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ONE-POLE APPROXIMATION FOR HIGHER WAVES AND THE CORRECT THRESHOLD BEHAVIOUR OF PHASE SHIFTS

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Treating the nonrelativistic elastic scattering, the Jost function is approximated by one pole and it is emphasized that this leads to the threshold behaviour of the phase shifts $\eta_l(k)$ expressed exactly by the formula $k \cot g \, \eta_l = a_l + b_l k^2$ (the a_l's and b_l's are constants) for any physical $l \geqq 0$. This does not correspond to the elastic scattering for $l \geqq 1$. The correct behaviour of the phase shifts for low energies expressed by the effective range theory generalized for higher waves $k^{2l+1} \cot g \, \eta_l = \alpha_l + \beta_l k^2 + \ldots$ can be obtained by considering more poles even when in the last series expansion only the first two terms are under consideration. However, in this case the pole parameters must obey some conditions. The connection between the number of poles and the correct threshold behaviour of the phase shift for a given angular momentum is discussed. Lastly the Jost function approximated by more poles for higher waves is treated. The derivation of potentials and phase shifts is performed by solving the Gel'fand-Levitan equation.

I. INTRODUCTION

The solutions of the inversion problem of the nonrelativistic scattering theory can be schematically divided into three groups: The solutions using a) the integral equations, b) a particular Ansatz usually based on the supposed analytic properties and c) the first Born approximation or its improvements. All three groups are often generalized and extended to more complicated cases. The presented paper belongs to the first group of mentioned methods and its results are in close connection with the ones of the second group.

By searching for an appropriate model to the given scattering data one often proceeds from the one-pole approximation for a function under consideration. We use a one-pole approximation for the higher-wave Jost function and by means of the Gel'fand-Levitan equation the potentials and the phase shifts are found. This problem was partially discussed in [1] with the main result that the potentials with a finite number of-non-negative point eigenvalues induce the zero phase shifts. However we distinguish here explicitly two cases corresponding to the symmetrical and asymmetrical Gel'fand-Levitan kernel $P_l(r, \xi)$ and for the assymetrical case we emphasize the behaviour of the potentials both near the origin and near the infinity. Generalizing the last case it can be shown that the potentials can have an assymptotic tail e.g. r^{-3} and r^{-2} respectively; however in recent literature there is absent a detailed investigation of the behaviour of the potentials resulting from the solution of the inversion problem in scattering theory within the bounds of the first group mentioned above.

In order to be self-contained we derive in this paper after establishing some basic relations, in Sec. 3 the Gel'fand-Levitan equation for higher waves by a slightly adapted and generalized Levinson's way used for s-scattering in [2], [3]. In Sec. 4 we solve the Gel'fand-Levitan equation for the Jost function approximated by one pole. The expressions for potentials are explicitly given in various cases. The problems in which the Jost function is approximated by more poles can be solved by similar methods (see Sec. 5). The potentials obtained are special cases of the Barmann's potentials for higher waves. From this point of view the present paper can be also considered as a contribution to the theory of Bargmann's potentials; the models based on the potentials are continually used (see e.g. [4]). We discuss also the connection between the number of approximating poles in the Jost function and the correct behaviour of the phase shifts for low energies.

II. BASIC RELATIONS

The Schrödinger equation for the radial part of the wave function can be written in the form

(1) $$\frac{d^2u_l(k, r)}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2} - V\right] u_l(k, r) = 0$$

where $k^2 = E$ is the energy ($h/2\pi = 2M = 1$) and $V = V(r)$ is the potential. The repulsive barrier is explicitly written down.

We denote

$$zj_l(z) = \mu_l(z),$$
$$zn_l(z) = \nu_l(z),$$
$$zh_l^{(2)}(z) = E_l^{(2)}(z)$$

where $j_l(z)$ and $n_l(z)$ are, respectively, the spherical Bessel and Neumann functions and $h_l^{(2)}(z)$ are the spherical Hankel functions of the second kind.

