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ON POWERS OF NON-NEGATIVE MATRICES

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Let A be a $n \times n$ matrix with non-negative entries. One of the main problems in studying such matrices is to study the distribution of zeros and „non-zeros" in the sequence

$$(1) \qquad A, A^2, A^3, \dots$$

In the paper [2] I have shown that there is a simple semigroup treatment of this problem which leads to a series of results without any mention of such notions as characteristic values, characteristic vectors, etc.

This semigroup treatment leads to some pertinent questions which will be partly solved in this paper.

For convenience of the reader I briefly recall the necessary notions introduced in [2].

Let $N = \{1, 2, \dots, n\}$. Consider the set of "$n \times n$ matrix-units", i.e. the set S of symbols $\{e_{ij} \mid i, j \in N\}$ together with a zero 0 adjoined: $S = \{e_{ij} \mid i, j \in N\} \cup \{0\}$.

Define in S a multiplication by

$$e_{ij}e_{ml} = \begin{cases} 0 \text{ for } j \neq m, \\ e_{il} \text{ for } j = m, \end{cases}$$

the zero 0 having the usual properties of a multiplicative zero. The set $S = S_n$ with this multiplication is a 0-simple semigroup. It contains exactly n non-zero idempotents, namely the elements $e_{11}, e_{22}, \dots, e_{nn}$.

Let $A = (a_{ij})$ be a non-negative $n \times n$ matrix. By the support of A we shall mean the subset of S containing 0 and all those elements $e_{ij} \in S$ for which $a_{ij} > 0$.

The support of A will be denoted by C_A. For typographical reasons we shall write occasionally $C_A = C(A)$.

For any two $n \times n$ non-negative matrices we clearly have $C_{A+B} = C_A \cup C_B$.

Consider further the set $\mathfrak{S} = \mathfrak{S}_n$ of all subsets of $S = S_n$ and define a multiplication in $\mathfrak{S}$ as the multiplication of complexes in S, i.e. if $C', C'' \in \mathfrak{S}$, then $C'C'' = \{c_1c_2/c_1 \in C', c_2 \in C''\}$. Then $\mathfrak{S}$ is again a finite semigroup containing exactly 2^{n^2} different elements.(¹)

If A, B are two non-negative matrices it is easy to see that $C_{AB} = C_A \,.\, C_B$. In particular, the supports of the elements of the sequence (1) are given by the sequence

$$(2) \qquad C_A, C_A^2, C_A^3, \ldots$$

Though (1) may contain an infinity of different elements, the sequence (2) contains only a finite number of different elements. The correspondence $A \to C_A$ is a homomorphic mapping of the semigroup of all non-negative matrices onto the semigroup $\mathfrak{S}$. [If we consider the union of sets as the second binary operation in $\mathfrak{S}$, we have even a homomorphic mapping of the semiring of all non-negative $n \times n$ matrices onto the semiring $\mathfrak{S}$.]

The following facts easily follow from the elements of the theory of finite semigroups.

Let A be a fixed $n \times n$ matrix. Let k be the least integer such that $C_A^k = C_A^l$ for some $l > k$. Let further $l = k + d$ $(d \geqq 1)$ be the least integer satisfying this relation. Then the sequence (2) is of the form

$$C_A, \ldots, C_A^{k-1} \mid C_A^k \ldots, C_A^{k+d-1}, \mid C_A^k, \ldots, C_A^{k+d-1} \mid \ldots$$

Denote by $\mathfrak{S}_A$ the subsemigroup of $\mathfrak{S}$ generated by C_A. Then $\mathfrak{S}_A$ has exactly $k + d - 1$ different elements and we have

$$(3) \qquad \mathfrak{S}_A = \{C_A, \ldots, C_A^{k-1}, C_A^k, \ldots, C_A^{k+d-1}\}.$$

For any $\alpha \geqq k$ and every $\beta \geqq 0$ we clearly have

$$(4) \qquad C_A^\alpha = C_A^{\alpha+\beta d}.$$

It is well known that $\mathfrak{G}_A = \{C_A^k, \ldots, C_A^{k+d-1}\}$ is a cyclic group of order d (subgroup of $\mathfrak{S}_A$). The unit element of the group $\mathfrak{G}_A$ is C_A^ϱ with a suitably chosen ϱ satisfying $k \leqq \varrho \leqq k + d - 1$. Let τ be the uniquely determined integer such that $k \leqq \tau d \leqq k + d - 1$. Then $\varrho = \tau d$. To show this it is sufficient to show that $C_A^{\tau d}$ is an idempotent. In fact we have (by (4) with $\alpha = \tau d$, $\beta = \tau$) $C_A^{2\tau d} = C_A^{\tau d + \tau d} = C_A^{\tau d}$.

In the following we shall consequently write $\varrho = \tau d$, so that C_A^ϱ is the (unique) idempotent $\in \mathfrak{S}_A$. Clearly, we also have $\mathfrak{G}_A = \{C_A^\varrho, C_A^{\varrho+1}, \ldots, C_A^{\varrho+d-1}\}$.

Note explicitly that to every non-negative matrix A we have associated three integers $k = k(A)$, $\varrho = \varrho(A)$ and $d = d(A)$ satisfying $k \leqq \tau d = \varrho \leqq$

(¹) $\mathfrak{S}$ may be considered — of course — also as the Boolean algebra of $n \times n$ square matrices with elements 0 and 1 and the usual binary operations.

$\leqq \varrho + d - 1$, which depend only on the distribution of the zeros and non-zeros in A.

