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GENERALIZED GROUPS WITH THE WELL-ORDERED SET OF IDEMPOTENTS

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Dedicated to Professor Štefan Schwarz on the occasion of his fiftieth birthday

A semigroup G is called a *generalized group* (or an *inverse semigroup*) if for any $g \in G$ there exists a unique element $g^{-1} \in G$ (the generalized inverse of g) such that

$$gg^{-1}g = g, \qquad g^{-1}gg^{-1} = g^{-1}$$

holds.

Generalized groups have been introduced by V. V. Wagner (В. В. Вагнер) [1] in 1952. The definition given above is due to A. E. Lieber (A. E. Либер) [2]. In the following we suppose the basic properties of generalized groups to be known (see, e.g. [3], [4]).

A. H. Clifford considered in 1941 [6] a special class of generalized groups in which for any $g \in G$

$$gg^{-1} = g^{-1}g$$

holds. We shall call this class of generalized groups "*Cliffordian* generalized groups".

In every generalized group we can introduce a binary relation ω by:

$$(g_1, g_2) \in \omega \leftrightarrow g_1g_1^{-1} = g_1g_2^{-1}.$$

(See also [1], [5] for other definitions of ω.) ω is known to be an order relation. We use the notation $g_1 \prec g_2$ as synonymous with $(g_1, g_2) \in \omega$.

ω is stable and involutorily invariant, i.e.

$$g_1 \prec g_2, \quad g_3 \prec g_4 \rightarrow g_1g_3 \prec g_2g_4,$$
$$g_1 \prec g_2 \rightarrow g_1^{-1} \prec g_2^{-1}.$$

We shall call ω the *canonical order relation* of G.

Denote by I the set of all idempotents of G. It is known (and easy to prove) that I is a commutative subsemigroup of G and for every couple $i_1, i_2 \in I$ we have

$$i_1 \prec i_2 \leftrightarrow i_1i_2 = i_1. \tag{1}$$

The purpose of this paper is to prove the following

Theorem. *Let G be a generalized group. If the set I of all idempotents $\in G$ is well-ordered (in the canonical order relation), then G is a Cliffordian generalized group.*

Proof. Let I be well-ordered (in ω), g any element $\in G$ and $[g]$ the generalized subgroup generated by g. Clearly $[g] \subset G$.

Note first: If $gg^{-1} \neq g^{-1}g$ then (since I is linearly ordered and $gg^{-1}, g^{-1}g \in I$) we have either $gg^{-1} \prec g^{-1}g$ or $g^{-1}g \prec gg^{-1}$.

In the following we shall suppose $gg^{-1} \prec g^{-1}g$. (The second case can be treated similarly.) The relation (1) implies $g^{-1}ggg^{-1} = gg^{-1}$ and multiplying by g to the right we have $g^{-1}g(gg^{-1}g) = gg^{-1}g$, i.e.

$$g^{-1}g^2 = g. \tag{2}$$

It is easy to see (cf. [7], n. 1, 3, 1) that $[g]$ coincides with the subsemigroup of G generated by g and g^{-1} and every $x \in [g]$ can be written in the form

$$x = (g^{-1})^k g^l (g^{-1})^m \quad \text{with} \quad l \geqq k \geqq 0, \quad l \geqq m \geqq 0, \quad l > 0 \tag{3}$$

([7], Theorem 1,3). It should be noted that the representation of x in this form is not uniquely determined.

Lemma 1. *Under the suppositions mentioned above every $x \in [g]$ can be written in one of the following forms:*

a) *either* $x = g^l \cdot (g^{-1})^m$ *with* $l > 0$, $m \geqq 0$,
b) *or* $x = (g^{-1})^m$ *with* $m > 0$,
c) *or* $x = (g^{-1})^k \cdot g$ *with* $k > 0$.

