MATEMATICKO-FYZIKÁLNY ČASOPIS SAV, 14, 4, 1964

ON THE APPROXIMATIVE CONSTRUCTION OF THE EIGENVECTORS CORRESPONDING TO A PAIR OF COMPLEX CONJUGATED EIGENVALUES

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INTRODUCTION

In the numerical praxis of the last years there occur more and more non self-adjoint eigenvalue problems. The solution of practical problems makes demands, on the one hand, the theoretical analysis of the mentioned problem and, on the other hand, its numerical analysis. The problem of approximative construction of the eigenvalues does not seem to be satisfactorily solved yet in any of the directions mentioned instances, particularly in the case of complex eigenvalues. It is well known (see [5]), that for the construction of the eigenvalues of linear operators the iterative methods are advantageous. However, most papers concerned with the construction of eigenvalues demand the symmetry of operators considered or at least require the constructed eigenvalues to be real.

In the recent paper [1] there is considered the problem of the approximative construction of the eigenvectors corresponding to the pair of complex conjugated eigenvalues lying on the boundary of the spectral circle of a given real matrix and the problem of the construction of the eigenvectors mentioned. The absolute value and the argument of the sought eigenvalues are constructed in [1] step by step by iterations; the corresponding eigenvectors can, however, be obtained from the formulae given in [1] only in exceptional cases.

The purpose of our paper is to show in what way the knowledge of the approximations of the absolute value and the approximations of the arguments of the eigenvalues described above can be used for the construction of the corresponding eigenvectors. Contrary to the papers [1], [5] we do not assume that the spaces occuring in our considerations are finite-dimensional.

Some functional analytical methods, particularly the operational calculus in the algebra of a linear bounded operator of a Banach space into itself, are used. The approximations of the eigenvectors mentioned are constructed with help of iterations. The convergence of the sequence of iterations follows from the theorems on Cesaro iterations of a linear bounded operator. These theorems are published in the paper [3]. In the present paper we also prove some of the statements given in [3] without the proofs.

1. NOTATIONS AND DEFINITIONS

Let Y be a real Banach space and let X be the complex extension of the space Y, i.e. $z \in X \Leftrightarrow z = x + iy$, where $x, y \in Y$, $i^2 = -1$. The norm in the space Y will be denoted by the symbol $\| \ \|_Y$. We supply the space X with the norm defined by the following formula

$$\| z \|_X = \sup_{0 \leq \vartheta \leq 2\pi} \| x \cos \vartheta + y \sin \vartheta \|_Y,$$

or with some equivalent norm. Further let Y' be the space of the continuous linear forms on Y and let $[Y]$ be the space of bounded linear operators mapping Y into itself. The norms in Y' and in $[Y]$ are defined as follows:

$$\| y' \|_{Y'} = \sup_{\| y \|_Y = 1} | y'(y) |, \qquad y \in Y, \qquad y' \in Y';$$

$$\| T \|_{[Y]} = \sup_{\| y \|_Y = 1} \| Ty \|_Y, \qquad y \in Y, \qquad T \in [Y],$$

where $| y'(y) |$ is the absolute value of the number $y'(y)$. In cases where it does not cause a misunderstanding, the indices of the norms will be omitted.

The complex number α we shall write as $\alpha = \varrho \exp\{i\varphi\}$ so that the complex conjugated number $\bar{\alpha}$ has the following form: $\bar{\alpha} = \varrho \exp\{-i\varphi\}$.

The object of our considerations will be an operator $T \in [Y]$ about which we shall assume that in its spectrum $\sigma(T)$ there lie at least two eigenvalues μ_1, μ_2 and that the relations

$$\bar{\mu}_1 = \mu_2, \qquad |\lambda| < \mu \; (|\mu_1| = |\mu_2|) \tag{1.1}$$

hold for $\lambda \in \sigma(T)$, $\lambda \neq \mu_j$, $j = 1, 2$.

The operator $T \in [Y]$ can be extended from Y onto the whole space X by the formula $Tz = Tx + iTy$, where $z = x + iy$. By the symbol $[X]$ we denote the space of linear bounded operators mapping X into itself with the norm

$$\| T \|_{[X]} = \sup_{\| x \|_X = 1} \| Tx \|_X, \qquad x \in X, \qquad T \in [X].$$

We denote by the symbol Θ the zero-operator and the identity-operator by the symbol I. We assume further that the eigenvalues μ_1, μ_2 are simple poles of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$ (λ – a complex number).