For further purposes we mention also the following facts proved in [2]. If A is any $n \times n$ non-negative matrix, then

$$C_A^{n+1} \subset C_A \cup C_A^2 \cup \ldots \cup C_A^n.$$

Hence the set $C_A \cup C_A^2 \cup \ldots \cup C_A^n$ is always a subsemigroup of $S = S_n$.

A non-negative matrix A is called reducible if there is a permutation matrix P such that $P^{-1}AP$ is of the form

$$P^{-1}AP = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}.$$

Otherwise it is called irreducible. An $n \times n$ non-negative matrix A is irreducible if and only if

$$C_A \cup C_A^2 \cup \ldots \cup C_A^n = S_n.$$

It should be mentioned in advance that in this paper the emphasis is rather on the reducible case.

Consider now the semigroup $\mathfrak{S}_A$ as given in (3). The elements of $\mathfrak{S}_A$ are subsets of S. At least one of the elements $\in \mathfrak{S}_A$ (namely C_A^ϱ) is itself a subsemigroup of S. The first problem treated in this paper concerns the following question. Under what conditions concerning A and s may it happen that the set C_A^s is a subsemigroup of S. The second problem is to find a "good" characterization of the number $d = \text{card}\, \mathfrak{G}_A$. It will turn out that both questions are intimately connected.

I.

Lemma 1. *Let A be any $n \times n$ non-negative matrix. Suppose that C_A^s is a subsemigroup of $S = S_n$. Then*

a) $C_A^\varrho \subset C_A^s$;

b) *C_A^s contains all idempotents $\in S$ contained in the union $C_A \cup C_A^2 \cup \ldots \cup C_A^n$.*

Proof. a) The sequence

$$C_A^s, C_A^{2s}, C_A^{3s}, \ldots$$

contains a unique idempotent C_A^ϱ. Hence there is an integer v such that $C_A^{vs} = C_A^\varrho$. Since C_A^s is a semigroup, we have $C_A^s \supset C_A^{2s}$, which implies

$$C_A^s \supset C_A^{2s} \supset C_A^{3s} \supset \ldots \supset C_A^{vs} = C_A^\varrho.$$

b) Let $E_A = \{e_{\alpha\alpha}/\alpha$ running through a subset of $N\}$ be the set of all non-zero idempotents $\in S$ contained in $C_A \cup C_A^2 \cup \ldots \cup C_A^n$. If $e_{\alpha\alpha} \in C_A^h$ $(1 \leqq h \leqq n)$, then $e_{\alpha\alpha} \in C_A^{ht}$ for any integer $t \geqq 1$. Since some power of C_A^h is C_A^ϱ, we have $e_{\alpha\alpha} \in C_A^\varrho$, hence $E_A \subset C_A^\varrho \subset C_A^s$.

Theorem 1. *The group $\mathfrak{G}_A = \{C_A^k, \dots, C_A^{k+d-1}\}$ contains exactly one element which is itself a subsemigroup of S.*

Remark. This is — of course — the idempotent C_A^ϱ.

Proof. Suppose that C_A^s, $k \leqq s \leqq k+d-1$ is a semigroup. By Lemma 1 we have $C_A^\varrho \subset C_A^s$. Multiplying by C_A^s we have $C_A^\varrho . C_A^s \subset C_A^{2s} \subset C_A^s$. But since C_A^ϱ is the unit element $\in \mathfrak{G}_A$, $C_A^\varrho . C_A^s = C_A^s$. Now $C_A^s \subset C_A^{2s} \subset C_A^s$ implies $C_A^s = C_A^{2s}$, i.e. C_A^s is an idempotent contained in $\mathfrak{G}_A$. Hence $C_A^s = C_A^\varrho$, q.e.d.

Remark. If $k > 1$, the set $\{C_A, C_A^2, \dots, C_A^{k-1}\}$ may contain subsemigroups of S. Let f.i. A be a non-negative 3×3 matrix with the support (in an obvious notation([2]))

$$C_A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then

$$C_A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $C_A^3 = \{0\}$. Hence all elements C_A, C_A^2, C_A^3 are subsemigroups of S_3.

Theorem 2. *Let A be a non-negative $n \times n$ matrix for which $C_A \cup C_A^2 \cup \dots \cup C_A^n$ contains all non-zero idempotents $\in S$, i.e. the set $E_A = \{e_{11}, e_{22}, \dots, e_{nn}\}$. Then $\mathfrak{S}_A$ contains exactly one element that is itself a subsemigroup of S.*

Proof. Let be $1 \leqq s \leqq k+d-1$ and C_A^s a subsemigroup of S. By Lemma 1 we have $\{e_{11}, \dots, e_{nn}\} \subset C_A^s$. If A is any subset of S we always have $A\{e_{11}, \dots, e_{nn}\} = A$. In particular (in our case) we have

$$C_A^s = C_A^s\{e_{11}, \dots, e_{nn}\} \subset C_A^{2s}.$$

The "inequalities" $C_A^s \subset C_A^{2s}$ and $C_A^{2s} \subset C_A^s$ (describing the semigroup property of C_A^s) imply $C_A^s = C_A^{2s}$. Since there is a unique idempotent $\in \mathfrak{S}_A$ we have $C_A^s = C_A^\varrho$, q.e.d.