To prove this we proceed as follows. Let l be the least positive integer such that $x = (g^{-1})^k g^l (g^{-1})^m$. If $k = 0$, we have the form a) and there is nothing more to prove. Suppose therefore $k > 0$

a) If $l > 1$ and $m \geqq 0$, we have

$$x = (g^{-1})^k g^l (g^{-1})^m = (g^{-1})^{k-1} g^{-1} g^2 \cdot g^{l-2} (g^{-1})^m =$$
$$= (g^{-1})^{k-1} \cdot g \cdot g^{l-2} (g^{-1})^m = (g^{-1})^{k-1} g^{l-1} (g^{-1})^m,$$

contrary to the choice of l. Hence for $k > 0$ the number l cannot be > 1.

b) Let $l = 1$ and $m \geqq 1$.
Then

$$x = (g^{-1})^{k-1} g^{-1} g g^{-1} (g^{-1})^{m-1} = (g^{-1})^{k-1} \cdot g^{-1} \cdot (g^{-1})^{m-1} = (g^{-1})^{k+m-1},$$

hence x is of the form b).

c) Let $l = 1$ and $m = 0$. Then $x = (g^{-1})^k g$ is of the form c).

Lemma 2. *Any idempotent $x \in [g]$ is of the form*

A. *either* $x = g^{-1}g$,
B. *or* $x = g^l (g^{-1})^l$, $l \geqq 1$.

Before proving this recall that if $x = (g^{-1})^k g^l (g^{-1})^m$, then $x^{-1} = g^m \cdot (g^{-1})^l g^k$.
Note further: If $m \geqq 2$ then

$$(g^{-1})^m \cdot g^m = (g^{-1})^{m-1} \cdot g^{-1} \cdot g^2 g^{m-2} = (g^{-1})^{m-1} \cdot g^{m-1} = \ldots = g^{-1} \cdot g.$$

Let now be x an idempotent $\in [g]$. Then $x = xx^{-1}$.

In the case a) we have $x = g^l (g^{-1})^m g^m (g^{-1})^l$. If $m = 0$, $x = g^l \cdot (g^{-1})^l$. If $m = 1$, $x = g^l \cdot g^{-1} g (g^{-1})^l = g^l (g^{-1} g g^{-1})(g^{-1})^{l-1} = g^l g^{-1} (g^{-1})^{l-1} = g^l \cdot (g^{-1})^l$. If $m \geqq 2$, $x = g^l \cdot g^{-1} \cdot g (g^{-1})^l = g^l (g^{-1})^l$.

In the case b) we have $x = xx^{-1} = (g^{-1})^m g^m = g^{-1}g$.

In the case c) we have $x = (g^{-1})^k g \cdot g^{-1} g^k = (g^{-1})^k (g g^{-1} g) g^{k-1} = (g^{-1})^k g g^{k-1} = (g^{-1})^k \cdot g^k = g^{-1} g$.

This proves our lemma.

Now it can be easily verified that

$$\ldots g^{l+1} \cdot (g^{-1})^{l+1} \prec g^l \cdot (g^{-1})^l \prec \ldots \prec gg^{-1} \prec g^{-1}g,$$

i.e. the idempotents $\in [g]$ form a decreasing chain. The well-ordering of I implies that this chain breaks up, i.e. $[g]$ possesses a finite number of idempotents. Now $g^{-1}g$ being the greatest idempotent $\in [g]$ is an indentity of $[g]$. It follows $gg^{-1} = g^{-1}g$. ([2], Theorem 8.) Thus G is Cliffordian.

Corollary 1. *If the set of all idempotents of a generalized group is finite and linearly ordered, then the generalized group is Cliffordian.*

Corollary 2. ([2], Theorem 9.) *Every generalized group having only two idempotents is Cliffordian.*

We conclude with the following conjecture (the converse of our Theorem) which we are not able to prove or disprove:

Let I be an idempotent generalized group having the property: If I is isomorphic to the generalized group of all idempotents of a generalized group G, then G is Cliffordian. Then the canonical order of I is the well-order.

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ОБОБЩЕННЫЕ ГРУППЫ С ВПОЛНЕ УПОРЯДОЧЕННЫМИ МНОЖЕСТВАМИ ИДЕМПОТЕНТОВ

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Резюме

Доказана теорема: Если множество всех идемпотентов обобщенной группы вполне упорядочено каноническим отношением порядка этой обобщенной группы, то данная обобщенная группа является клиффордовой (т. е. удовлетворяет условию $gg^{-1} = g^{-1}g$ для любого элемента g).