Let

$$R(\lambda, T) = \sum_{k=0}^{\infty} (\lambda - \mu_j)^k T_{kj} + \sum_{k=1}^{\infty} (\lambda - \mu_j)^{-k} B_{kj}, \qquad j = 1, 2, \tag{1.2}$$

be the Laurent expansions of the resolvent $R(\lambda, T)$ in neighbourhoods of the poles μ_1, μ_2. It is well known (see [6] p. 306) that

$$B_{1j} = \frac{1}{2\pi i} \int_{C_j} R(\lambda, T) \, d\lambda, \qquad j = 1, 2, \tag{1.3}$$

$$B_{k+1,j} = (T - \mu_j I) B_{kj}, \qquad k = 1, 2, \dots;$$

where $C_j = \{\lambda \mid |\lambda - \mu_j| = \varrho_j\}$ and the radius ϱ_j is such that – with the exception of μ_j – there does not lie another point of the spectrum $\sigma(T)$ either on C_j or in the interior of C_j.

From the assumptions and from the spectral theorem ([6] p. 304, theorem 5 . 71-D) the operator T can be expressed as

$$T = \mu_1 B_{11} + \bar{\mu}_1 B_{12} + \frac{1}{2\pi i} \int_C \lambda R(\lambda, T) \, d\lambda,$$

where $C = \{\lambda \mid |\lambda| < \varrho_3, \varrho_3 < \mu\}$ assuming that in the interior of C there lies the set $\sigma(T) - \{\mu_1, \mu_2\}$. From the same theorem it follows that for any integer $n \geq 0$

$$T^n = \mu_1^n B_{11} + \bar{\mu}_1^n B_{12} + \frac{1}{2\pi i} \int_C \lambda^n R(\lambda, T) \, d\lambda. \tag{1.4}$$

For any $n \geq 1$ we put

$$S_{jn} = \frac{1}{n} \sum_{k=1}^{n} \mu_j^{-k} T^k, \tag{1.5}$$

$$U_{jn}^s = \frac{1}{n} \sum_{k=s}^{n} \binom{k}{s} \mu_j^{-k} T^k. \tag{1.6}$$

Let $y_j' \in Y'$ and $x^{(0)} \in Y$ be such that

$$y_j'(B_{1j} x^{(0)}) \neq 0 \tag{1.7}$$

hold for $j = 1, 2$.

2. AUXILIARY STATEMENTS

In this and in the next paragraphs we denote by symbols $c_1, c_2, \dots$ the constants independent of n, where $n = 1, 2, \dots$

Lemma 1. *There exists a constant c_1 such that*

$$\| S_{jn} - B_{1j} \| \leq c_1 n^{-1}, \qquad j = 1, 2. \tag{2.1}$$

Proof. First we prove that the sequence of the operators

$$W_{jn} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2\pi i} \int_C \left(\frac{\lambda}{\mu_j}\right)^k R(\lambda, T)\, d\lambda$$

converges in the norm of the space $[X]$ to the zero-operator Θ when $n \to \infty$.

From the assumption it follows that for $\lambda \in C$ it holds $|\lambda \mu_j^{-1}| = \gamma_j < 1$, so that

$$\left|\sum_{k=1}^{n} \left(\frac{\lambda}{\mu_j}\right)^k\right| \leqq \sum_{k=1}^{n} \gamma_j^k \leqq \frac{\gamma_j}{1-\gamma_j}$$

and thus

$$\| W_{jn} \| \leqq \frac{1}{n} \cdot \frac{\gamma_j}{1-\gamma_j}\, \mu \sup_{\lambda \in C} \| R(\lambda, T) \|. \tag{2.2}$$

In the second part of the proof we shall consider the sequence of the operators

$$V_{jn} = S_{jn} - W_{jn} = \sum_{r=1}^{2} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{\mu_r}{\mu_j}\right)^k B_{1r} =$$

$$= B_{1j} + \frac{1}{n} \sum_{k=1}^{n} \left(\frac{\bar{\mu}_j}{\mu_j}\right)^k B_{1,3-j}. \tag{2.3}$$