If S is irreducible, then $C_A \cup C_A^2 \cup \dots \cup C_A^n = S$, so that the suppositions of Theorem 2 are satisfied and we obtain:

Corollary 1. *If A is irreducible, then C_A^ϱ is the unique element $\in \mathfrak{S}_A$ which is itself a subsemigroup of S.*

([2]) We shall occasionally use this obvious notation by puting **1** on those places (i, k) or which $e_{ik} \in C_A$. F.i. in our example the "Boolean matrices" C_A, C_A^2, C_A^3 denote $C_A = \{0, e_{21}, e_{31}, e_{32}\}$, $C_A^2 = \{0, e_{31}\}$, $C_A^3 = \{0\}$.

Corollary 2. *If A is any $n \times n$ non-negative matrix and C_A^s is a semigroup containing $\{e_{11}, \dots, e_{nn}\}$, then $C_A^s = C_A^\varrho$.*

Proof. By supposition $C_A^s = C_A^s\{e_{11}, \dots, e_{nn}\} \subset C_A^{2s}$. On the other hand $C_A^{2s} \subset C_A^s$, hence $C_A^s = C_A^{2s}$; therefore $C_A^s = C_A^\varrho$, q.e.d.

Remark. In Corollary 2 the supposition that C_A^s is a semigroup cannot be omitted. Let f.i. A be a 3×3 matrix with

$$C_A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then

$$C_A^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

contains $\{e_{11}, e_{22}, e_{33}\}$, but C_A^3 is not the idempotent $\in \mathfrak{S}_A$. (The idempotent $\in \mathfrak{S}_A$ is C_A^5.)

The next two Lemmas will enable us to locate, so to say, the semigroups in the sequence (2) and to find at the same time a new characterization of the number d.

Lemma 2. *Let s be an integer such that C_A^s is a subsemigroup of S. We then have:*

a) $C_A^\varrho = C_A^{\varrho+s}$;

b) $d \mid s$;

c) $C_A^\varrho \subset C^{s+td}$ *for any integer* $t \geqq 0$.

Proof. a) We have $C_A^{\varrho+s} \in \mathfrak{G}_A$. Further $C_A^{\varrho+s}$ is a subsemigroup of S since $C_A^{2(\varrho+s)} = C_A^{2\varrho} . C_A^{2s} = C_A^\varrho . C_A^{2s} \subset C_A^\varrho . C_A^s = C_A^{\varrho+s}$. Hence by Theorem 1 $C_A^{\varrho+s} = C_A^\varrho$.

b) Suppose that $d \nmid s$ and write $s = \alpha d + \beta$, where $\alpha \geqq 0$ is an integer and $0 < \beta < d$. Since for any integer α we have $C_A^{\varrho+\alpha d} = C_A^\varrho$ the relation $C_A^\varrho = C_A^{\varrho+s}$ implies

$$C_A^\varrho = C_A^{\varrho+\alpha d+\beta} = C_A^{\varrho+\alpha d} C_A^\beta = C_A^\varrho . C_A^\beta = C_A^{\varrho+\beta}.$$

The relation $C_A^\varrho = C_A^{\varrho+\beta}$ contradicts to the fact that the group $\mathfrak{G}_A = \{C_A^\varrho, C_A^{\varrho+1}, \dots, C_A^{\varrho+d-1}\}$ is of order d.

c) By Lemma 1, we have $C_A^\varrho \subset C_A^s$, hence $C_A^{\varrho+td} \subset C_A^{s+dt}$ and since $C_A^{\varrho+td} = C_A^\varrho$, we obtain $C_A^\varrho \subset C_A^{s+td}$. This proves our Lemma.

Lemma 3. *If C_A^s is a semigroup, then none of the sets $C_A^{s+1}, C_A^{s+2}, \dots, C_A^{s+d-1}$ can be a semigroup.*

Proof. If $C_A^{s+\lambda}$, $1 \leqq \lambda \leqq d-1$, were a semigroup, then Lemma 3b) would imply that $d \mid s$ and $d \mid s+\lambda$, which is impossible.

Let s_0 be the least integer s such that C_A^s is a semigroup. Then $s_0 \leqq \varrho$ and we may arrange the set of powers in the following way:

$$(5)\qquad \begin{array}{llll} C_A, C_A^2, \ldots, C_A^{s_0-1}, C_A^{s_0}, & C_A^{s_0+1}, & \ldots, & C_A^{s_0+d-1} \\ C_A^{s_0+d}, & C_A^{s_0+d+1}, & \ldots, & C_A^{s_0+2d-1}, \\ C_A^{s_0+2d}, & C_A^{s_0+2d+1}, & \ldots, & C_A^{s_0+3d-1}, \\ C_A^{\varrho}, & C_A^{\varrho+1}, & \ldots, & C_A^{\varrho+d-1}. \end{array}$$

Since $d \mid \varrho$ and $d \mid s_0$ there is necessarily an integer t such that $\varrho = s_0 + td$. We get exactly $t+1$ rows. The last of them contains at least one element $\in \mathfrak{G}_A$ which does not occur in the foregoing row. (This means: It may happen that to obtain all different elements $\in \mathfrak{S}_A$ it is not necessary to consider the whole last row, but certainly at least the first element contained in it.)

The idempotent C_A^ϱ is necessarily contained in the column $\{C_A^{s_0}, C_A^{s_0+d}, \ldots\}$ and (by Lemma 2c) C_A^ϱ is a subset of each element of this column.

