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ON A PRODUCT OF SEMIGROUPS

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To Professor A. D. Wallace on the occasion of his 60th birthday

The aim of the presented paper is to study the structure of semigroups from Definition 1. Some special cases of semigroups of the studied type are given in Theorems 26—30. The direct product of semigroups is also a special case (Remark 2).

Lemma 1. *Let Γ be a semigroup. To each element α of Γ we assign a semigroup P_α. Moreover let a set $\mathfrak{G}$ of homomorphisms $\varphi^\alpha_\beta(\alpha, \beta \in \Gamma)$ be given: φ^α_β is a homomorphism of P_α into P_β. Let $\varphi^\alpha_\beta \in \mathfrak{G}$ iff $\alpha = \beta$, or there exists $\gamma \in \Gamma$ with $\alpha\gamma = \beta$ or $\gamma\alpha = \beta$. Let the set $\mathfrak{G}$ have the following properties: 1. For $\alpha \in \Gamma$ φ^α_α denotes the identical mapping of P_α onto P_α. 2. If for $\alpha, \beta, \gamma \in \Gamma$ there exist in $\mathfrak{G}$ homomorphisms φ^α_β, φ^β_γ, φ^α_γ then $\varphi^\beta_\gamma\varphi^\alpha_\beta = \varphi^\alpha_\gamma$. Elements of P_α will be denoted by (x, α). Let P be a set-theoretical sum of all sets P_α. Define a multiplication on P as follows: $(x, \alpha)(x, \beta) = \varphi^\alpha_{\alpha\beta}(x, \alpha)\varphi^\alpha_{\alpha\beta}(x, \beta)$. The set P with this multiplication is a semigroup.*

The proof is easy.

Definition 1. *The semigroup P from Lemma 1 will be called the product of semigroups P_α over the semigroup Γ.*

Remark 1. In case P_α have idempotents, such a construction is always possible. It suffices to take for φ^α_β that mapping under which the image P_α is an idempotent of P_β.

Remark 2. A special case of the product from Definition 1 is the direct product $Q \times \Gamma$ of the semigroups Q and Γ. This can be obtained by taking (in Lemma 1) the semigroup Q for P_α for all $\alpha \in \Gamma$ and the identical mapping Q onto Q for the homomorphisms φ^α_β.

Remark 3. Evidently Γ must be a semigroup in order that the set P in Lemma 1, with the multiplication as indicated, be a semigroup. On the other hand, it is not necessary that P_α be semigroups.

Example. Let $\Gamma = \{\alpha, \beta\}$, where $\alpha\alpha = \alpha\beta = \beta\alpha = \beta\beta = \beta$. Let P_α be a groupoid, which is not a semigroup. Let P_β be a semigroup. Moreover,

let there exist a homomorphism φ^α_β of P_α into P_β. Then P from Lemma 1 is a semigroup.

In what follows let Γ be a semigroup of idempotent elements. The elements of Γ will be denoted by e (with indices, if needed). In this case, each semigroup P_e is a subsemigroup of P.

We shall obtain now some of the properties of P.

Theorem 1. *Let J be a left ideal of P. Then $J \cap P_e$ is a left ideal of P_e. Therefore, the ideal J is the union of left ideals (and thus of semigroups) of the semigroups P_e.*

Proof. We shall denote $J \cap P_e = J_e$. Since $J_e \subset P_e$ and P_e is a subsemigroup of P, $P_eJ_e \subset P_e$. Since $P_eJ_e \subset J$, $P_eJ_e \subset J_e$. Hence J_e is a left ideal of P_e.

Similar assertion holds for a right ideal of P.

Theorem 2. *Let each semigroup P_e have a unique idempotent (i, e). Let L be a left ideal of Γ. Let J_e be a left ideal of $P_e(e \in L)$, where J_e is a finite semigroup. Then we can construct at least one semigroup which is the product of P_e over Γ, where $\bigcup_{e \in L} J_e$ is a left ideal.*

Proof. Let $e_i \in L$, then for $e_k \in \Gamma$ we have $e_ke_i = e_n \in L$. We need to assume $\varphi^{e_i}_{e_n}$, $\varphi^{e_k}_{e_n}$; let these homomorphisms be $\varphi^{e_i}_{e_n}(x, e_i) = e_n$, $\varphi^{e_k}_{e_n}(y\, e_k) = e_n$. Since J_e are finite, $e_i \in J_{e_i}$. It follows that $PJ_{e_i} \subset J_{e_i}$. Hence $\bigcup_{e \in L} J_e$ is a left ideal of P.