Let us estimate the norms of the operators

$$Q_{jn} = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{\bar{\mu}_j}{\mu_j}\right)^k B_{1,3-j}.$$

Evidently we have

$$\| Q_{jn} \| \leqq \frac{1}{n} \| B_{1,3-j} \| \cdot \left| \sum_{k=1}^{n} \left(\frac{\bar{\mu}_j}{\mu_j}\right)^k \right|.$$

Let us put $(\bar{\mu}_j/\mu_j) = \exp\{i\beta_j\}$, where β_j, $j = 1, 2$, are real. We then get

$$\| Q_{jn} \| \leqq \frac{1}{n} \| B_{1,3-j} \| \cdot \left| \frac{1 - e^{in\beta_j}}{1 - e^{i\beta_j}} \right| \leqq \frac{1}{n} \cdot \frac{2}{|\sin \beta_j|} \| B_{1,3-j} \|. \tag{2.4}$$

From (2.2), (2.3) and from (2,4) there follows (2.1).

Similarly we can prove the following lemma:

Lemma 2. *There exists a constant c_2 dependent neither of n nor of s such that*

$$\| U_{jn}^s \| \leqq c_2 n^s, \qquad j = 1, 2, \tag{2.5}$$

Proof. According to (1.4) for a given integer $s \geqq 0$ we have

$$U_{jn}^s = L_{jn} + K_{jn}, \qquad L_{jn} = \frac{1}{n} \sum_{r=1}^{2} \sum_{k=s}^{n} \binom{k}{s} \left(\frac{\mu_r}{\mu_j}\right)^k B_{1r},$$

$$K_{jn} = \frac{1}{n} \sum_{k=s}^{n} \frac{1}{2\pi_i} \int_C \left(\frac{\lambda}{\mu_j}\right)^k \binom{k}{s} R(\lambda, T)\, d\lambda.$$

We get easily the estimate

$$\| K_{jn} \| \leqq n^{s-1} \varrho_3 \sup_{\lambda \in C} \left\{ \| R(\lambda, T) \| \cdot \sum_{k=s}^{n} \left|\frac{\lambda}{\mu_j}\right|^k \right\} \leqq$$

$$\leqq n^{s-1} \varrho_3 \frac{\gamma_j}{1-\gamma_j} \cdot \sup_{\lambda \in C} \| R(\lambda, T) \|.$$

We further have

$$L_{jn} = \frac{1}{n} \sum_{k=s}^{n} \binom{k}{s} B_{1j} + \frac{1}{n} \sum_{k=s}^{n} \binom{k}{s} \left(\frac{\bar{\mu}_j}{\mu_j}\right)^k B_{1,3-j},$$

so that we obtain the norm-estimate

$$\| L_{jn} \| \leqq n^s \{ \| B_{1j} \| + \| B_{1,3-j} \| \}.$$

The estimates of K_{jn}, L_{jn} lead to the wanted estimate (2.5).

Remark. Lemmas 1 and 2 hold also if more general assumptions than those made in the first paragraph are fulfilled. Some simple generalizations are given in the two following lemmas 3 and 4.

Lemma 3. *Let us assume that the operator $T \in [X]$ has the property that on the boundary of the spectral circle there lies a finite but otherwise arbitrary number of simple poles of the resolvent $R(\lambda, T)$. Then the estimates (2.1) and (2.5) hold.*

For multiple eigenvalues we have:

Lemma 4. *Let us assume that on the boundary of the spectral circle of the operator $T \in [X]$ there lie p mutually different eigenvalues $\mu_1, \ldots, \mu_p$. Let $q_1, \ldots, q_p$ be multiplicities of the poles $\mu_1, \ldots, \mu_p$ of the resolvent $R(\lambda, T)$. Let $1 \leqq r \leqq p$, $q_r \geqq q_j$ for $j = 1, \ldots, p$. Then we have*

$$\left\| \frac{1}{n} \sum_{k=1}^{n} k^{-q_r+1} \mu_r^{-k} T^k - \frac{\mu_r^{-q_r+1}}{(q_r-1)!} B_{q_r,r} \right\| \leqq c_3 \frac{\log n}{n}.$$

The proof of the lemma 4 we shall not give, because it is possible to prove lemma 4 in the same way as theorem 4, which is to a certain degree a generalization of lemma 4.