Also (by Lemma 2b) all elements $\in \mathfrak{S}_A$ which are themselves subsemigroups of S are located in the column $\{C_A^{s_0}, C_A^{s_0+d}, C_A^{s_0+2d}, \ldots, C_A^\varrho\}$. Hence the semigroups contained in the sequence (2) are some of the powers

$$C_A^{s_0}, C_A^{s_0+d}, \ldots, C_A^{s_0+(t-1)d}$$

and all the following

$$C_A^\varrho = C_A^{s_0+td} = C_A^{s_0+(t+1)d} = C_A^{s_0+(t+2)d} = \ldots$$

Now since $d \mid s_0$, the number d is the greatest common divisor of the sequence of integers

$$s_0, s_0 + d, s_0 + 2d, \ldots$$

We have proved:

Theorem 3. *The number $d = \operatorname{card} \mathfrak{G}_A$ is the greatest common divisor of all such integers s for which C_A^s is a semigroup (subsemigroup of S).*

We make some supplementary remarks to the "tableau" (5).

Remark 1. None of the sets $C_A^\varrho, \ldots, C_A^{\varrho+d-1}$ is contained as a proper subset in another, i.e. $C_A^{\varrho+u} \subset C_A^{\varrho+v}$ implies $C_A^{\varrho+u} = C_A^{\varrho+v}$.

Proof. We first prove that $C^\varrho \subset C_A^{\varrho+u}$, $0 \leqq u \leqq d-1$, implies $C_A^\varrho = C_A^{\varrho+u}$. Note that by Lemma 2 a $C_A^\varrho = C_A^{\varrho+\lambda s_0}$ for any integer $\lambda \geqq 0$. The relation $C_A^\varrho \subset C_A^{\varrho+u}$ implies

$$C_A^\varrho \subset C_A^{\varrho+u} \subset C_A^{\varrho+2u} \subset \ldots \subset C_A^{\varrho+s_0u} = C_A^\varrho.$$

Hence $C_A^\varrho = C_A^{\varrho+u}$. Suppose now

$$(6)\qquad C_A^{\varrho+u} \subset C_A^{\varrho+v}$$

for some $u, v \geqq 0$. Since $C_A^{\varrho+u} \in \mathfrak{G}_A$, there is a $C_A^{\varrho+u'}$ such that $C_A^{\varrho+u} \cdot C_A^{\varrho+u'} = C_A^\varrho$. Here $u + u' \equiv 0 \pmod d$ Multiplying (6) by $C_A^{\varrho+u'}$ we have $C_A^\varrho \subset$

$\subset C_A^{\varrho+v+u'}$, hence $C_A^\varrho = C_A^{\varrho+v+u'}$, so that $v + u' \equiv 0 \pmod d$. Therefore $u - v \equiv 0 \pmod d$ and $C_A^{\varrho+u} = C_A^{\varrho+v}$, q.e.d.

Remark 2. The statement just proved implies that none of the elements $C_A^{s_0}, C_A^{s_0+1}, \ldots, C_A^{s_0+d-1}$ can be contained (as a proper subset) in an another. For $C^{s_0+i} \subset C^{s_0+l}$, $0 \leqq i$, $l \leqq d-1$, $i \neq l$ multiplied by C_A^{td} would imply $C_A^{s_0+dt+i} \subset C_A^{s_0+dt+l}$, i.e. $C_A^{\varrho+i} \subset C_A^{\varrho+l}$, hence $C_A^{\varrho+i} = C_A^{\varrho+l}$, which is not true. An analogous statement holds for the remaining rows.

Remark 3. In [2] we have proved that for an irreducible matrix the intersection $T_A = C_A^\varrho \cap C_A^{\varrho+1} \cap \ldots \cap C_A^{\varrho+d-1}$ is $\{0\}$. [Even the intersection of any two of these sets is $\{0\}$.] This is not necessarily true in the case of a reducible matrix. Consider f.i. a 3×3 matrix A with $C_A = \{e_{12}, e_{21}, e_{33}, 0\}$. Then $C_A^2 = \{e_{11}, e_{22}, e_{33}, 0\}$ and $\mathfrak{G}_A = \{C_A, C_A^2\}$. Here $T_A = C_A \cap C_A^2 = \{e_{33}, 0\}$.

But it is easy to show that T_A is always a subsemigroup of S. For let be $a \in T_A$, $b \in T_A$. Then $a \in C_A^{\varrho+k}$ for any $k = 0, 1, \ldots, d-1$ and $b \in C_A^{\varrho+l}$ for any $l = 0, 1, \ldots, d-1$. Hence $ab \in C_A^{\varrho+k+l}$. If k, l run through a residue system $\pmod d$ so does $k + l$ so that $ab \in \bigcap_{m=0}^{d-1} C_A^{\varrho+m}$; hence $ab \in T_A$, q.e.d.

Remark 4. For an irreducible matrix A we have $s_0 = \varrho$ and we always have $C_A^d \subset C_A^{s_0}$. Again this is not necessarily true for a reducible matrix. This is shown on the following example. Let A be a matrix with

$$C_A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Here $d = 1$ and $\mathfrak{G}_A$ is the one-point group $\mathfrak{G}_A = \{C_A^2\}$, where

$$C_A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We have $s_0 = 2$ and $C_A \subset C_A^2$ does not hold.

Example. We conclude this section with a simple example of a matrix with $\operatorname{card} \mathfrak{G}_A > 1$ and $s_0 < \varrho$. Let A be a matrix with

$$C_A = \left[\begin{array}{cc|cccc} 0 & 1 & & & & \\ 1 & 0 & & \multicolumn{2}{c}{\mathbf{0}} & \\ \hline & & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ \multicolumn{2}{c|}{\mathbf{0}} & 1 & 1 & 0 & 0 \\ & & 1 & 1 & 1 & 0 \end{array}\right].$$

Then

$$C_A^2 = \left[\begin{array}{cc|cccc} 1 & 0 & & & & \\ 0 & 1 & & \multicolumn{2}{c}{0} & \\ \hline & & 0 & 0 & 0 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & 1 & 1 & 0 & 0 \end{array}\right], \qquad C_A^3 = \left[\begin{array}{cc|cccc} 0 & 1 & & & & \\ 1 & 0 & & \multicolumn{2}{c}{0} & \\ \hline & & 0 & 0 & 0 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \end{array}\right],$$

$$C_A^4 = \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right), \qquad C_A^5 = \left(\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right).$$

Here $\mathfrak{S}_A$ has 5 different elements, $\mathfrak{G}_A = \{C_A^4,\ C_A^5\}$, $d = 2$, $s_0 = 2$, while $\varrho = 4$.