In what follows the following relations will be used:

1. The free motion functions:

a) the regular solution $\varphi_l(k, r) \equiv \varphi_l^{(0)}(k, r)$

(2) $$\varphi_l^{(0)}(k, r) = \frac{(2l+1)!!}{k^{l+1}} \mu_l(kr);$$

b) the Jost solution $f_l(k, r) \equiv f_l^{(0)}(k, r)$

$$f_l^{(0)}(k, r) = i^{-(l+1)} E_l^{(2)}(kr);$$

c) the Jost function $f_l(k) \equiv f_l^{(0)}(k)$

(3) $$f_l^{(0)}(k) = \frac{(2l+1)!!}{(ik)^l}.$$

2. The asymptotic form of the regular solution

(4) $$\varphi_l(k, r) \underset{r\to\infty}{\sim} \frac{|f_l(k)|}{k} \sin\left(kr - \frac{\pi l}{2} + \eta_l\right).$$

Introducing the function $F_l(k)$ (which is often referred to as the Jost function [5], [6])

(5) $$F_l(k) = \frac{f_l(k)}{f_l^{(0)}(k)}$$

we express the phase shifts $\eta_l(k)$ in the form

(6) $$\operatorname{cotg} \eta_l(k) = \frac{\operatorname{Re}F_l(k)}{\operatorname{Im}F_l(k)}.$$

The relation between the elements of the S-matrix $S_l(k)$ and the Jost function $f_l(k)$ is the following

$$S_l(k) = e^{i\pi l} \frac{f_l(k)}{f_l(-k)} = e^{2i\eta_l(k)}.$$

We have

$$f_l(k) = |f_l(k)| e^{i\delta_l(k)}, \quad \delta_l(k) = \eta_l(k) - \frac{\pi l}{2}.$$

3. The completeness relation for the regular solutions

(7) $$\int_{-\infty}^{+\infty} \varphi_l(k, r)\varphi_l(k, r') d\varrho_l(E) = \delta(r - r').$$

For a given angular momentum l, the spectral function $\varrho_l(E)$ in the case when there exists besides the continuous spectrum for positive energies also a discrete spectrum with eigenvalues $k = -i\varkappa_s$, $\varkappa_s > 0$ $(s = 1, 2, \ldots, n_l)$ is expressed in the form

$$\frac{d\varrho_l(E)}{dE} = \begin{cases} = \dfrac{1}{\pi} \cdot \dfrac{k}{|f_l(k)|^2} & \text{for } E > 0, \\ = \sum\limits_{s=1}^{n_l} M_s^2 \cdot \delta(E - E_s) & \text{for } E \leqq 0 \end{cases}$$

(let $\varrho_l(-\infty) = 0$); the normalizing constants M_s are given by $M_s^{-2} = \int\limits_0^\infty [\varphi_l(-i\varkappa_s, r)]^2 \, dr$. For the free particle case we have $\dfrac{d\varrho_l(E)}{dE} = \dfrac{d\varrho_l^{(0)}(E)}{dE} = \dfrac{1}{\pi} \cdot \dfrac{k}{|f_l^{(0)}(k)|^2}$ for $E > 0$, $\dfrac{d\varrho_l^{(0)}(E)}{dE} = 0$ for $E \leqq 0$. We introduce an auxiliary function $\sigma_l(E)$

(8) $$\frac{d\sigma_l(E)}{dE} = \frac{d\varrho_l(E)}{dE} - \frac{d\varrho_l^{(0)}(E)}{dE} = \begin{cases} = \dfrac{k}{\pi} \cdot \left[\dfrac{1}{|f_l(k)|^2} - \dfrac{1}{|f_l^{(0)}(k)|^2} \right] & \text{for } E > 0 \\ = \dfrac{d\varrho_l(E)}{dE} & \text{for } E \leqq 0 . \end{cases}$$

For the free motion the completeness relation (7) has the form

(9) $$\int\limits_{-\infty}^{+\infty} \varphi_l^{(0)}(k, r) \varphi_l^{(0)}(k, r') d\varrho_l^{(0)}(E) = \frac{2}{\pi} \int\limits_0^\infty \mu_l(kr) \mu_l(kr') dk = \delta(r - r').$$

4. The regular solution $\varphi_l(k, r)$ can be expressed by means of that for the free particle case (2) in the form

(10) $$\varphi_l(k, r) = \varphi_l^{(0)}(k, r) + \int\limits_0^r K_l(r, t) \varphi_l^{(0)}(k, t) dt .$$

It can be shown that the kernel $K_l(r, t)$ satisfies the following conditions

(11) $$\frac{\partial^2 K_l(r, t)}{\partial r^2} - \frac{\partial^2 K_l(r, t)}{\partial t^2} - \left[\frac{l(l+1)}{r^2} - \frac{l(l+1)}{t^2} \right] K_l(r, t) = V(r) K_l(r, t)$$

(12) $$2 \frac{dK_l(r, r)}{dr} = V(r)$$

and

(13′) $$\lim_{t \to 0} \left[\mu_l(kt) \frac{\partial K_l(r, t)}{k \partial t} - K_l(r, t) \frac{\partial \mu_l(kt)}{k \partial t} \right] = 0$$

or

(13) $$K_l(r, 0) = 0 .$$

The conditions (11), (12) and (13) are equivalent to the Schrödinger equation (1) with boundary condition for the regular solution.