Lemma 5. *Let us assume that for the terms of the sequence $\{\lambda_{jn}\}$ the following inequalities*

$$|\lambda_{jn} - \mu_j| \leqq c_4 n^{-1-\delta} \tag{2.6}$$

hold for $j = 1, 2$; $n = 1, 2, \ldots$, where $\delta > 0$. Then the sequence defined as

$$x_{jn} = \frac{1}{n} \sum_{k=1}^{n} \lambda_{jn}^{-k} T^k x^{(0)} \tag{2.7}$$

converges in the norm of the space X to the vector X_j. Further it holds that

$$T x_j = \mu_j x_j, \qquad x_j \neq 0, \tag{2.8}$$

$$\| x_{jn} - x_j \| \leqq c_3 n^{-\omega} \tag{2.9}$$

where $\omega = \min(1, \delta)$.

Proof. Evidently the following expression is true

$$x_{jn} - x_j = (S_{jn} - B_{1j})\, x^{(0)} + \frac{1}{n} \sum_{k=1}^{n} (\lambda_{jn}^{-k} - \mu_j^{-k})\, T^k x^{(0)}.$$

From lemma 1 it follows that

$$\| S_{jn} x^{(0)} - B_{1j} x^{(0)} \| \leqq \frac{c_1}{n} \| x^{(0)} \|, \tag{2.10}$$

so that it suffices if we consider the vectors $\frac{1}{n} \sum_{k=1}^{n} (\lambda_{jn}^{-k} - \mu_j^{-k})\, T^k x^{(0)}$, or the operators

$$Z_{jn} = \frac{1}{n} \sum_{k=1}^{n} (\lambda_{jn}^{-k} - \mu_j^{-k})\, T^k.$$

According to the assumption (2.6) we have $|\lambda_{jn}| \geqq c_6 > 0$. There exist the functions $c_{j7} = c_{j7}(n)$ such that

$$\frac{\mu_j}{\lambda_{jn}} = 1 + \frac{c_{j7}(n)}{n^{1+\delta}}, \qquad |c_{j7}(n)| \leqq c_8.$$

From this expression it follows that

$$\left(\frac{\mu_j}{\lambda_{jn}}\right)^k = 1 + \binom{k}{1} \frac{c_{j7}(n)}{n^{1+\delta}} + \binom{k}{2} \frac{c_{j7}^2(n)}{n^{2+2\delta}} + \dots,$$

so that

$$Z_{jn} = \frac{1}{n} \sum_{k=1}^{n} \sum_{s=1}^{k} \binom{k}{s} \frac{c_{j7}^s(n)}{n^{s(1+\delta)}}\, \mu_j^{-k} T^k = \\ = \frac{1}{n} \sum_{s=1}^{n} \frac{c_{j7}^s(n)}{n^{s(1+\delta)}} \cdot \sum_{k=s}^{n} \binom{k}{s} \mu_j^{-k} T^k.$$

According to lemma 3

$$\| Z_{jn} \| \leqq \sum_{s=1}^{n} \frac{c_8^s}{n^{s(1+\delta)}} \cdot c_2 n^s = c_2 c_8 n^{-\delta} \cdot \left| \frac{1 - [c_8 \cdot n^{-\delta}]^{n-1}}{1 - c_8 n^{-\delta}} \right| \leqq c_9 n^{-\delta}$$

which together with the estimate (2.10) gives the estimate (2.9).

To prove (2.8) it is sufficient to remark that from (1.7) there follows the relation $B_{1j} x^{(0)} \neq 0$, so that the vector $x_j = B_{1j} x^{(0)}$ is an eigenvector of the operator T corresponding to the eigenvalue μ_j. Since, according to (1.1) we have

$$(T - \mu_j I)\, x_j = (T - \mu_j I)\, B_{1j} x^{(0)} = B_{2j} x^{(0)} = 0 \qquad (\text{since } B_{2j} = \Theta).$$

The validity of (2.8) is proved and thus the proof of lemma 5 is accomplished.

3. ITERATIVE PROCESSES

The purpose of this paragraph is the proof of the convergence of some iterative methods for the construction of the eigenvalues $\mu_1, \mu_2 = \bar{\mu}_1$ and the eigenvectors x_1, x_2 corresponding to these eigenvalues.