II.

The result of Theorem 2 may be formulated in a somewhat other way by introducing the notion of the normal form of a non-negative matrix M.

Let M be a non-negative matrix (of order n). It is well known that there is a permutation matrix P (of order n) such that $PMP^{-1} = A$ is of the form

$$(7) \qquad A = \begin{pmatrix} A_{11}, & 0, & \dots, & 0 \\ A_{21}, & A_{22}, & \dots, & 0 \\ \hline A_{r1}, & A_{r2}, & \dots, & A_{rr} \end{pmatrix},$$

where A_{ii} $(1 \leqq i \leqq r)$ are irreducible matrices (including the case that some of the A_{ii}'s may be zero matrices of order 1).

Consider the sequences

$$(8) \qquad C_M,\ C_M^2,\ C_M^3,\ \dots$$

$$(9) \qquad C_A,\ C_A^2,\ C_A^3,\ \dots$$

The semigroups $\mathfrak{S}_A$ and $\mathfrak{S}_M$ are clearly isomorphic. If C_M^s is a semigroup, then so is C_A^s since

$$C_A^{2s} = C_P C_M^s C_{P^{-1}} \,.\, C_P C_M^s C_{P^{-1}} = C_P C_M^{2s} C_{P^{-1}} \subset C_P C_M^s C_{P^{-1}} = C_A^s,$$

and conversely. In particular, if C_M^ϱ is the idempotent $\in \mathfrak{S}_M$, then $C_P C_M^\varrho C_{P^{-1}}$ is the idempotent $C_A^\varrho \in \mathfrak{S}_A$, so that $\varrho(A) = \varrho(M)$. Hence instead of studying the sequence (8) we may restrict ourselves to the study of the sequence (9).

We shall use the following notations. d_i will denote the order of the group $\mathfrak{G}_{A_i}$, ϱ_i will denote the least integer for which $C_{A_i}^{\varrho_i}$ is an idempotent $\in \mathfrak{S}_{A_i}$.

If C_A^ϱ is the idempotent $\in \mathfrak{G}_A$, then $C_{A_i}^\varrho$ is necessarily the idempotent $\in \mathfrak{G}_{A_i}$.

If $\varrho = \varrho(A)$ has the meaning introduced from the beginning (i.e. the smallest integer for which C_A^ϱ is an idempotent $\in \mathfrak{S}_A$), then ϱ is necessarily of the form $\varrho = \varrho_1 + x_1 d_1 = \varrho_2 + x_2 d_2 = \dots = \varrho_r + x_r d_r$, with suitably chosen non-negative integers $x_1, x_2, \dots, x_r$. Since $\varrho_i = \tau_i d_i$, we have $\varrho = d_i(\tau_i + x_i)$, $i = 1, 2, \dots, r$. Denote $d^* = [d_1, d_2, \dots, d_r]$ the least common multiple of the integers $d_1, \dots, d_r$. The relation $d_i \mid \varrho$ implies $d^* \mid \varrho$. We have proved: there is an integer τ^* such that $\varrho(A) = \tau^* d^*$.

In what follows it is often of decisive importance whether in the normal form (7) there is among the A_{ii}' s a zero matrix (of order 1) or not. If none of the A_{ii}' s is a zero matrix, then

$$C_A^\varrho = C_A^{\tau^* d^*} \subset C_A \cup C_A^2 \cup \dots \cup C_A^n$$

contains $\{e_{11}, e_{22}, \dots, e_{nn}\}$. With respect to Theorem 2 we have

Theorem 4. *If a matrix A written in the normal form (7) has no zero matrix in the main diagonal, then C_A^ϱ is the unique semigroup contained in the sequence* (9).

The condition mentioned in this Theorem is not necessary. There are classes of non-negative matrices with zeros in the main diagonal having the same property. We prove f.i.:

Theorem 5. *Let*

$$A = \begin{pmatrix} A_1 & 0 \\ R & 0 \end{pmatrix},$$

where A_1 is irreducible and not the zero matrix of order 1. Then C_A^s is a semigroup if and only if it is the idempotent $\in \mathfrak{S}_A$.

Proof. Let A_1 be a $m \times m$ matrix (so that R is a $(n - m) \times m$ rectangular matrix). Denote $E = \{e_{11}, e_{22}, \dots, e_{mm}\}$. The support of

$$A^s = \begin{pmatrix} A_1^s & 0 \\ RA_1^{s-1} & 0 \end{pmatrix}$$

is a semigroup if and only if

$$(10) \qquad C_{A_1}^{2s} \subset C_{A_1}^s, \quad C(RA_1^{2s-1}) \subset C(RA_1^{s-1}).$$

Now $C_{A_1}^s$ is a semigroup if and only if $C_{A_1}^s = C_{A_1}^{2s}$ is the idempotent $\in \mathfrak{S}_{A_1}$ and $C_{A_1}^s$ contains then E. Hence we have

$$C_R = C_R \,.\, \{e_{11}, e_{22}, \dots, e_{mm}\} \subset C(RA_1^s).$$

Now if C_A^s is a semigroup, (10) implies

$$C(RA_1^{s-1}) \supset C(RA_1^{2s-1}) = C(RA_1^s) C(A_1^{s-1}) \supset C(R) C(A_1^{s-1}) = C(RA_1^{s-1}).$$

Hence $C(RA_1^{s-1}) = C(RA_1^{2s-1})$. Therefore $C_A^s = C_A^{2s}$, q.e.d.