We shall now introduce convenient definitions for the principal ideals and F-classes.

Definition 2. *The set $(x, e)_L = P(x, e) \cup \{(x, e)\}$ is said to be the principal left ideal of P, generated by (x, e).*

Similarly we define the principal right ideal of P.

The set of all elements which generate the same principal ideal (left $(x, e)_L$, right $(x, e)_R$) is called F-class (left $F_L(x, e)$, right $F_R(x, e)$).

Hereafter analogous results as given for left ideals hold for right ideals.

Theorem 3. $(x, e)_L = \bigcup (P_{e_i}\varphi^e_{e_i}(x, e) \cup \{(x, e)\})$, *where $e_i \in (e)_L$ of Γ.*

Proof. According to Theorem 1 $(x, e)_L \cap P_{e_i} = J_{e_i}$ for $e \neq e_i$, where J_{e_i} is a left ideal of P_{e_i}. Since $J_{e_i} \subset (x, e)_L$, for $(y, e_i) \in J_{e_i}$ we have $(y, e_i) = (z, e_k)(x, e)$. Hence according to Lemma 1, $(y, e_i) = \varphi^{e_k}_{e_i}(z, e_k)\varphi^e_{e_i}(x, e)$; this means that $(y, e_i) \in P_{e_i}\varphi^e_{e_i}(x, e)$. It follows that $(x, e)_L \cap P_{e_i} \subseteq P_{e_i}\varphi^e_{e_i}(x, e)$. At the same time, according to Lemma 1, $e_i = e_ke$, hence $e_ie = e_i$. This shows that $e_i \in (e)_L$ of Γ. Since $e_i \in (e)_L$ of Γ, $e_ie = e_i$, thus $P_{e_i}(x, e) = (\varphi^{e_i}_{e_ie}P_{e_i})(\varphi^e_{e_ie}(x, e)) = P_{e_i}\varphi^e_{e_ie}(x, e)$. However, since $P_{e_i}(x, e) \subset (x, e)_L$, $P_{e_i}\varphi^e_{e_ie}(x, e) \subset (x, e)_L$. This means that $P_{e_i}\varphi^e_{e_i}(x, e) \subset (x, e)_L \cap P_{e_i}$. By the result above, this gives $(x, e)_L \cap P_{e_i} = P_{e_i}\varphi^e_{e_i}(x, e)$, proving the assertion.

Definition 3. $F_L(x, e_1) \leqq F_L(x, e_2)$ *iff* $(x, e_1)_L \subseteq (x, e_2)_L$. *(Similarly for F_R-classes.)*

By analogy we shall introduce the relation $\leqq$ for F_L- and F_R-classes of Γ.

Remark. The set of $F_L(F_R)$-classes is partially ordered with respect to the relation $\leqq$.

Lemma 2. *The ideal $(x, e)_L$ is the union of all $F_L(y, e_i)$ for which $F_L(y, e_i) \leqq F_L(x, e)$ is true.*

Proof. Let $(y, e_i) \in (x, e)_L$. Then $(y, e_i)_L \subset (x, e)_L$, therefore $F_L(y, e_i) \subset \subset (x, e)_L$, where $F_L(y, e_i) \leqq F_L(x, e)$. Clearly for $F_L(j, e_i)$ with $F_L(j, e_i) \leqq F_L(x, e)$ we have $F_L(j, e_i) \subset (x, e)_L$.

In the following, the assertions are valid if we replace $(x, e)_L$ by $(x, e)_R$ and $F_R(x, e)$ by $F_L(x, e)$.

Theorem 4. a) *Let $(e_2)_L \subseteq (e_1)_L$ in Γ. Then for each (x, e_1) there exists (x, e_2) such that $F_L(x, e_2) \leqq F_L(x, e_1)$.* b) *Let $F_L(x, e_2) \leqq F_L(x, e_1)$. Then $(e_2)_L \subseteq (e_1)_L$.*

Proof. a) We suppose $(e_2)_L \subseteq (e_1)_L$, then we have $e_2 = e_3 e_1$ for some e_3; thus $e_2 e_1 = e_2$. According to Lemma 1, for (z, e_2) we have $(z, e_2)(x, e_1) = (x, e_2) \in (x, e_1)_L$, whence $(x, e_2)_L \subseteq (x, e_1)_L$. By Definition 3 this means that $F_L(x, e) \leqq F_L(x, e_1)$. b) According to Definition 3 and Lemma 1, $e_2 e_1 = e_2$, therefore $(e_2)_L \subseteq (e_1)_L$.