Let

$$x^{(n)} = T x^{(n-1)}, \qquad n = 1, 2, \dots \tag{3.1}$$

$$\Delta_j^n = [y_j'(x^{(n)})]^2 - y_j'(x^{(n+1)}) \cdot y_j'(x^{(n-1)}). \tag{3.2}$$

The elements of the sequences

$$y_j'(x^{(0)}), \qquad y_j'(x^{(1)}), \dots \tag{3.3}$$

are real numbers according to our assumption that $T \in [Y]$, $x^{(0)} \in Y$, $y_j' \in Y'$.

According to [1] we define the indices n_k^j as follows: The symbol n_k^j, $j = 1, 2$; $k = 0, 1, \dots$ denotes the index of such element of the sequence (3.3) for which the relations sign $y_j'(x^{(n-1)}) = -1$, sign $y_j'(x^{(n)}) = +1$ hold for the k-th time. Among the numbers (3.3) there can occur the null-elements. In that case the corresponding zero-element has the same sign as the first non-zero element, which follows after it.

We define further ([1])

$$P_j^k = n_{k+1}^j - n_k^j, \quad j = 1, 2; \quad k = 0, 1, \dots \tag{3.4}$$

With the help of (1.4) we get for the vector $x^{(n)}$ the following expression

$$x^{(n)} = \mu_1^n x_1 + \mu_1^{-n} x_2 + w^{(n)}, \tag{3.5}$$

where $x_1 = B_{11} x^{(0)}$, $x_2 = B_{12} x^{(0)}$, $w^{(n)} = (1/2\pi i) \int_C \lambda^n R(\lambda, T)\, d\lambda x^{(0)}$, so that

$$\| w^{(n)} \| \leqq c_{10} \varrho_3^n \tag{3.6}$$

where $\varrho_3 = \mu q$, $0 < q < 1$ is the radius of the circle C.

The eigenvalues μ_1, μ_2 can be expressed in the following form

$$\mu_1 = \mu e^{i\varphi}, \qquad \mu_2 = \mu e^{-i\varphi}, \qquad 0 < \varphi < 2\pi. \tag{3.7}$$

Further let be

$$y_j'(x_1) = \gamma_j e^{i\alpha_j}, \qquad y_j'(x_2) = \gamma_j e^{-i\alpha_j}. \tag{3.8}$$

Theorem 1. *Let us assume the validity of* (3.8). *Then there exists a constant* c_{11} *such that*

$$\left| \frac{\Delta_j^{n+1}}{\Delta_j^n} - \mu^2 \right| \leqq c_{11} q^n, \qquad j = 1, 2. \tag{3.9}$$

Proof. We evidently have

$$y_j'(x^{(n)}) = \mu^n \gamma_j \exp\{in\varphi + i\alpha_j\} + \mu^n \gamma_j \exp\{-in\varphi - i\alpha_j\} + \eta_{jn},$$

where $\eta_{jn} = y_j'(w^{(n)})$, so that

$$|\eta_{jn}| \leqq c_{12}\mu^n q^n. \tag{3.10}$$

Easily we get that

$$y_j'(x^{(n)}) = 2v_j\mu^n \cos(n\varphi + \alpha_j) + \eta_{jn}. \tag{3.11}$$

From this expression it follows that

$$\Delta_j^n = 4v_j\mu^{2n}\sin^2\varphi + \zeta_{jn}, \tag{3.12}$$

where

$$\zeta_{jn} = \eta_{jn}^2 + 4v_j\mu^2\eta_{jn}\cos(n\varphi + \alpha_j) - 2v_j\mu^{n-1}\eta_{j,n+1}\cos[(n-1)\varphi + \alpha_j] - - 2v_j\mu^{n+1}\eta_{j,n-1}\cos[(n+1)\varphi + \alpha_j] - \eta_{j,n-1}\eta_{j,n+1}.$$

Thus there exists a constant c_{13} with the following property

$$|\zeta_{jn}| \leqq c_{13}\mu^{2n}q^n, \quad j = 1, 2. \tag{3.13}$$

The identities

$$\frac{\Delta_j^{n+1}}{\Delta_j^n} = \mu^2 \frac{1 + \dfrac{\zeta_{j,n+1}}{4v_j^2\mu^{2n+2}\sin^2\varphi}}{1 + \dfrac{\zeta_{jn}}{4v_j^2\mu^{2n}\sin^2\varphi}}$$

follow from the relations (3.12) and the estimates are then consequences of the inequalities (3.13).