Theorem 5 may be generalized as follows:

Theorem 6. *Let*

$$A = \begin{pmatrix} A_1 & 0 \\ R & A_2 \end{pmatrix},$$

with A_1 irreducible and not the zero matrix of order 1. If C_A^s is a semigroup, then C_A^s is the idempotent $\in \mathfrak{S}_A$ if and only if $C_{A_1}^s = C_{A_1}^{2s}$.

Proof. Denote

$$A^s = \begin{pmatrix} A_1^s & 0 \\ R_s & A_2^s \end{pmatrix}$$

and $R_1 = R$. Then

$$A^{2s} = \begin{pmatrix} A_1^{2s} & 0 \\ R_s A_1^s + A_2^s R_s & A_2^{2s} \end{pmatrix}.$$

The set C_A^s is a semigroup if and only if

$$C_{A_1}^{2s} \subset C_{A_1}^s, \quad C_{A_2}^{2s} \subset C_{A_2}^s,$$
$$C(R_s A_1^s) \cup C(A_2^s R_s) \subset C(R_s).$$

Since A_1 is irreducible, we conclude $C_{A_1}^{2s} = C_{A_1}^s$ and the diagonal of $C_{A_1}^s$ is positive, i.e. if A_1 is a $m \times m$ matrix, we have $\{e_{11}, e_{22}, \dots e_{mm}\} \subset C_{A_1}^s$ so that $C(R_s) = C(R_s)\{e_{11}, \dots, e_{mm}\} \subset C(R_s)C(A_1^s)$.
The relation

$$C(R_s A_1^s) \cup C(A_2^s R_s) \subset C(R_s) \subset C(R_s A_1^s)$$

implies

$$C(R_s A_1^s) \cup C(A_2^s R_s) = C(R_s) = C(R_s A_1^s).$$

Therefore $C(A^s) = C(A^{2s})$ if and only if $C(A_2^s) = C(A_2^{2s})$, q.e.d.

III.

In this last section we shall deal with some special types of matrices for which card $\mathfrak{G}_A = 1$.

Let A be the matrix of the form (7). The question arises what can be said about card $\mathfrak{G}_A$ by knowing card $\mathfrak{G}_{A_i} = d_i$.

The following Lemma holds.

Lemma 4. *If $d^* = [d_1, \dots, d_r]$, then* card $\mathfrak{G}_A = d^*$.

The proof of this Lemma (which has been known to the author for some time) is given in the recent paper of Ю. И. Любич (Ju. I. Ljubič) [see [1], Lemma 2, p. 344].

A non-negative irreducible matrix A is called primitive if some power of A is positive. This is the case if and only if $d(A) = 1$. In this case $\mathfrak{G}_A$ is a one-point group, namely the idempotent $\in \mathfrak{S}_A$.

If A is reducible of the form (7) then Lemma 4 implies card $\mathfrak{G}_A = 1$ if and only if $d_1 = d_2 = \dots = d_r = 1$. Hence:

Theorem 7. *If A is of the form (7), then $\mathfrak{G}_A$ is a one point group if and only if the matrices A_{ii} are either primitive or zero matrices of order 1.*

Remark. There are some special cases in which we may decide that $\mathfrak{G}_A$ is a one-point group without reference to the normal form (7).

Assertion 1. *If C_A is a semigroup, then* card $\mathfrak{G}_A = 1$.

Proof. By Lemma 2 $d = d(A)$ divides every s for which C_A^s is a semigroup. Since in our case we may put $s = 1$, we conclude $d = 1$.

Assertion 2. *If A is any non-negative $n \times n$ matrix and C_A contains $E = \{e_{11}, \dots, e_{nn}\}$, then* card $\mathfrak{G}_A = 1$.

Proof. By supposition $C_A = C_A \cdot E \subset C_A C_A = C_A^2$. Hence $C_A \subset C_A^2 \subset \subset \dots \subset C_A^m \subset C_A^{m+1}$. On the other hand we always have $C_A^{m+1} \subset C_A \cup C_A^2 \cup \dots \cup \cup C_A^m$, i.e. $C_A^{m+1} \subset C_A^m$. Hence $C_A^m = C_A^{m+1}$. This implies that C_A^m is the idempotent $\in \mathfrak{S}_A$ and, moreover, card $\mathfrak{G}_A = 1$.

A special class of matrices with $d(A) = 1$ is the class of lower triangular non-negative matrices, i.e. matrices of the following form:

$$A = \begin{pmatrix} a_{11}, & 0, & 0, & \dots, & 0 \\ a_{21}, & a_{22}, & 0, & \dots, & 0 \\ \hline a_{n1}, & a_{n2}, & a_{n3}, & \dots, & a_{nn} \end{pmatrix} \tag{11}$$

where a_{ik} (for $i \geqq k$) are non-negative elements, while all elements above the main diagonal are zeros.