III. THE GEL'FAND-LEVITAN EQUATION FOR HIGHER WAVES

We generalize here briefly for the higher waves the slightly modified version [3] of the Levinson's derivation [2] of the Gel'fand-Levitan equation which was performed for s-scattering.

Similarly to (10) we can write

(14) $$\varphi_l^{(0)}(k, r') = \varphi_l(k, r') + \int\limits_0^{r'} N_l(r, t) \varphi_l(k, t) dt$$

where the kernel $N_l(r, t)$ has similar properties as $K_l(r, t)$. We multiply (14) by $\varphi_l(k, r)$ and integrate (according to Stieltjes) with respect to $\varrho_l(E)$. Using the completeness relation (7) we obtain for $0 \leqq r' < r$

(15) $$\int\limits_{-\infty}^{+\infty} \varphi_l^{(0)}(k, r') \varphi_l(k, r) d\varrho_l(E) = 0 .$$

In (15) we express the regular solution $\varphi_l(k, r)$ in accordance with (10). We get

$$\int\limits_{-\infty}^{+\infty} \varphi_l^{(0)}(k, r') \varphi_l^{(0)}(k, r) d\varrho_l(E) + \int\limits_0^r dt K_l(r, t) \int\limits_{-\infty}^{+\infty} \varphi_l^{(0)}(k, r') \varphi_l^{(0)}(k, t) d\varrho_l(E) = 0 .$$

We replace the spectral function $\varrho_l(E)$ by means of $\sigma_l(E)$ (8). Taking into account (9) we get

(16) $$\int\limits_{-\infty}^{+\infty} \varphi_l^{(0)}(k, r') \varphi_l^{(0)}(k, r) d\sigma_l(E) + \int\limits_0^r dt K_l(r, t) \int\limits_{-\infty}^{+\infty} \varphi_l^{(0)}(k, r) \varphi_l^{(0)}(k, r') d\sigma_l(E) + K_l(r, r') = 0 .$$

We introduce the function

(17a) $$P_l(r, r') = \int\limits_{-\infty}^{+\infty} \varphi_l^{(0)}(k, r') \varphi_l^{(0)}(k, r) d\sigma_l(E)$$

for

$$r' \leqq r. \tag{17b}$$

By means of (17) we obtain from (16) the Gel'fand-Levitan equation in the form

$$P_l(r, r') + \int_0^{r'} K_l(r, \xi) P_l(r', \xi) d\xi + \int_{r'}^{r} K_l(r, \xi) P_l(\xi, r') d\xi + \\ + K_l(r, r') = 0. \tag{18}$$

If the kernel $P_l(r, r')$ is symmetrical we have

$$P_l(r, r') + \int_0^r K_l(r, \xi) P_l(\xi, r') d\xi + K_l(r, r') = 0. \tag{19}$$

If we consider for instance the problem mentioned above, (8) ,we have

$$P_l(r, r') = \sum_{s=1}^{m} M_s^2 \varphi_l^{(0)}(-i\varkappa_s, r) \varphi_l^{(0)}(-i\varkappa_s, r') + \\ + \frac{2}{\pi} \int_0^\infty \left[\frac{1}{|f_l(k)|^2} - \frac{1}{|f_l^{(0)}(k)|^2} \right] \varphi_l^{(0)}(k, r) \varphi_l^{(0)}(k, r') k^2 dk \tag{20}$$

where the free particle regular solutions $\varphi_l^{(0)}$ are given in (2).