Corollary 1. *The following inequalities hold:*

$$\left|\sqrt{\frac{\Delta_j^{n+1}}{\Delta_j^n}} - \mu\right| \leqq c_{14}q^n, \quad j = 1, 2; \quad n > n_0 \tag{3.14}$$

where $c_{14} = \sup_n \left(\dfrac{\Delta_j^n}{\Delta_j^{n+1}}\right)^{\frac{1}{2}} c_{11}$ *and where n_0 denotes some positive integer.*

Proof. According to (3.9) we get

$$\frac{\Delta_j^{n+1}}{\Delta_j^n} \geqq c_{15} > 0$$

for n sufficiently large, say $n > n_0$ and thus according to the identity

$$\left(\sqrt{\frac{\Delta_j^{n+1}}{\Delta_j^n}} - \mu\right) = \left(\frac{\Delta_j^{n+1}}{\Delta_j^n} - \mu^2\right)\left(\sqrt{\frac{\Delta_j^{n+1}}{\Delta_j^n}} + \mu\right)^{-1}$$

we obtain (3.14) with $c_{14} = \sup_n \left(\dfrac{\Delta_j^n}{\Delta_j^{n+1}}\right)^{\frac{1}{2}} c_{11}$.

Theorem 2 [1]. *The relations*

$$\frac{1}{n}\sum_{k=1}^{n} P_j^k = \frac{2\pi}{\varphi} + c_{16}(n)\frac{1}{n} \tag{3.15}$$

hold for the sequence $\{P_j^k\}$ of the numbers P_j^k defined above, where $|c_{16}(n)| \leqq c_{17}$.

The proof can be carried out in the same way as the proof of the corresponding theorem in the case of the finite dimensional space. In this case $y_j'(x)$ can be, for example, the value of one of the coordinates of the finite dimensional vector $x = (x_1, \ldots, x_l)$. The mentioned proof is given in paper [1].

Combining theorems 1 and 2 and lemma 5 we obtain the following theorem:

Theorem 3. *The sequence of the numbers $\{\lambda_{jn}\}$, where $\lambda_{jn} = \mu_{jn}\exp\{i\varphi_{jn}\}$ and where*

$$\mu_{jn} = \sqrt{\frac{\Delta_j^{n+1}}{\Delta_j^n}}, \quad \varphi_{jn} = \frac{2\pi}{\dfrac{1}{n}\sum_{k=1}^{n} P_j^k}, \tag{3.16}$$

converges to the eigenvalue μ_j of the operator T and we have the estimate

$$|\lambda_{jn} - \mu_j| \leqq c_{18}\frac{1}{n}, \quad j = 1, 2. \tag{3.17}$$

The sequence $\{x_{jn}\}$, where

$$x_{jn} = \frac{1}{n}\sum_{k=1}^{n} \lambda_{jn^2}^{-k} T^k x^{(0)},$$

converges in the norm of the space X to the eigenvector x_j corresponding to the eigenvalue μ_j of the operator T.

Proof. It is sufficient to prove the validity of the inequalities (3.17). From corollary 1 it follows that $|\mu_{jn} - \mu| \leqq c_{14}q^n$ or $\mu_{jn} = \mu + O(q^n)$ and from theorem 2 we can obtain the expression

$$\mu_{jn}\exp\{i\varphi_{jn}\} = [\mu + O(q^n)]\exp\left\{\frac{2\pi i}{P_j + O\left(\dfrac{1}{n}\right)}\right\},$$

where $P_j = \lim_{n\to\infty}\dfrac{1}{n}\sum_{k=1}^{n} P_j^k$. In other words $\lambda_{jn} = \mu_{jn}\exp\{i\varphi_{jn}\} = \mu\exp\{i\varphi\} + O\left(\dfrac{1}{n}\right)$ and thus the validity of the estimate (3.17) is proved.

From (3.17) it follows immediately that

$$|\lambda_{jn^2} - \mu\exp\{i\varphi\}| \leqq c_{18}n^{-2}$$

so that according to lemma 5 with $\delta = 1$ the sequence $\{x_{jn}\}$,

$$x_{jn} = \frac{1}{n}\sum_{k=1}^{n} \lambda_{jn^2}^{-k} T^k x^{(0)},$$

converges in the norm of the space X to the vector x_j and the relations $Tx_j = \mu_j x_j$, $j = 1, 2$ are valid.