Theorem 5. $(e_2)_L \subseteq (e_1)_L$ *iff* $(e_1 e_2)_L = (e_2 e_1)_L = (e_2)_L$.

Proof. For $(e_2)_L \subseteq (e_1)_L$ we have $e_2 e_1 = e_2$, therefore $e_1 e_2 e_1 = e_1 e_2$. Hence $(e_1 e_2)_L \subseteq (e_2 e_1)_L$. Since Γ is a semigroup of idempotents, we have from the foregoing $e_2 e_1 e_2 = e_2 e_1 e_2 e_1 = e_2 e_1 = e_2$, thus $(e_2 e_1)_L \subseteq (e_1 e_2)_L$. This, together with $(e_1 e_2)_L \subseteq (e_2 e_1)_L$ proves that $(e_1 e_2)_L = (e_2 e_1)_L = (e_2)_L$.

The second part of the proof is evident.

As a consequence we have proved the following

Theorem 6. *$F_L(e_i)$-classes in Γ $(e_i \in M \subset \Gamma)$ form a chain under the relation $\leqq$ iff there exists in P a chain of F_L-classes with at least one F_L-class from each P_{e_i}.*

Theorem 7. *Let $(e_1)_L = (e_2)_L$ in Γ. Then for each (x_1, e_1) there exists a desceding chain of F_L-classes $\ldots$ $F_L(x_3, e_1) \leqq F_L(y_2, e_2) \leqq F_L(x_2, e_1) \leqq F_L(y_1, e_2) \leqq F_L(x_1, e_1)$ in which the classes of P_{e_1} and P_{e_2} appear alternately. In case P_{e_1} and P_{e_2} are the union of F_L-classes, this chain is infinite. In case the chain is finite, for some (x, e_1), (y, e_2) the relation $F_L(x, e_1) = F_L(y, e_2)$ is true.*

Proof. Since $(e_1)_L = (e_2)_L$, $e_1 e_2 = e_1$, $e_2 e_1 = e_2$. In a similar manner as in the proof of Theorem 4, for (y, e_2) we obtain $(y, e_2)(x_1, e_1) = (y_1, e_2) \in \in (x_1, e_1)_L$. Hence $F_L(y_1, e_2) \leqq F_L(x_1, e_1)$ and further $(x_1, e_1)(y_1, e_2) = = (x_2, e_1) \in (y_1, e_2)_L$. This implies $F_L(x_2, e_1) \leqq F_L(y_1, e_2) \leqq F_L(x_1, e_1)$. Continuing in this way, we obtain further elements of the chain. The last statement of the theorem is evident from Theorems 4 and 5.

Theorem 8. *Let $(e_1)_L = (e_2)_L$ in Γ. Then P_{e_1} and P_{e_2} are isomorphic semigroups.*

Proof. By $(e_1)_L = (e_2)_L$ we have $e_1 e_2 = e_1$, $e_2 e_1 = e_2$. On the other hand, by Lemma 1 there exist homomorphisms $\varphi_{e_2}^{e_1}$ and $\varphi_{e_1}^{e_2}$ and so for (z, e_1) we have $(z, e_1) = \varphi_{e_1}^{e_1}(z, e_1) = \varphi_{e_1}^{e_2}\varphi_{e_2}^{e_1}(z, e_1)$. Hence $\varphi_{e_1}^{e_2}$ is a homomorphism of P_{e_2} onto P_{e_1}. In the same way we can prove that $\varphi_{e_2}^{e_1}$ is a homomorphism of P_{e_1} onto P_{e_2}. It follows that P_{e_1} and P_{e_2} are isomorphic semigroups.