Solving consequentially the inversion problem we should proceed as follows: Postulating the basic integral equation (18) we have to determine the properties of the solutions and potentials obtained. This, however, is rather hard to accomplish. We consider the potentials obtained in the one-pole approximation of the Jost function for higher waves. Some of their properties are being emphasized. However, for the correct threshold behaviour of the phase shifts it is necessary to consider more poles in the Jost functions. Namely for a given angular momentum l we must take into account at least $l + 1$ poles and between them l conditions are to be satisfied in order to obtain the correct behaviour of the phase shifts for low energies ($\eta_l \underset{k \to 0}{\sim} k^{2l+1}$). However the potentials obtained by considering more poles in the Jost function are in the higher-wave case more extensive and cumbersome; for this reason we give for $l \geqq 1$ only the solution of the Gel'fand-Levitan equation (18) assuming there are no bound states.

IV. ONE-POLE APPROXIMATION

In what follows the discussion leads us to the Jost function of the form $f_l(k) = f_l^{(0)}(k) \cdot (k - i\beta)/(k - i\alpha)$ for various values of α and β. In particular cases the phase shifts can be determined with respect to (6) or by direct computation; this is performed mainly when the potential induced does not satisfy the usually imposed conditions (e.g. the existence of the first and second absolute moments).

A. Symmetrical degenerate kernel

As the first example of the use of the Gel'fand-Levitan equation in the case of the general (integer non-negative) value of the angular momentum we consider the simplest case of a symmetrical degenerate kernel $P_l(r, r')$. Let

$$P_l(r, \xi) = A_l(r) A_l(\xi)$$

and with respect to (13)

$$A_l(0) = 0. \tag{21}$$

Using the Gel'fand-Levitan equation (19) we get

$$K_l(r, \xi) = \frac{-A_l(r) A_l(\xi)}{1 + \int_0^r [A_l(t)]^2 dt}. \tag{22}$$

If we put (22) into (11) we obtain

$$\frac{d^2 A_l(r)}{dr^2} + \frac{l(l+1)}{r^2} A_l(r) = \text{const.}\ A_l(r) \equiv -p^2 \cdot A_l(r).$$

Taking into account the boundary condition (21) there are the following possibilities:

1a) $p^2 < 0$ }
1b) $p^2 > 0$ } $p^2 \neq 0$, then $A_l(r) \sim \mu_l(pr)$;
2. $p^2 = 0$, then $A_l(r) \sim r^{l+1}$.

Ad 1. Let

$$P_l(r, \xi) = N \mu_l(pr) \mu_l(p\xi)$$

(N = const). Considering (22) we have

$$K_l(r, \xi) = \frac{-2Np \cdot \mu_l(pr) \mu_l(p\xi)}{2p + Npr[\mu_l^2(pr) - \mu_{l-1}(pr) \cdot \mu_{l+1}(pr)]}.$$

One can obtain the potential by means of (12)

$$V(r) = -4Np \frac{d}{dr} \frac{\mu_l^2(pr)}{2p + Npr[\mu_l^2(pr) - \mu_{l-1}(pr)\mu_{l+1}(pr)]}$$

and the regular solution $\varphi_l(k, r)$ by means of (10). From the asymptotic expansion of this regular solution we get the phase shifts $\eta_l(k)$ in accordance with (4). The computation is straightforward but a little tedious. We obtain for the case 1a): Let $p = -i\tau$, $\tau > 0$. Using the formula

$$\mu_l(z) \underset{z\to\infty}{\sim} \sin \omega_l + \frac{l(l+1)}{2} \cdot \frac{\cos \omega_l}{z} + \dots$$

where $\omega_l = z - \frac{\pi}{2} l$, we have for any l exactly the relation

(23) $$k \operatorname{cotg} \eta_l = -\frac{1}{2}\tau + \frac{1}{2\tau} k^2 .$$

With respect to (5) the relation (23) corresponds to the Jost function

$$f_l(k) = f_l^{(0)}(k) \frac{k + i\tau}{k - i\tau};$$

for the case 1b): The use of the familiar formulae leads to $\operatorname{tg} \eta_l \equiv 0$ i.e. $f_l(k) = f_l^{(0)}(k)$ (almost everywhere). There exists one bound state with positive energy, the continuous spectrum is absent (similarly as in [1] and [3] for s-scattering).

Ad 2. We put

$$P_l(r, \xi) = N r^{l+1} \cdot \xi^{l+1}$$

(N = const). Denoting $\lambda = (2l + 3)/N$ we get

$$K_l(r, \xi) = \frac{-N r^{l+1} \xi^{l+1}}{1 + \frac{1}{\lambda} \cdot r^{2l+3}}$$

and

$$V = 2(2l + 3) r^{2l+1} \frac{r^{2l+3} - 2\lambda(l+1)}{(r^{2l+3} + \lambda)^2} .$$

This potential was explicitly obtained also in [1]. Making again use of the asymptotic expansion of the regular solution we obtain $\operatorname{tg} \eta_l \equiv 0$ i.e. $F_l(k) \equiv \equiv \frac{f_l(k)}{f_l^{(0)}(k)} = 1$. In this case there exists a bound state with zero energy.

