i}{m} =$$

(16)
$$= \begin{cases} (-1)^n (x+n+1) & \text{for } m = n+1, \\ (-1)^n & \text{for } m = n, \\ 0 & \text{for } m < n, \end{cases}$$

or

(16′)
$$\frac{m!}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{x+m+n-i}{m} =$$

$$= \begin{cases} x+n+1 & \text{for } m = n+1, \\ 1 & \text{for } m = n. \\ 0 & \text{for } m < n. \end{cases}$$

These formulas are all special cases of the combinatorial relation

(16″)
$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{x+i}{l} = (-1)^n \binom{x}{l-n}.$$

See [2].

c) We put

$$a_i = ai + i^2, \; x_j = -x - j$$

a and x being arbitrary complex numbers. Then

$$(a_i - x_1)(a_i - x_2)\dots(a_i - x_m) = (x + m + ai + i^2)_m,$$

$$(a_i - a_0)(a_i - a_1)\dots(a_i - a_{i-1}) = i!\,(a + 2i - 1)_i,$$

$$(a_i - a_{i+1})(a_i - a_{i+2})\dots(a_i - a_n) = $$

$$= (-1)^{n-i}(n-i)!\,(a + n + i)_{n-i}$$

so that

$$S(m, n) = (-1)^n \sum_{i=0}^{n} (-1)^i \frac{(x + m + ai + i^2)_m}{i!(n-i)!(a+2i-1)_i (a+n+i)_{n-i}} =$$

$$= \frac{(-1)^n m!}{(n!)^2} \sum_{i=0}^{n} (-1)^i \frac{\binom{n}{i}^2 \binom{x+m+ai+i^2}{m}}{\binom{a+2i-1}{i}\binom{a+n+i}{n-i}}.$$

Moreover

$$\sum_{i=0}^{n} a_i - \sum_{j=1}^{n+1} x_j = (n+1)\left\{x + \frac{an}{2} + \frac{n^2+2n+3}{3}\right\}.$$

The equations (1), (2), (3) give the following results:

(17)
$$\sum_{i=0}^{n} (-1)^i \frac{\binom{n}{i}^2 \binom{x+n+1+ai+i^2}{n+1}}{\binom{a+2i-1}{i}\binom{a+n+i}{n-i}} =$$

$$= (-1)^n n! \left\{x + \frac{an}{2} + \frac{n^2+2n+3}{3}\right\},$$

$$\sum_{i=0}^{n} (-1)^i \frac{\binom{n}{i}^2 \binom{x+n+ai+i^2}{n}}{\binom{a+2i-1}{i}\binom{a+n+i}{n-i}} = (-1)^n n!$$

$$\sum_{i=0}^{n} (-1)^i \frac{\binom{n}{i}^2 \binom{x+m+ai+i^2}{m}}{\binom{a+2i-1}{i}\binom{a+n+i}{n-i}} = 0\,, \; m < n.$$

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