Theorem 9. *Let $F_L(x, e_2) \cap P_{e_1} \neq \emptyset$. Then $(e_2)_L = (e_1)_L$.*

Proof. By hypothesis, for $(x, e_1) \in F_L(x, e_2)$ and for some (x, e_3) we have $(x, e_2) = (x, e_3)(x, e_1)$. Similarly we obtain $(x, e_1) = (x, e_4)(x, e_2)$ for some (x, e_4). Hence, according to Lemma 1, $e_3 e_1 = e_2$, $e_4 e_2 = e_1$, that is $(e_2)_L \subseteq \subseteq (e_1)_L$, $(e_1)_L \subseteq (e_2)_L$, thus $(e_1)_L = (e_2)_L$.

Theorem 10. $\varphi_{e_1}^{e_2} F_L(x, e_2) \subset F_L \varphi_{e_1}^{e_2}(x, e_2)$.

Proof. Let $(y, e_2) \in F_L(x, e_2)$, that is $(y, e_2)_L = (x, e_2)_L$. We wish to show that $(\varphi_{e_1}^{e_2}(y, e_2))_L = (\varphi_{e_1}^{e_2}(x, e_2))_L$. By hypothesis, for some (x, e_k) we have $(x, e_2) = (x, e_k)(y, e_2)$, where $e_k e_2 = e_2$. Then $\varphi_{e_1}^{e_2}(x, e_2) = \varphi_{e_1}^{e_2}[\varphi_{e_k e_2}^{e_k}(x, e_k)\,\varphi_{e_2}^{e_2}(y, e_2)] = \varphi_{e_1}^{e_k}(x, e_k)\varphi_{e_1}^{e_2}(y, e_2)$, whence $\varphi_{e_1}^{e_2}(x, e_2) \in (\varphi_{e_1}^{e_2}(y, e_2))_L$, therefore $(\varphi_{e_1}^{e_2}(x, e_2))_L \subset \subset (\varphi_{e_1}^{e_2}(y, e_2))_L$. In a similar manner we can prove $(\varphi_{e_1}^{e_2}(y, e_2))_L \subset \subset (\varphi_{e_1}^{e_2}(x, e_2))_L$ and so $(\varphi_{e_1}^{e_2}(y, e_2))_L = (\varphi_{e_1}^{e_2}(x, e_2))_L$. Hence $\varphi_{e_1}^{e_2} F_L(x, e_2) \subset \subset F_L \varphi_{e_1}^{e_2}(x, e_2)$.

Theorem 11. *Let $(e_1)_L = (e_2)_L$. Then $\varphi_{e_1}^{e_2} F_L(x, e_2) = F_L(\varphi_{e_1}^{e_2}(x, e_2))$.*

Proof. After considering Theorem 10 there remains to be shown that: if $(y, e_i) \in F_L \varphi_{e_1}^{e_2}(x, e_2)$ then $(y, e_1) = \varphi_{e_1}^{e_2}(z, e_2)$, where $(z, e_2) \in F_L(x, e_2)$. Using the proof of Theorem 10 we can see that $(z, e_2) = (\varphi_{e_2}^{e_1}(y, e_1))_L = (\varphi_{e_2}^{e_1}\varphi_{e_1}^{e_2}(x, e_2))_L = = (x, e_2)_L$.

Remark. Clearly, if $(e_1)_L = (e_2)_L$, the ideal $(x, e_1)_L$ is isomorphic to $(\varphi_{e_2}^{e_1}(x, e_1))_L$ (which follows from Theorem 8 as well).

Theorem 12. a) *Let the $F_L(e)$-class in Γ consist of a unique element. Then P_e is the union of F_L-classes in P.* b) *Let Γ be a commutative semigroup. Then P_e are the union of F_L classes in P.*

Proof. a) Let $F_L(x, e) \cap P_{e_i} \neq \emptyset$, then, according to Theorem 9, $(e)_L = (e_i)_L$ — a contradiction. b) Suppose that $(e_1)_L = (e_2)_L$ in Γ; hence $e_1 = e_2$. Then a) implies b).

Theorem 13. *Let $F_L(x, e_2) \cap P_{e_1} \neq \emptyset$; then $F_R(x, e_2) \cap P_{e_i} = \emptyset$ for all (x, e_2).*

Proof. Let $(y, e_1) \in F_L(x, e_2)$. Then for some (z, e_3) we have $(x, e_2) = (z, e_3)(y, e_1)$ and according to Lemma 1, $e_3 e_1 = e_2$, that is $e_2 e_1 = e_2$. On the other hand, let $(v, e_1) \in F_R(x, e_2)$. Similarly we can show that $e_2e_1 = e_1$. Finally we have $e_1 = e_2$ (clearly, we consider only $e_1 \neq e_2$).