B. Asymmetrical degenerate kernel

We express the Jost function in the form

(24) $$f_l(k) = f_l^{(0)}(k) \frac{k - i\beta}{k - i\alpha} .$$

Let us suppose there is no bound state and $\alpha > \beta > 0$. Using (24) we obtain from (20)

$$P_l(r, \xi) = (-1)^{l+1} \cdot \frac{\alpha^2 - \beta^2}{\beta} \cdot E_l^{(2)}(-i\beta r) \mu_l(i\beta\xi).$$

Further putting

$$K_l(r, \xi) = B_l(r) \mu(i\alpha\xi)$$

and using the formula (A 2) we obtain from (18)

$$B_l(r) = \frac{1}{D_l} \cdot i(\alpha^2 - \beta^2) E_l^{(2)}(-i\beta r)$$

where

$$D_l \equiv \beta \mu_l(i\alpha r) E_{l-1}^{(2)}(-i\beta r) + \alpha \mu_{l-1}(i\alpha r) E_l^{(2)}(-i\beta r).$$

This leads to the Eckart's potentials for higher waves

(25) $$V_E \equiv V(r) = \frac{2(\alpha^2 - \beta^2)}{(D_l)^2} \left[-\frac{2il}{r} \mu_l \cdot E_l \cdot D_l + (\beta\mu_l)^2 \cdot (E_l^2 + E_{l-1}^2) - (\alpha E_l)^2 \cdot (\mu_l^2 + \mu_{l-1}^2) \right]$$

where $\mu_l \equiv \mu_l(i\alpha r)$, $E_l \equiv E_l^{(2)}(-i\beta r)$ and $\mu_l^2 \equiv (\mu_l)^2$, $E_l^2 \equiv (E_l^{(2)})^2$. For the potential (25) the following relation holds

$$V_E(r \to 0) = \frac{-2}{2l + 1} (\alpha^2 - \beta^2) + O(r)$$

and

(26) $$V_E(r \to \infty) = (-1)^{l+1} 8\alpha^2 \frac{\alpha - \beta}{\alpha + \beta} e^{-2\alpha r} + O\left(\frac{e^{-2\alpha r}}{r}\right).$$

This potential exponentially falls down at infinity. We have further

$$k \operatorname{cotg} \eta_l = \frac{\alpha\beta}{\alpha - \beta} - \frac{1}{\alpha - \beta} k^2 .$$

We consider another rather pathological case when

$$f_l(k) = f_l^{(0)}(k)\,\frac{k - i\beta}{k}$$

and $\operatorname{cotg} \eta_l = -\frac{k}{\beta}$ (the Jost function has the behaviour $f_l(k) \underset{k\to 0}{\sim} k^{-(l+1)}$).

We have

$$P_l(r, \xi) = (-1)^l \,.\, \beta \,.\, E_l^{(2)}(-i\beta r)\mu_l(i\beta\xi)$$

and taking $K_l(r, \xi) = C_l(r) \,.\, (i\beta\xi)^{l+1}$ we get

$$C_l(r) = \frac{i\beta E_l^{(2)}(-i\beta r)}{(i\beta r)^{l+1} \,.\, E_{l+1}^{(2)}(-i\beta r)}.$$

In this case the potential is given by

$$V_x \equiv V(r) = -\frac{2\beta^2}{E_{l+1}^2}\cdot\left[\frac{2(l+1)}{i\beta r} E_{l+1} \,.\, E_l + E_{l+1}^2 + E_l^2\right]; \tag{27}$$

there is $E_l \equiv E_l^{(2)}(-i\beta r)$ and $E_l^2 \equiv [E_l^{(2)}(-i\beta r)]^2$. For the potential (27) it is true

$$V_x(r \to 0) = \frac{2\beta^2}{2l+1} + O(r)$$

and

$$V_x(r \to \infty) = \frac{2(l+1)}{r^2} + O\left(\frac{1}{r^3}\right). \tag{28}$$

The potential (27) in the s-scattering case was obtained in [5], [3].

