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ON TOPOLOGICAL REPRESENTATION OF SEMIGROUPS AND SMALL CATEGORIES

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A category $\mathfrak{K}$ is called small if its morphisms form a set. The cardinal of this set is called the cardinal of $\mathfrak{K}$, and is denoted by $|\mathfrak{K}|$. Every semigroup S^1 with a unity element may be considered in a natural way as a small category (morphisms e.g. all left translations) with the cardinal $|S^1|$. Hence, representing small categories we represent at the same time semigroups with unity elements.

By a topological representation of a category $\mathfrak{K}$ we mean an isomorphism of $\mathfrak{K}$ onto a category $\mathfrak{L}$, where the objects of $\mathfrak{L}$ are topological spaces and a certain topologically defined class of continuous mappings forms the morphisms of $\mathfrak{L}$.

In [1] J. de Groot has proved, among other results, the following theorem concerning a topological representation of groups:

Let G be an arbitrary group. Then there exists a Hausdorff space T such that the group of all auto(homeo)morphisms of T (under composition) is isomorphic with G. The space T can be chosen to fulfil some other conditions, e. g. to be metric or compact.

The above quoted paper gave us the idea for this article. We prove here, using some of our earlier results (see [2, 3, 4, 5]), a similar theorem for semigroups and small categories, namely:

Theorem 1. *Let $\mathfrak{K}$ be a small category, $|\mathfrak{K}|$ being less than the first inaccessible cardinal. Then $\mathfrak{K}$ is isomorphic with a category $\mathfrak{L}$, the objects of $\mathfrak{L}$ are Hausdorff topological spaces and the morphisms all their quasi-coverings.*(1) *All the spaces in $\mathfrak{L}$ may be chosen either metric or locally compact.*

We remark that, avoiding almost all topological considerations, the following theorem was proved in [5]:

(1) Let X, Y be Hausdorff spaces. $X_1 \subset X$ is said to be regularly closed in X, if X_1 is the closure of its interior in X. $f: X \to Y$ is called quasi-covering (of $f(X)$), if it is continuous, and if for each $x \in X$ there exists a regularly closed set X_1, $x \in X_1$, such that $f(X_1)$ is regularly closed in Y and f / X_1 is a homeomorphism of X_1 onto $f(X_1)$.

Theorem 2. *Let $\mathfrak{K}$ be a small category, $|\mathfrak{K}|$ being less than the first inaccessible cardinal. Then there exists a category $\mathfrak{L}$, the objects of $\mathfrak{L}$ are T_0-topological spaces and its morphisms all their local homeomorphisms, such that $\mathfrak{K}$ is isomorphic with $\mathfrak{L}$.*

Further, we state here a simple condition ($\mathcal{F}$ ($\mathfrak{k}$)) (depending on a cardinal $\mathfrak{k}$), the proof of which for some higher cardinals would enable to increase the cardinal of $\mathfrak{K}$ in theorems 1 and 2.

Proving his theorem, J. de Groot proceeded in three main stages

(i) replacing an abstract group G by the same group (up to isomorphism) of all automorphisms of a graph,

(ii) finding a suitable rigid space,

(iii) replacing every edge of the graph by a copy of the rigid space.

Our proof of the above theorem has the same pattern and uses sometimes the same spaces and constructions as J. de Groot did. Therefore we shall assume that the reader is familiar with the paper [1].

(i) To simplify our considerations we shall speak about relations instead of directed graphs. Evidently, the matter is the same.

Let X, Y be sets, $R \subset X \times X$, $S \subset Y \times Y$ (to show it explicitely we write $R = (R, X)$, $S = (S, Y)$). A mapping f: $X \to Y$ is called RS-compatible if xRx' implies $f(x)Sf(x')$ for all x, $x' \in X$ (we write often xRx' instead of $(x, x') \in R$). If (R, X) is a relation, we denote by $C(R, X)$ the set of all RR-compatible transformations of X. Evidently, $C(R, X)$ is a semigroup under composition with the identity transformation as the unity element. (R, X) is said to be rigid, if $C(R, X) = 1$. If $\mathfrak{k}$ is a cardinal, we denote by $\mathcal{F}$ ($\mathfrak{k}$) the following assertion: There exists a rigid relation (R, X) such that $|X| \geqq \mathfrak{k}$.

