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NOTE ON A DOUBLE COSET DECOMPOSITION OF SEMIGROUPS DUE TO ŠTEFAN SCHWARZ

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In a recent paper in this journal, Štefan Schwarz [1] proved the interesting theorem that any homomorphism φ of a completely simple semigroup S onto a group G can be described by a double coset decomposition

$$S = H \cup HaH \cup HbH \cup \dots \qquad (a, b, \dots \in S) \tag{1}$$

of S with respect to the kernel H of φ. The double cosets appearing in (1) are mutually disjoint, and HaH consists precisely of those elements of S mapped by φ into $\varphi(a)$. It is natural to inquire when this happens in general, and the purpose of this note is to take a small step in this direction.

Theorem. *Let S be a regular semigroup, and let φ be a homomorphism of S onto a group G. Let e be the identity element of G, and let $H = \varphi^{-1}(e)$ be the kernel of φ. Then*

$$\varphi^{-1}\varphi(a) = HaH \qquad (\text{for all } a \text{ in } S) \tag{2}$$

if and only if H is simple.

Proof. Assuming (2), let $a \in H$. Then

$$HaH = \varphi^{-1}\varphi(a) = \varphi^{-1}(e) = H,$$

so H is simple. (We did not need the regularity of S for this.)

Conversely, assume that H is simple, and let $a \in S$. Since

$$\varphi(HaH) = \varphi(H)\,\varphi(a)\,\varphi(H) = e\varphi(a)\,e = \varphi(a),$$

we clearly have $HaH \subseteq \varphi^{-1}\varphi(a)$. To prove the opposite inclusion, let $b \in \varphi^{-1}\varphi(a)$, so that $\varphi(b) = \varphi(a)$. Since S is regular, there exists c in S such that $bcb = b$. Then $\varphi(b)\,\varphi(c)\,\varphi(b) = \varphi(b)$ in G, so that

$$\varphi(c) = \varphi(b)^{-1} = \varphi(a)^{-1},$$
$$\varphi(ac) = e = \varphi(bc).$$

Thus ac and bc belong to H. Since H is simple, there exist x and y in H such that $bc = xacy$. Hence

$$b = bcb = xa(cyb).$$

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Now

$$\varphi(cyb) = \varphi(c)\,\varphi(y)\,\varphi(b) = \varphi(a)^{-1}\,e\varphi(a) = e,$$

whence $cyb \in H$, and we conclude that $b \in HaH$. This proves our theorem.

The following example shows that there exist regular semigroups with the Schwarz property that are not completely simple. Let r, s, t be mappings of the set of non-zero integers into itself defined as follows.

$$r(x) = -|x|; \quad s(x) = \begin{cases} x & \text{if } x > 0, \\ -x+1 & \text{if } x < 0; \end{cases}$$

$$t(x) = \begin{cases} x & \text{if } x > 0, \\ 1 & \text{if } x = -1, \\ -x-1 & \text{if } x < -1. \end{cases}$$

Setting $p = rs$, $q = rt$, $e_0 = pq$, $e_1 = qp$, we find:

$$r^2 = e_0 r = r, \quad s^2 = st = se_0 = s, \quad t^2 = ts = te_0 = t,$$

$$e_0^2 = e_0, \quad e_1^2 = e_0 e_1 = e_1 e_0 = e_1 \neq e_0.$$

The semigroup S generated by r, s, and t can be shown to be regular and simple (in fact bisimple); but it is not completely simple since the idempotent e_0 is not primitive. (p and q generate a so-called "bicyclic" subsemigroup B of S, and one can show that

$$S = B \cup Br \cup sB \cup tB \cup sBr \cup tBr.)$$

Since S is generated by idempotents, the only homomorphic group image of S is the group of order one, and S has the Schwarz property by virtue of being itself simple. For an apparently less trivial example, let $T = G \times S$, where G is any group. Then the kernel of any homomorphism of T onto a group has the form $N \times S$, where N is a normal subgroup of G, and every such $N \times S$ is simple.

REFERENCE

[1] Štefan Schwarz, *Homomorphisms of a completely simple semigroup onto a group*, Matematicko-fyz. časopis SAV *12* (1962), 293—300.

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ЗАМЕТКА О РАЗЛОЖЕНИИ ПОЛУГРУПП ПО ДВОЙНОМУ МОДУЛЮ

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Резюме

В статье доказывается следующая теорема:

Пусть S регулярная полугруппа и φ гомоморфизм S на группу G. Пусть e единица группы G и $H = \varphi^{-1}(e)$ — ядро φ. Потом $\varphi^{-1}\varphi(a) = HaH$ (для всякого $a \in S$) имеет место тогда и только тогда, если H — простая полугруппа.

На примере показано, что существует регулярная полугруппа S, которая не является вполне простой так, что $\varphi^{-1}\varphi(a) = HaH$ для всякого $a \in S$ и для всякого гомоморфизма φ полугруппы S на группу.