Theorem 14. *Let $F_L(x, e_2) \leqq F_L(y, e_1)$ for $e_1 \neq e_2$. Then either $F_R(x, e_2) \leqq F_R(y, e_1)$, or $F_R(x, e_2)$, $F_R(y, e_1)$ are incomparable.*

Proof. By hypothesis, for some (x, e_3) we get $(x, e_2) = (x, e_3)(y, e_1)$, then $e_3e_1 = e_2$, that is $e_2e_1 = e_2$. Let $F_R(y, e_1) \leqq F_R(x, e_2)$. Similarly we obtain $e_2e_1 = e_1$. Finally we have $e_1 = e_2$ — a contradiction.

Theorem 15. *Let $(e_1)_L = (e_2)_L$, $e_1 \neq e_2$. Then $F_R(x, e_1)$, $F_R(x, e_2)$ are incomparable.*

Proof. Theorem 4 for F_R classes may now be applied to show that $F_R(x, e_2) \leqq F_R(x, e_1)$ implies $(e_2)_R \subseteq (e_1)_R$, which is to say that $e_1e_2 = e_2$. Since $(e_1)_L = (e_2)_L$, $e_1e_2 = e_1$. Finally we have $e_1 = e_2$ — a contradiction. In a similar manner it can be shown that $F_R(x, e_1) \leqq F_R(x, e_2)$ does not hold.

Remark. Theorem 15 (according to Lemma 2) may be interpreted as follows: if $(e_1)_L = (e_2)_L$ $(e_1 \neq e_2)$ then $(x, e_1)_R \cap Pe_2 \neq \emptyset$ for all (x, e_1).

Theorem 16. *Let (i, e) be an idempotent of P. Then a left ideal L has the right identity (i, e) iff $L = (i, e)_L$.*

Proof. Let L be a left ideal in P and (i, e) its right identity, which means $(x, e)(i, e) = (x, e)$ for $(x, e) \in L$, whence $L \subset (i, e)_L$. Since $(i, e) \in L$ (is its right identity), $(i, e)_L \subset L$. According to the foregoing result $(i, e)_L = L$.

Theorem 17. *Let (i, e) be an idempotent. Then $(x, e)(i, e) = (x, e)$ holds for $(x, e) \in F_L(i, e)$.*

Proof. The statement is clear, since $(x, e) = (y, e)(i, e)$ (because $(x, e)_L = (i, e)_L$).

Remark. For $(x, e) \in F_R(i, e)$ we have $(x, e) = (i, e)(x, e)$.

Theorem 18. *Let P_{e_1}, P_{e_2} have the unique idempotents (i, e_1), (i, e_2). Let $(x, e_1)_L = (x, e_2)_L$. Then $(i, e_1)_L = (i, e_2)_L$.*

Proof. By hypothesis and Theorem 9 we have $(e_1)_L = (e_2)_L$. Using Theorem 11 we can prove our assertion.

Theorem 19. *Let P_{e_1}, P_{e_2} have the unique idempotents (i, e_1), (i, e_2). Then $(i, e_1)_L = (i, e_2)_L$ iff $(e_1)_L = (e_2)_L$.*

Proof. Let $(i, e_1)_L = (i, e_2)_L$. Using Theorem 9 we can see that $(e_1)_L = (e_2)_L$. Let $(e_1)_L = (e_2)_L$; from Theorem 11 it follows $(i, e_1)_L = (i, e_2)_L$.

Theorem 20. *Let P_e have a unique idempotent (i, e). Let $F_L(i, e) = F_R(i, e)$. Then $F_L(i, e)$ is the maximal subgroup of P.*

Proof. According to Theorem 17 (i, e) is an identity in $F_L(i, e)$. Moreover $(y, e)_L = (x, e)_L = (i, e)_L$ implies $((x, e)(y, e))_L = ((i, e)(y, e))_L = (y, e)_L$.