In [4] it was proved that $\mathcal{F}$ ($\mathfrak{k}$) holds for every cardinal $\mathfrak{k}$ less than the first inaccessible cardinal.

We denote by $\mathfrak{R}$ the following category: the objects of $\mathfrak{R}$ are couples (R, X), and, if (R, X) and (S, Y) are objects, all morphisms from (R, X) into (S, Y) are exactly all RS-compatible mappings from X into Y.

The following theorem was proved in [5]: Let $\mathfrak{K}$ be a small category, and let $\mathcal{F}$ ($|\mathfrak{K}|$) holds. Then $\mathfrak{K}$ is isomorphic with a *full* subcategory of $\mathfrak{R}$.

We remark that the last theorem is an analogon with replacing a color directed graph by a graph with the same automorphism group.

(ii) Generalizing the theorem by J. de Groot for semigroups there arises the question: what semigroup of continuous transformations ought to replace the group of auto(homeo)morphisms. We did not succeed with local homeomorphisms or open continuous mappings. The quasi-coverings seem to be most convenient even if they have a rather surprising property, i. e. they do not form, in general, a semigroup.

Let $f : X \to Y$ be a quasi-covering. $x \in X$ is called regular if there exists

an open set U, $x \in U$, $U \subset X$, such that $f \mid U$ is a homeomorphism of U onto an open set $f(U)$ in Y. If $f : X \to Y$ is a quasi-cover, then the regular points in X form an open dense set. This implies that the space P_n ([1], § 3, p. 88) is rigid ([1], § 3, p. 86) for quasi-coverings, where the trivial mapping is only the identity transformation.

Take P_n, $n > 3$, and let h^0 be a point on the boundary of the disk D, h^1 a point on any of the propellers except the center of the propeller. Then $P_n - \{h^0, h^1\}$ is connected and rigid for quasi-coverings. We denote by H the space P_n and by $\tilde{H}$ the space $P_n - \{h^0, h^1\}$.

(iii) Let (R, X) be a relation. Similarly as in [1], we can replace every $\alpha = (x^0, x^1) \in R$, x^0, $x^1 \in X$, by a copy H_α of H replacing x^i by h^i ($i = 0,1$). All H_α are homeomorphic to each other and disjoint with the possible exception of their „vertices". Into the union of all H_α,

$$M = \cup \{H_\alpha | \alpha \in R\},$$

we introduce topologies in two different ways.

(a) We define a metric ϱ in the same way as in ([1], § 7, p. 97) (under the assumption that there exists a finite chain connecting two arbitrary points). We denote the metric space obtained in this way by (R, X, ϱ).

(b) We define a topology on M defining the system of all open sets in M. Let $U \subset M$. U is said to be open, if and only if for every $x' \in U$ it is true: if x' is not a „vertex", i.e. $x' \in H_\alpha$ for exactly one α, there exists $\delta > 0$ such that all $x \in M$, for which $\varrho(x, x') < \delta$, $x \in H_\alpha$ (the metric is considered as in H_α) belong to U; if x' is a „vertex", then for almost all copies H_α, $x' \in H_\alpha$, all points $x \in H_\alpha$ such that $\varrho(x, x') \leqslant \varrho(h^0, h^1)/3$ belong to U, and for the remaining H_α, $x' \in H_\alpha$, there exists $\delta_\alpha > 0$ such that all $x \in H_\alpha$, $\varrho(x, x') < \delta_\alpha$, belong to U. This topological space will be denoted by (R, X, T).