Theorem 8. *For a lower triangular non-negative matrix A of order n the set C_A^n is the idempotent $\in \mathfrak{S}_A$.*

Proof. a) We first prove that $C_A^n \subset C_A^{n+1}$. Any element $\alpha \in C_A^n$ is the product of n elements $\in C_A$ of the form $\alpha = e_{i_1 i_2} e_{j_1 j_2} \dots e_{w_1 w_2}$. Such a product is certainly zero if the subscripts do not follow in the following order

$$(i_1, i_2), (i_2, i_3), \dots, (i_n, i_{n+1}). \tag{12}$$

Suppose $\alpha \neq 0$. Then by supposition we have $i_1 \geqq i_2 \geqq \dots \geqq i_n \geqq i_{n+1}$. The integers $i_1, i_2, \dots, i_{n+1}$ cannot be all different. There is therefore a couple, say i_j, i_{j+1}, such that $i_j = i_{j+1}$. The sequence (12) is of the form

$$(i_1, i_2) \dots (i_{j-1}, i_j)(i_j, i_j)(i_j, i_{j+2}) \dots (i_n, i_{n+1})$$

and α may be written as the product

$$\alpha = e_{i_1 i_2} \dots e_{i_{j-1} i_j} \cdot e_{i_j i_j} \cdot e_{i_j i_{j+2}} \dots e_{i_n i_{n+1}}. \tag{13}$$

But then we may write also

$$\alpha = e_{i_1 i_2} \ldots e_{i_{j-1} i_j} e^2_{i_j i_j} \ldots e_{i_n i_{n+1}},$$

so that $\alpha \in C_A^{n+1}$. Hence $C_A^n \subset C_A^{n+1}$.

b) On the other hand if $\alpha \in C_A^n$ and $\alpha \neq 0$, α is of the form (13) and we may omit $e_{i_j i_j}$ in α (without changing the value of α) so that

$$\alpha = e_{i_1 i_2} \ldots e_{i_{j-1} i_j} e_{i_j i_{j+2}} \ldots e_{i_n i_{n+1}}.$$

Hence $C_A^n \subset C_A^{n-1}$.

The last relation implies $C_A^{n+1} \subset C_A^n$. Both "inequalities" $C_A^n \subset C_A^{n+1} \subset C_A^n$ imply $C_A^n = C_A^{n+1}$ and $C_A^n = C_A^{n+1} = \ldots = C_A^{2n}$, q.e.d.

Remark 1. The exponent n is sharp since for a matrix with n zeros along the main diagonal and all elements below the main diagonal equal to 1 we have $C_A^{n-1} \neq 0$, but $C_A^n = 0$.

Remark 2. Also the exponent n in the relation $C_A^n \subset C_A^{n-1}$ (proved in *b*) cannot be in general replaced by a smaller one. Take f.i. the matrix A with

$$C_A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $C_A^3 \subset C_A^2$, but it is not true that $C_A^2 \subset C_A$, since $C_A \subsetneq C_A^2 = \{e_{11}, e_{21}, e_{22}, e_{31}, e_{32}, 0\}$ holds.

Theorem 9. *For a lower triangular matrix of the type* (11) *and* $n \geqq 2$ *there is always a number* $s \leqq n - 1$ *such that* C_A^s *is a semigroup.*

Proof. In Theorem 8 we have proved $C_A^{n-1} \supset C_A^n = C_A^{n+1} = \ldots$. Since for $n \geqq 2$ we have $2n - 2 \geqq n$, we conclude $C_A^{n-1} \supset C_A^{2(n-1)}$.

We now give a non-trivial generalization of Theorem 8 concerning a larger class of matrices with $d(A) = 1$.

Theorem 10. *Let* A *be a matrix of the form*

$$(14) \qquad A = \begin{pmatrix} A_{11}, 0, & \ldots, 0 \\ A_{21}, A_{22}, & \ldots, 0 \\ \hline A_{r1}, A_{r2}, & \ldots, A_{rr} \end{pmatrix},$$

where A_{ii} *is either a positive square matrix or a zero matrix of order* 1. *Then* C_A^{2r-1} *is the idempotent* $\in \mathfrak{S}_A$.

Proof. Denote — for typographical reasons — $C(A_{ij})$ by C_{ij}.

We first prove that $C_{\varrho\sigma} C_{\tau\varkappa} = 0$ for $\sigma \neq \tau$. Let n_i be the order of A_{ii}. Then, if $e_{\varrho_0\sigma_0} \in C_{\varrho\sigma}$, we have $n_1 + \ldots + n_{\sigma-1} < \sigma_0 \leqq n_1 + \ldots + n_\sigma$. If $e_{\tau_0\lambda_0} \in C_{\tau\lambda}$, we have $n_1 + \ldots + n_{\tau-1} < \tau_0 \leqq n_1 + \ldots + n_\tau$. If $\sigma > \tau$, then $\tau_0 \leqq n_1 + \ldots n_\tau \leqq n_1 + \ldots n_{\sigma-1} < \sigma_0$, hence $\sigma_0 \neq \tau_0$, and $e_{\varrho_0\sigma_0} e_{\tau_0\lambda_0} = 0$. If $\sigma < \tau$, then $\sigma_0 \leqq n_1 + \ldots + n_\sigma \leqq n_1 + \ldots + n_{\tau-1} < \tau_0$, hence $\sigma_0 \neq \tau_0$, and $e_{\varrho_0\sigma_0} e_{\tau_0\lambda_0} = 0$. Therefore the product $C_{\varrho\sigma} C_{\tau\lambda}$ can be different from zero only if it is of the form $C_{\varrho\sigma} C_{\sigma\lambda}$ (and of course $\varrho \geqq \sigma \geqq \lambda$).