Hence $F_L(i, e)$ is a semigroup. We shall prove now that $(x, e)_L = (i, e)_L$ implies the existence of (z, e) with $(z, e)(x, e) = (i, e)$. Because $(x, e)_R = (i, e)_R$, for some (v, e) we have $(x, e) = (i, e)(v, e)$, whence $(z, e)(x, e) = (z, e)(i, e)(v, e)$, therefore $(i, e)_R \subset ((z, e)(i, e))_R$. Similarly we can prove $((z, e)(i, e))_R \subset (i, e)_R$; thus $(i, e)_R = ((z, e)(i, e))_R$. Hence $(z, e)(i, e) \in F_R(i, e) = F_L(i, e)$. Thus $[(z, e)(i, e)](x, e) = (z, e)[(i, e)(x, e)] = (z, e)(x, e) = (i, e)$ as required. This shows that $F_L(i, e)$ is a group.

It is evident that the elements of the group generate the same principal left (right) ideal. Therefore $F_L(i, e)$ is a maximal subgroup of P.

We derive next (Theorem 21—25) some of the properties of semigroups with identity (hypogroup).

Theorem 21. *Let the semigroup P be the product of semigroups P_α over the semigroup Γ. In this case P will be a hypogroup iff Γ and P_α are hypogroups (where Γ is a semigroup of idempotents). Moreover, if e is the identity in Γ and (j, e) the identity in P_e, then (j, e) is the identity in P.*

Proof. Let P be the product of hypogroups over the hypogroup of idempotents Γ. Since Γ is isomorphic to the semigroup of identity elements in P_e $(e \in \Gamma)$ (because the image of identity is an identity), we can see, using Lemma 1, that P is a hypogroup.

Let P be the hypogroup which is the product of the semigroups P_α over the semigroup Γ. Let (j, e) be the identity in P. Since (j, e) is an idempotent, according to Lemma 1, $e \in \Gamma$ is an idempotent as well. In P we have $(x, e_i)(j, e) = (j, e)(x, e_i) = (x, e_i)$, this means (by Lemma 1) $e_ie = ee_i = e_i$ and so e is an identity in Γ. Hence Γ is a hypogroup. Moreover; according to Lemma 1, it follows that $\varphi^e_{e_i}(j, e)$ is an idempotent in P_{e_i}, therefore e_i is an idempotent in Γ. As for each $e_i \in \Gamma$ we have $ee_i = e_i$, Γ is a hypogroup of idempotents. But since the image of identity is an identity, $\varphi^e_{e_i}(j, e)$ is the identity in P_{e_i}. This means that P_{e_i} is a hypogroup. This completes the proof.

Theorem 22. *The necessary and sufficient condition for hypogroup P to be the product of semigroups over the semigroup Γ is that there exist on P a congruence, the classes of which are hypogroups, while their identity elements form a subsemigroup of P.*

Proof. We denote the classes of the congruence by S_{e_i} $(i = 1, 2, \ldots)$. (j, e_i) is the identity in S_{e_i}. It follows that $(x, e_1)(y, e_2) \in S_{e_1e_2}$. We shall show that the mapping $(x, e_1) \to (x, e_1)(j, e_2)$ is a homomorphism of S_{e_1} into $S_{e_1e_2}$ and $(y, e_2) \to (j, e_1)(j, e_2)$ is a homomorphism of S_{e_2} into $S_{e_1e_2}$. The following holds: $((x_1, e_1)(j, e_2))((x_2, e_1)(j, e_2)) = ((x_1, e_1)(j, e_1)(j, e_2))(x_2, e_1)(j, e_2) = (x_1, e_1)((j, e_1)(j, e_2)(x_2, e_1)(j, e_2)) = (x_1, e_1)(x_2, e_1)(j, e_2)$ (because $(x_2, e_1)(j, e_2) \in S_{e_1e_2}$, hence $((j, e_1)(j, e_2))((x_2, e_1)(j, e_2)) = (x_2, e_1)(j, e_2)$ as required. Similarly it can be shown that $(y, e_2) \to (j, e_1)(y, e_2)$ is a homomorphism of S_{e_2} into $S_{e_1e_2}$. We denote these homomorphisms by $\varphi_{12}^1, \varphi_{12}^2$. Clearly $\varphi_3^{12}\varphi_{12}^1 = \varphi_3^1$. Using such homomorphisms we can construct the product of the semigroups S_{e_i} over the semigroup Γ, where Γ is a semigroup isomorphic to the semigroup of identity elements in P; in this way we obtain exactly P.