Now we are able to present the proof of theorem 1. Let $\mathfrak{R}'$ be a full subcategory of $\mathfrak{R}$ such that $\mathfrak{K}$ is isomorphic with $\mathfrak{R}'$. Every object of $\mathfrak{R}'$ is a couple (R, X), R being a relation on a set X. We may associate with every (R, X) the space (R, X, ϱ) ((R, X, T), respectively). Consider a class $\mathfrak{L}$ of all spaces (R, X, ϱ) ((R, X, T), resp.) as objects and all quasi-coverings of these spaces as morphisms. Then $\mathfrak{L}$ is a category. It is easy to establish an isomorphism of $\mathfrak{L}$ onto $\mathfrak{R}'$, using the facts that each „vertex" must be mapped under quasi-covering into a „vertex" and that $\tilde{H}$ is connected. We omit the details of the proof as it runs in the same way as the proof of theorem 7 in [1].

Using the metric spaces in our construction (we remark that all considered (R, X) have the „finite chain property" as follows from the proofs in [5]), the objects of $\mathfrak{L}$ are bounded metric spaces, using (R, X, T), the objects of $\mathfrak{L}$ are locally compact Hausdorff spaces.

Now we state explicitly some corollaries:

Corollary 1. *Let $\mathfrak{K}$ be a small category such that $\mathscr{F}$ ($|\mathfrak{K}|$) holds. Then there exists a category $\mathfrak{L}$, the objects of $\mathfrak{L}$ are Hausdorff topological spaces, morphisms all their quasi-coverings, such that $\mathfrak{K}$ is isomorphic with $\mathfrak{L}$. All the spaces could be chosen either metric or locally compact.*

Corollary 2. *Let S^1 be a semigroup, and let $\mathscr{F}$ ($|S^1|$) holds. Then there exists a Hausdorff topological space T such that the set of all its quasi-coverings is a semigroup under composition isomorphic with S^1.*

Corollary 3. *Let G be a group, and let $\mathscr{F}$ ($|G|$) holds. Then there exists a Hausdorff topological space T such that the semigroup of all its local homeomorphisms forms a group under composition isomorphic with G.*

Proof. We can find T such that all quasi-coverings form a group isomorphic with G. In this case every quasi-covering is an autohomeomorphism. As every local homeomorphism is quasi-covering, corollary 3 follows.

We remark that our proofs could be modified for groups of homeomorphisms without of the restriction of cardinals, as every well ordering relation is rigid for compatible 1-1-transformations with the compatible inverse.

Finally, we formulate in relation with our above remark two problems which seem to be open:

Problem 1. Does the assertion $\mathscr{F}$ ($\mathfrak{k}$) hold for all cardinals $\mathfrak{k}$?(²)

Problem 2. Is it possible to use in theorem 1 compact spaces? The Čech-Stone compactification does not work immediately in the same way as for homeomorphisms.

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(²) Added in proof: The problem 1 has been solved positively in [6].

О ТОПОЛОГИЧЕСКОМ ПРЕДСТАВЛЕНИИ ПОЛУГРУПП И МАЛЫХ КАТЕГОРИЙ

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Резюме

Под малой категорией (мощности $\mathfrak{m}$) мы подразумеваем всякую категорию, класс морфизмов которой — множество (мощности $\mathfrak{m}$).

Под квази-покрытием мы будем понимать всякое непрерывное отображение $f: X \to Y$ (X, Y — отделимые топологические пространства), обладающие следующим свойством: для всякого $x \in X$ существует регулярно замкнутое множество U (т. е. множество, являющееся замыканием своей внутренней части) такое, что $x \in U$, $f \mid U$ — гомеоморфное отображение в Y, и $f(U)$ регулярно замкнутое в Y.

В работе доказана следующая

Теорема 1. *Пусть $\mathfrak{K}$ — малая категория мощности меньше первого недостижимого кардинального числа. Тогда $\mathfrak{K}$ изоморфна категории $\mathfrak{L}$, объектами которой являются отделимые топологические пространства, а морфизмами — все их квази-покрытия. При этом $\mathfrak{L}$ можно выбрать так, что все ее объекты — метрические пространства, или так, что все они локально компактны.*

Работа тесно связана с работой [1]; использованы результаты работ [1], [4], [5].