We shall now study the behaviour of the powers of $C_A = \bigcup_{i > j} C_{ij}$.

The set C_A^r is a union of products of the form $C_{i_1 i_2} C_{j_1 j_2} \ldots C_{w_1 w_2}$. Such a product can be non-zero only if the subscripts follow in the order indicated in the product

$$C_{i_1 i_2} C_{i_2 i_3} \ldots C_{i_r i_{r+1}}.$$

Suppose that this product is non-zero. Since $i_1 \geqq i_2 \geqq \ldots \geqq i_{r+1}$, there is necessarily a couple, say i_j, i_{j+1}, such that $i_j = i_{j+1}$, and each of the non-zero summands in the set C_A^r is of the form

$$C_{i_1 i_2} \ldots C_{i_{j-1} i_j} C_{i_j i_j} C_{i_j i_{j+2}} \ldots C_{i_r i_{r+1}}.$$

But since $C^2_{i_j i_j} = C_{i_j i_j}$ (and $C_{i_j i_j}$ is not zero) this is the same as

$$C_{i_1 i_2} \ldots C_{i_{j-1} i_j} C^2_{i_j i_j} C_{i_j i_{j+2}} \ldots C_{i_r i_{r+1}},$$

which belongs to the set C_A^{r+1}. Hence $C_A^r \subset C_A^{r+1}$.

We next show that $C_A^{2r} \subset C_A^{2r-1}$. Each non-zero summand of C_A^{2r} is of the form

$$C_{i_1 i_2} \ldots C_{i_j i_{j+1}} \ldots C_{i_{2r} i_{2r+1}}.$$

The non-increasing sequence of $2r + 1$ integers

$$i_1 \geqq i_2 \geqq \ldots \geqq i_j \geqq i_{j+1} \ldots \geqq i_{2r+1}$$

contains at most r integers different one from the other. Hence there must be at least one triple such that $i_j = i_{j+1} = i_{j+2}$. (For if each of the r numbers appeared at most twice, the system would contain at most $2r$ members.) Hence any non-zero summand of C_A^{2r} may be written in the form

$$C_{i_1 i_2} \ldots C_{i_{j-1} i_j} C_{i_j i_j} C_{i_j i_j} C_{i_j i_{j+3}} \ldots C_{i_{2r} i_{2r+1}}.$$

Now since $C^2_{i_j i_j} = C_{i_j i_j}$, this product is yet contained in C_A^{2r-1}. Hence $C_A^{2r} \subset C_A^{2r-1}$.

Now the relation $C_A^r \subset C_A^{r+1}$ implies $C_A^{2r-1} \subset C_A^{2r}$. This combined with $C_A^{2r} \subset C_A^{2r-1}$ gives $C_A^{2r-1} = C_A^{2r}$, which proves our Theorem. (By the way the last result proves again that $\mathfrak{S}_A$ is a one-point group.)

Remark 1. In general the exponent $2r - 1$ cannot be replaced by a smaller one. This is shown on the following example. Let A be a matrix with

$$C_A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad C_A^2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Here $r = 2$, C_A^2 is not an idempotent, while C_A^3 is the idempotent $\in \mathfrak{S}_A$.

Remark 2. This example shows at the same time that it is in general not true that $C_A^r \subset C_A^{r-1}$ as one could expect by analogy with the proof of Theorem 8. On the other hand we cannot prove $C_A^{r-1} \subset C_A^r$ since, for instance, for the matrix A with $C_A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ we have $r = 2$ and $C_A \underset{\neq}{\supset} C_A^2 = \{0\}$.

The next theorem gives an information concerning the semigroups in the sequence

$$C_A, C_A^2, C_A^3, \ldots, \tag{15}$$

with A given by (14).

Theorem 11. *If C_A^r is not a semigroup, then the sequence (15) contains a unique subsemigroup of S_n (namely the idempotent $C_A^e \in \mathfrak{S}_A$). If C_A^r is a semigroup, then it is at the same time the idempotent $\in \mathfrak{S}_A$ and (15) contains at most r different elements.*

Proof. Let s_0 be the least integer for which $C_A^{s_0}$ is a semigroup.

a) Let first $s_0 > r$. Since $C_A^r \subset C_A^{r+1}$, we have $C_A^r \subset C_A^{r+1} \subset \ldots \subset C_A^{s_0} \ldots \subset C_A^{2s_0}$. The semigroup property implies $C_A^{2s_0} \subset C_A^{s_0}$. Hence $C_A^{s_0} = C_A^{2s_0}$ and the idempotent $\in \mathfrak{S}_A$ is the unique semigroup contained in the sequence (15).

b) Let $s_0 \leqq r$. Then $C_A^{2s_0} \subset C_A^{s_0}$ implies (multiplied by $C_A^{r-s_0}$) $C_A^{s_0+r} \subset C_A^r$. But $C_A^r \subset C_A^{r+1}$ implies $C_A^r \subset C_A^{r+s_0}$. Hence $C_A^{s_0+r} = C_A^r$. Now a power of C_A which occurs in the sequence (15) more than once is contained in $\mathfrak{G}_A$. Since $\mathfrak{G}_A$ is a one-point group, we conclude that C_A^r is the idempotent $\in \mathfrak{S}_A$. Moreover in this case the sequence (15) has at most r different members.

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СТЕПЕНИ НЕОТРИЦАТЕЛЬНЫХ МАТРИЦ

Штефан Шварц

Резюме

В статье изучаются некоторые свойства последовательности $A, A^2, A^3, \ldots,$ где A — неотрицательная разложимая матрица.