The necessary condition is evident by considering that the image of identity is identity.

The following holds for hypogroups of Theorem 21 (hereafter the identity in P_e is denoted by (j, e)).

Lemma 3. *For each* (x, e) *the relation* $F_L(x, e) \leqq F_L(j, e)$ *is true.*

Theorem 23. *Let* $(j, e_1)_L \neq (j, e_2)_L$. *If* $F_L(x, e_1) \leqq F_L(y, e_2)$ *for some* (x, e_1), (y, e_2), *then:* a) $F_L(j, e_1) \leqq F_L(j, e_2)$, b) $F_L(z, e_2) \leqq F_L(j, e_1)$ *is not true for any* (z, e_2).

Proof. a) By hypothesis, there exists such an element (z, e_3) that $(x, e_1) = (z, e_3)(y, e_2)$ and so $(x, e_1) = (z, e_3)(y, e_2)(j, e_2) = (x, e_1)(j, e_2)$. According to Lemma 1 $(j, e_1)(j, e_2) = (j, e_1)$ which is to say that $(j, e_1)_L \subseteq (j, e_2)_L$. b) Let $F_L(z, e_2) \leqq F_L(j, e_1)$ for some (z, e_2). According to a), $F_L(j, e_2) \leqq F_L(j, e_1)$. On the other hand, by hypothesis and by a) we have $F_L(j, e_1) \leqq F_L(j, e_2)$; thus we obtain $(j, e_1)_L = (j, e_2)_L$ — a contradiction.

Theorem 24. *For* $F_L(j, e)$ *and* $F_R(j, e)$ *the following are true:* a) $F_L(j, e)$ *and* $F_R(j, e)$ *are semigroups.* b) *For each* $(x_1, e) \in F_L(j, e)$ *there exists* $(s_2, e) \in F_R(j, e)$ *such that* $(s_2, e)(x_1, e) = (j, e)$. *Similarly for each* $(x_2, e) \in F_R(j, e)$ *there exists* $(s_1, e) \in F_L(j, e)$ *such that* $(x_2, e)(s_1, e) = (j, e)$. c) *Each element* $(s, e) \in F_L(j, e) \cup F_R(j, e)$ *can be written in the form* $(s, e) = (z_2, e)(z_1, e)$ *where* $(z_1, e) \in F_L(j, e)$, $(z_2, e) \in F_R(j, e)$ *for* (z_1, e) *or* (z_2, e) *given before.*

Proof. a) Let $(j, e)_L = (j, e_1)_L = (x, e)_L = (x, e_1)_L$. Then $((x, e)(x, e_1))_L = ((j, e_1)(x, e_1))_L = (x, e_1)_L$. Similarly $((x, e_1)(x, e))_L = (x, e)_L$. b) $(x_1, e)_L = (j, e)_L$ implies $(s, e_i)(x_1, e) = (j, e)$ for some (s, e_i). According to Lemma 1, $e_ie = e$. We denote $\varphi_e^{e_i}(s, e_i) = (s_2, e)$. Evidently $(s_2, e) \in (j, e)_R$. But $(j, e) = (s_2, e)(x_1, e)$, therefore $(j, e) \in (s_2, e)_R$, hence $(j, e)_R = (s_2, e)_R$, thus $(s_2, e) \in F_R(j, e)$. The second assertion can be proved in the same way. c) Let $(s_1, e) \in F_L(j, e)$ and let $(z_2, e) \in F_R(j, e)$. Then according to b) for some $(u_1, e) \in F_L(j, e)$ we have $(j, e) = (z_2, e)(u_1, e)$, hence $(s_1, e) = (z_2, e)(u_1, e)(s_1, e)$, whence according to a) $(u_1, e)(s_1, e) = (z_1, e) \in F_L(j, e)$, thus $(s_1, e) = (z_2, e)(z_1, e)$. Similarly the second part of the assertion can be proved.

Theorem 25. *The left ideal* L *has the identity* (j, e) *iff* $L = (j, e)_L$ *and* $(e)_L$ *has an identity.*

Proof. Let $L = (j, e)_L$. Since $ee_k = e_k$ (because e is the identity in $(e)_L$,

$\varphi_{ee_k}^e(j, e) = (j, e_k)$, hence $(j, e)(l, e_k) = (l, e_k)$ for $(l, e_k) \in L$. Thus (j, e) is the identity in L.