Let us once more turn to the case, where there lie on the boundary of the spectral circle an arbitrary but finite number, in general, of multiple poles of the resolvent $R(\lambda, T)$. We shall assume that $\mu_1, \dots, \mu_p$ are these poles and that $q_1, \dots, q_p$ are the corresponding multiplicities. If the value μ_r, $1 \leqq r \leqq p$ is known, not only approximately but exactly and if $s \geqq q_j$ for $j = 1, \dots, p$, $j \neq r$, then similarly as in lemma 5 it is possible to obtain the corresponding eigenvector using the formula

$$x_{rn} = \frac{1}{n}\sum_{k=1}^{n} k^{-s+1}\mu_r^{-k}T^k x_r^{(0)}. \tag{3.18}$$

We assume that

$$B_{sr}x_r^{(0)} \neq 0, \qquad B_{s+1,r}x_r^{(0)} = 0, \tag{3.19}$$

where B_{sr}, $B_{s+1,r}$ are defined by the Laurent expansion

$$R(\lambda, T) = \sum_{k=0}^{\infty} (\lambda - \mu_r)^k T_{kr} + \sum_{k=1}^{\infty} B_{kr}(\lambda - \mu_r)^{-k}.$$

Theorem 4. *The sequence* (3.18) *converges in the norm of the space X to the eigenvector x_r corresponding to the eigenvalue μ_r of the operator T, i.e., $Tx_r = \mu_r x_r$.*

Proof. According to the assumption of the theorem:

$$x_{rn} = \frac{1}{n}\sum_{k=1}^{n} k^{-s+1}\left\{\sum_{j=1}^{p} H_k[\mu_j, T]\, x_r^{(0)} + \frac{1}{2\pi i}\int_C \left(\frac{\lambda}{\mu_r}\right) R(\lambda, T)\,\mathrm{d}\lambda x_r^{(0)}\right\},$$

where the interior of the circle $C = \{\lambda \mid |\lambda| = |\mu_r|\, q,\ q < 1\}$ contains the set $\sigma(T) - \{\mu_1, \dots, \mu_p\}$,

$$H_k[\mu_j, T] = \frac{1}{2\pi i}\int_{C_j} \left(\frac{\lambda}{\mu_j}\right)^k R(\lambda, T)\,\mathrm{d}\lambda,$$

where $C_j = \{\lambda \mid |\lambda - \mu_j| = \varrho_j\}$ and ϱ_j is such that neither in the interior of C_j not on the C_j there lies another point of the spectrum $\sigma(T)$ besides μ_j.

Evidently we have

$$H_k[\mu_j, T] = \sum_{h=1}^{q_j} \frac{f_{jk}^{(h-1)}(\mu_j)}{(h-1)!} B_{hj},$$

where $f_{jk}(\lambda) = (\lambda/\mu_r)^k$, $f_{jk}^{(h)}(\mu_j) = (\mathrm{d}/\mathrm{d}\lambda) f_{jk}^{(h-1)}(\lambda)|_{\lambda=\mu_j}$. Thus

$$H_k[\mu_j, T] = B_{1j} + \sum_{h=2}^{q_j} k(k-1)\dots(k-h+2)\frac{\mu_r^{-h+1}}{(h-1)!}\cdot\left(\frac{\mu_j}{\mu_r}\right)^{k-h+1} B_{hj}.$$

From this expression there follows according to (3.19) and according to that $s \geqq q_j$, $j \neq r$ the validity of the expression

$$k^{-s+1}H_k[\mu_j, T] = k^{-s+q_j}\frac{\mu_r^{-q_j+1}}{(q_j-1)!}\left(\frac{\mu_j}{\mu_r}\right)^{k-s+1} B_{q_j j}x_r^{(0)} + z_{jk},$$

where z_{jk} contains the elements w_{jk}, for which $\| w_{jk} \| \leqq O(k^{-1})$. Since $\| z_{jk} \| \leqq$ $\leqq c_{19}k^{-1}$ with some constant c_{19} independent of k, we get the estimate

$$\left\| \frac{1}{n}\sum_{k=1}^{n} z_{jk} \right\| \leqq O\left(\frac{\log n}{n}\right).$$