V. THE JOST FUNCTION WITH MORE POLES

In the case without bound states we consider the Jost function(1) in the form

$$f_l(k) = f_l^{(0)}(k) \cdot \prod_{s=1}^{m_l} \frac{k - i\beta_s}{k - i\alpha_s} \tag{29}$$

(1) Solving the inversion problem for higher waves by direct substitution of the assumed Jost solution in the Schrödinger equation (using an adopted Bargmann's method, belonging to the second group of methods mentioned in the introduction) it is more convenient to take into account the Jost function of the form (see e.g. [7] eq. (22))

$$f_l(k) = f_l^{(0)}(k) + \sum_{s=1}^{m_l} \frac{A_s}{k + iB_s}$$

where all $\beta_s > 0$. We get

$$P_l(r, \xi) = (-1)^{l+1} \sum_{s=1}^{m_l} \sigma_s \,.\, E_l^{(2)}(-i\beta_s r) \,.\, \mu_l(i\beta_s\xi)$$

where

$$\sigma_s = 2i \operatorname*{Res}_{k=K_s} \left|\frac{f_l^{(0)}(k)}{f_l(k)}\right|^2;$$

the K_s' s represent the zeros of $f_l(k)$ in the upper half of the momentum plane k. Using (29) we get

$$\sigma_s = \frac{\alpha_s^2 - \beta_s^2}{\beta_s} \cdot \prod_{\substack{t=1\\ t\neq s}}^{m_l} \frac{\alpha_t^2 - \beta_s^2}{\beta_t^2 - \beta_s^2}.$$

The solution of the Gel'fand-Levitan equation (18) has the form

$$K_l(r, \xi) = \sum_{s=1}^{m_l} X_s(r) \,.\, \mu_l(i\alpha_s\xi).$$

We use (A 2) and the relation

$$\sum_{v=1}^{m_l} \frac{\sigma_v\beta_v}{\alpha_s^2 - \beta_v^2} \equiv \sum_{v=1}^{m_l} \prod_{\substack{t=1\\ t\neq v}}^{m_l} \frac{\alpha_v^2 - \beta_v^2}{\alpha_s^2 - \beta_v^2} \cdot \frac{\alpha_t^2 - \beta_v^2}{\beta_t^2 - \beta_v^2} = 1; \quad s = 1, 2. \ldots, m_l.$$

We obtain a system of linear algebraic equations for the functions $X_s(r)$

$$\sum_{t=1}^{m_l} X_t(r) \cdot \frac{1}{\alpha_t^2 - \beta_s^2} [\beta_s \mu_l(i\alpha_t r) E_{l-1}^{(2)}(-i\beta_s r) + \alpha_t \mu_{l-1}(i\alpha_t r) E_l^{(2)}(-i\beta_s r)] -$$

$$- iE_l^{(2)}(-i\beta_s r) = 0 \quad (s = 1, 2, \ldots, n)$$

which is a high-wave generalization of the system of equations obtained for s-scattering in [8].

In the case under discussion it is possible by an appropriate choise of the parameters to obtain potentials with an asymptotic tail r^{-n} where $n \geq 3$; it was mentioned also in [5]. However, if one has e.g. $V \underset{r\to\infty}{\sim} r^{-4}$, the comparison with the exactly solvable Schrödinger equation for the case $V = \text{const}/r^4$ [9] is rather cumbersome.

VI. DISCUSSION

The Jost function approximated by one pole leads to the phase shifts expressed exactly in the form $k \operatorname{cotg} \eta_l = a_l + b_l k^2$. However this is not in agreement with the theory of the elastic scattering; the theory of the effective range of nuclear forces gives in this case (e.g. [10], p. 267)

$$k^{2l+1} \operatorname{cotg} \eta_l = \alpha_l + \beta_l k^2 + \dots \tag{30}$$

With respect to (6) one concludes that the higher the waves the higher number of poles must be considered. In this case, in general, the kernel (17) is not symmetrical and the expressions obtained by means of the Gel'fand-Levitan equation are not so simple as they are for a one-pole case. From the expression (29) it follows that in order to satisfy the relation (30) [having in mind the relation (6)] it is necessary for a given angular momentum l a) to consider at least $l + 1$ poles and b) all pole parameters are not free as the l conditions must be fulfilled.