Let L possess the identity (j, e). Clearly, the semigroup of elements $(j, e_i) \in L$ is the left ideal in the subsemigroup of all identity elements of P, and is isomorphic to the subsemigroup of Γ, which is, therefore, a left ideal in Γ. This ideal evidently possesses the identity e. It is therefore $(e)_L$. According to Theorem 16, $L = (j, e)_L$.

Remark. From the hypothesis of the theorem stating that $(e)_L$ has an identity, it follows that $F_L(j, e) \cap P_{e_i} = \emptyset$ for $e_i \neq e$.

Finally, we mention some examples of semigroups which are the product of semigroups over a semigroup, the properties of which have already been studied.

From [1] it follows:

The semigroup S is said to admit relative inverses if to each $a \in S$ there exists an element $e \in S$ such that $ae = ea = a$ and an element $a' \in S$ such that $a'a = aa' = e$. Then following holds:

Theorem 26. *Each semigroup admitting relative inverses in which every pair of idempotents commute is a product of groups over a semigroup of idempotents.*

From [2] and [3] we have:

Theorem 27. *Each finite simple semigroup* S *without zero, having at least one minimal left ideal and at least one minimal right ideal, while the idempotents form a semigroup in* S, *is the product of isomorphic groups over the simple semigroup without zero.*

From [4] can be deduced:

In a periodic semigroup let the set of elements x with $x^n = e$ (for some n and for the idempotent e) be called K-class belonging to e.

Theorem 28. *The product of commutative periodic semigroups* P_e *over the commutative semigroup of idempotents (semilattice) is a commutative periodic semigroup in which the* K*-classes are exactly* P_e. *Moreover each commutative periodic semigroup, the* K*-classes of which are groups, is the product of commutative periodic groups over a semilattice.*

We shall say that the periodic semigroup S is partially commutative if for each $e \in S$ and each $x \in S$, $xe = ex$ is true.

From [5] it follows:

Theorem 29. *The product of partially commutative periodic semigroups having a unique idempotent over a commutative semigroup of idempotents (semilattice) is a partially commutative semigroup, the* K*-classes of which are exactly* P_e.

Theorem 30. (according to [6]). *Let each principal left ideal in the semigroup P contain an identity. Then P is the product of groups over the commutative semigroup of idempotents (semilattice) iff for each* $e \in \Gamma$, $(j, e)_L = (j, e)_R$ *where* (j, e) *is the identity in* P_e.

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ОБ ОДНОМ ПРОИЗВЕДЕНИИ ПОЛУГРУПП

Бланка Колибиарова

Резюме

Пусть Γ — полугруппа и пусть всякому элементу α из Γ поставлена в соответствие полугруппа P_α. Пусть дано множество $\mathfrak{G}$ гомоморфизмов φ_β^α $(\alpha, \beta \in \Gamma)$, где φ_β^α гомоморфное отображение P_α в P_β. Пусть при этом $\varphi_\beta^\alpha \in \mathfrak{G}$ тогда и только тогда, когда $\alpha = \beta$ или существует $\gamma \in \Gamma$ для которого $\alpha\gamma = \beta$ или $\gamma\alpha = \beta$. Пусть множество $\mathfrak{G}$ удовлетворяет условиям: 1. Для $\alpha \in \Gamma$ φ_α^α является тождественным отображением P_α на P_α. 2. Если для $\alpha, \beta, \gamma \in \Gamma$ существуют в $\mathfrak{G}$ гомоморфизмы φ_β^α, φ_γ^β, φ_γ^α тогда $\varphi_\gamma^\beta\varphi_\beta^\alpha = \varphi_\gamma^\alpha$. Обозначим элементы множества P_α через (x, α). Пусть P — теоретическо-множественное объединение множеств $P_\alpha (\alpha \in \Gamma)$. Определим в P умножение следующим образом: $(x, \alpha)(x, \beta) = \varphi_{\alpha\beta}^\alpha(x, \alpha)\varphi_{\alpha\beta}^\beta(x, \beta)$. Множество P с этим умножением является полугруппой. Назовем ее произведением полугрупп P_α над Γ.

В настоящей статье изучаются некоторые свойства этих произведений.