Finally we obtain the expression

$$x_{rn} = \frac{1}{n}\sum_{k=1}^{n}\left\{\sum_{\substack{j=1\\ j\neq r}}^{p} k^{-s+q_j}\left(\frac{\mu_j}{\mu_r}\right)^{k-s+1}\frac{\mu_r^{-q_j+1}}{(q_j-1)!}B_{q_j j}x_r^{(0)} + \frac{\mu_r^{-s+1}}{(s-1)!}B_{sr}x_r^{(0)} + v_{jn}\right\},$$

where

$$v_{jn} = \frac{1}{n}\sum_{k=1}^{n}\left\{z_{jk} + \frac{1}{2\pi i}\int_C\left(\frac{\lambda}{\mu_r}\right)^k R(\lambda, T)\,\mathrm{d}\lambda x_r^{(0)}\right\}.$$

Thus

$$\| v_{jn} \| \leqq O\left(\frac{1}{n}\log n\right). \tag{3.20}$$

We further have

$$\left\| \sum_{\substack{j=1\\ j\neq r}}^{p}\frac{1}{n}\sum_{k=1}^{n} k^{-s+q_j}\left(\frac{\mu_j}{\mu_r}\right)^{k-q_j+1}\frac{\mu_r^{-q_j+1}}{(q_j-1)!}B_{q_j j}x_r^{(0)} \right\| \leqq$$

$$\leqq \frac{1}{n}\sum_{\substack{j=1\\ j\neq r}}^{p}\left|\sum_{k=1}^{n}\left(\frac{\mu_j}{\mu_r}\right)^{k-q_j+1}\right|\cdot\left|\frac{\mu_r^{-q_j+1}}{(q_j-1)!}\right|\cdot\| B_{q_j j}x_r^{(0)} \| \leqq$$

$$\leqq \frac{1}{n}\sum_{\substack{j=1\\ j\neq r}}^{p}\left|\frac{2}{\sin\varepsilon_j}\right|\cdot\left|\frac{\mu_r^{-q_j+1}}{(q_j-1)!}\right|\cdot\| B_{q_j j}x_r^{(0)} \|, \tag{3.21}$$

where $\exp\{i\varepsilon_j\} = \mu_j/\mu_r$.

From (3.20) and (3.21) we obtain the estimate

$$\left\| x_{rn} - \frac{\mu_r^{-s+1}}{(s-1)!}B_{sr}x_r^{(0)} \right\| \leqq O\left(\frac{1}{n}\log n\right),$$

which shows the validity of the first part of theorem 4.

It remains to be proved that

$$x_r = \frac{\mu_r^{-s+1}}{(s-1)!}B_{sr}x_r^{(0)}$$

is an eigenvector corresponding to the eigenvalue μ_r of the operator T. But this assertion follows immediately from (3.19), since

$$(T - \mu_r I)\, x_r = \frac{\mu_r^{-s+1}}{(s-1)!} B_{s+1,r} x_r^{(0)} = 0.$$

Remark. The eigenvalues can be considered as known, if we know that they are solutions of a known algebraic equation which can be solved exactly. This is for instance the case of the cyclic kernels (see [4] p. 152) or the case of the stochastic matrices (see [2] chapter XII). In these cases the mentioned eigenvalues lie on the unit circle and are the roots of the binomial equation

$$\lambda^d - 1 = 0,$$

where d is so called index of imprimitivity ([2], p. 345).

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Received August 13, 1963.

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О ПРИБЛИЖЕННОМ ПОСТРОЕНИИ СОБСТВЕННЫХ ВЕКТОРОВ СООТВЕТСТВУЮЩИХ ПАРЕ КОМПЛЕКСНО-СОПРЯЖЕННЫХ СОБСТВЕННЫХ ЗНАЧЕНИЙ

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Резюме

В статье приводится метод приближенного построения собственных векторов соответствующих паре комплексно-сопряженных собственных значений линейного ограниченного оператора T, отображающего некоторое банахово пространство в себя, лежащих на границе круга $|\lambda| \leqq r(T)$, где $r(T)$ — спектральный радиус отображения T. Метод основан на некоторых свойствах последовательности операторов $\{n^{-1} \sum_{k=1}^{n} \mu_n^{-k} T^k\}$, где μ_n некоторые приближения одного из отмеченных собственных значений.