The potentials (25) and (27) with their generalizations are finite in origin and at infinity they can have an asymptotic tail r^{-2} and r^{-n} ($n \geqq 3$) respectively. It would be convenient to solve the inversion problem for singular potentials; in this case however the S-matrix has in the k plane an essential singularity in infinity. On the other hand, solving the inversion problem by means of the familiar methods one can try to find whether an inverse Carter's statement holds (²).

Lastly, it is obvious that the rational Jost function for every angular momentum leads to the potential $V = V_l(r)$ which induces in the Schrödinger equation $\hat{H}\psi \equiv (\hat{T} + \hat{U})\psi = E\psi$ a nonlocal interaction term $\hat{U}\psi$ of the form

$$\hat{U}\psi = \int_{\Omega_{\vec{n}}} \frac{d\Omega_{\vec{n}}}{4\pi} \mathscr{U}(r, \vec{n}_1 . \vec{n}) \psi(k, r; \vec{n} . \vec{n}_2) \quad \text{where} \quad \mathscr{U}(r, \vec{n}_1 . \vec{n}) = \sum_{l=0}^{\infty} (2l + 1) V_l(r)$$

$P_l(\vec{n}_1 . \vec{n})$ (the P_l' s are the Legendre polynomials).

APPENDIX

By solving the Gel'fand-Levitan equation and computing the given expressions, mainly the following relations were used:

$$E_l^{(2)}(z) \equiv z . h_l^{(2)}(z) = \mathrm{i}^{(l+1)} \mathrm{e}^{-\mathrm{i}z} \sum_{s=0}^{l} \frac{1}{2^s . s!} \cdot \frac{(l+s)!}{(l-s)!} \cdot \frac{1}{(\mathrm{i}z)^s}$$

(²) D. S. Carter proved (see [5] p. 333): If a potential satisfies the conditions: a) $\int_0^\infty r^{2l+2} |V(r)| \, dr < \infty$, b) there is no zero-energy resonance, i.e. $f_0(0) \neq 0$, then the following holds: $\lim_{k \to 0} \frac{1 - S_l(k)}{k^{2l+1}} < \infty$.

$$\mu_l(z) \equiv z j_l(z) \underset{z \to \infty}{\sim} \sin \omega_l + \frac{(l+1)!}{(l-1)!} \cdot \frac{\cos \omega_l}{2z} - \frac{(l+2)!}{(l-2)!} \cdot \frac{\sin \omega_l}{8z^2} - \frac{(l+3)!}{(l-3)!} \cdot \frac{\cos \omega^l}{48z^3} + \dots \tag{A 1}$$

where $\omega_l = z - \frac{\pi}{2} l$ and [11]

$$\mu_l(z) \nu_{l-1}(z) - \mu_{l-1}(z) \nu_l(z) = 1, \tag{A 2}$$

$$\int M_l(\alpha z) N_l(\beta z) \mathrm{d}z = \frac{1}{\alpha^2 - \beta^2} [\beta M_l(\alpha z) N_{l-1}(\beta z) - \alpha M_{l-1}(\alpha z) N_l(\beta z)],$$

$$\int [M_l(\alpha z)]^2 \mathrm{d}z = \frac{1}{2} \{z[M_l(\alpha z)]^2 - M_{l-1}(\alpha z) M_{l+1}(\alpha z)\}$$

where $M_l(z)$ and $N_l(z)$ are any linear combinations of $\mu_l(z)$ and $\nu_l(z)$ with coefficients independent on z and l.

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ОДНОПОЛЮСНОЕ ПРИБЛИЖЕНИЕ ПРИ ВЫСШИХ ВОЛНАХ И ПРАВИЛЬНОЕ ПОРОГОВОЕ ПОВЕДЕНИЕ ФАЗОВЫХ СМЕЩЕНИЙ

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Резюме

Работа занимается упругим рассеянием нерелятивистских частиц. Однополюсное приближение функции Йоста ведет к неправильному пороговому поведению фазовых смещений при угловых моментах l, больших нуля. Правильное поведение фазовых смещений при низких энергиях можно получить только многополюсным приближением, по меньшей мере $(l + 1)$-полюсным приближением, но полюсные параметры должны удовлетворять l условиям. Решая обратную задачу теории рассеяния для высших волн с помощью уравнения Гельфанда и Левитана, можно для многополюсной функции Йоста получить потенциалы, которые не удовлетворяют обычно налагаемым условиям (например, для них не обязательно существование первого и второго абсолютных моментов).