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HOMOMORPHISMS OF A COMPLETELY SIMPLE SEMIGROUP ONTO A GROUP

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The homomorphisms of a semigroup S onto a group have been studied in a great number of papers beginning with the general results of P. Dubreil and his school (see the literature in [2]). The special case of a completely simple semigroup (without zero) S has been studied by Л. М. Глускин (see the remark in his paper [3]), R. R. Stoll [6] and in a recent paper of G. B. Preston [5].

These authors use the Rees representation theorem and prove the existence of a maximum group homomorphic image of S.

Now I have found that there is a simple method of describing the homomorphisms of a completely simple semigroup S onto a group which gives a rather unexpected and elegant explicit description of the corresponding congruence classes, a description which is very close to that in the group case. The congruence classes are simply distinct classes of a double coset decomposition of S with respect to a subsemigroup H of S. Hereby the use of double cosets is an essential one (see the example below).

Moreover we do not need the Rees representation theorem. Our presentation is based on the rather elementary description of S by means of minimal one-sided ideals (as given in section 1 below).

Double coset decompositions of S modulo two subsemigroups of S seem to appear first in the paper [7]. They are used then in the study of the semigroup of measures on a compact semigroup. (See [8], [9], [10].)

The key for all the following considerations is Lemma 2, the other considerations being of a more or less straithforward nature.

1. We shall need the following preliminary results the proof of which can be found in [1] or [4].

A completely simple semigroup (without zero) S can be written in the form $S = \bigcup_{\alpha\in\Lambda_1} R_\alpha = \bigcup_{\beta\in\Lambda_2} L_\beta$, where R_α, L_β are minimal right and left ideals of S respectively. Also $R_\alpha L_\alpha = R_\alpha \cap L_\beta = G_{\alpha\beta}$ is a group, hence $S = \bigcup_{\alpha\in\Lambda_1} \bigcup_{\beta\in\Lambda_2} G_{\alpha\beta}$. We shall call the $G_{\alpha\beta}$, s group-components of S. They are all isomorphic one with another. Denote by $e_{\alpha\beta}$ the unit element of the group $G_{\alpha\beta}$. Then $\{e_{\alpha\beta} \mid \alpha \in \Lambda_1\}$ is the set of all idempotents contained in L_β each of them being a right unit of L_β. Analogously $\{e_{\alpha\beta} \mid \beta \in \Lambda_2\}$

is the set of all idempotents $\in R_\alpha$ each of them being a left unit of R_α. The elements of a group-component $G_{\alpha\beta}$ will be always denoted by the indices α, β. If $g_{\alpha\beta} \in G_{\alpha\beta}$, then $g_{\alpha\beta} G_{\gamma\delta} = G_{\alpha\delta}$, and analogously $G_{\gamma\delta} g_{\alpha\beta} = G_{\gamma\beta}$.

The following general results have been explicitly proved in the paper [7]. Let H be any simple subsemigroup of a completely simple semigroup S containing all idempotents of S. Then

I) H is itself a completely simple semigroup (without zero).

II) If $a, b \in S$, then $Ha \cap Hb \neq \emptyset$ implies $Ha = Hb$ and $HaH \cap HbH \neq \emptyset$ implies $HaH = HbH$.

III) S admits a (uniquely determined) decomposition into disjoint summands of the form

$$S = H \cup HaH \cup HbH \cup \ldots, \qquad a, b, \ldots \in S.$$

Hereby $HaH = HbH$ if and only if $b \in HaH$; in particular $H = HaH$ if and only if $a \in H$.

IV) If $G'_{\alpha\beta} = G_{\alpha\beta} \cap H$, then $HaH \cap G_{\alpha\beta} = G'_{\alpha\beta} a G'_{\alpha\beta}$ and we have

$$G_{\alpha\beta} = G'_{\alpha\beta} \cup G'_{\alpha\beta} a G'_{\alpha\beta} \cup G'_{\alpha\beta} b G'_{\alpha\beta} \cup \ldots$$

Note further: If a is any element $\in S$, then $a_{\alpha\beta} = e_{\alpha\beta}\, a\, e_{\alpha\beta} \in G_{\alpha\beta}$ and $Ha_{\alpha\beta}H = He_{\alpha\beta} a e_{\alpha\beta} H \subset HaH$, hence $Ha_{\alpha\beta}H = HaH$. This says that every class HaH has a non-empty intersection with any $G_{\alpha\beta}$ ($\alpha \in \Lambda_1$, $\beta \in \Lambda_2$) and every class HaH can be "generated" by means of an element a chosen from a fixed group, say G_{11}. (In the following we shall suppose always that $1 \in \Lambda_1 \cap \Lambda_2$.)

Finally we note: If $H \cap R_\alpha = R'_\alpha$, $H \cap L_\beta = L'_\beta$, then $H = \bigcup_{\alpha \in \Lambda_1} R'_\alpha = \bigcup_{\beta \in \Lambda_2} L'_\beta = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G'_{\alpha\beta}$, where R'_α, L'_β are the minimal right and left ideals of H respectively and $G'_{\alpha\beta}$ are the group-components of H.

2. Let now $\bar{G}$ be a group with the unit element $\bar{e}$, $\bar{G} = \{\bar{e}, \bar{a}, \bar{b}, \ldots\}$. Let φ be a homomorphism of S onto $\bar{G}$ and let be $\varphi^{-1}(\bar{e}) = H$. H is clearly a subsemigroup of S containing all idempotents $\in S$.

Lemma 1. *H is a simple subsemigroup of S.*

Proof. We first prove that $a \in aHa$ for every $a \in H$ (i. e. H is a regular semigroup). Let be $a \in H$. Then a is contained in some group, say $a \in G_{\alpha\beta}$. Denote by a^{-1} the element $\in G_{\alpha\beta}$ such that $aa^{-1} = e_{\alpha\beta}$. Now $\varphi(a)\,\varphi(a^{-1}) = \varphi(e_{\alpha\beta})$, i. e. $\bar{e} \cdot \varphi(a^{-1}) = \bar{e}$, implies $\varphi(a^{-1}) = \bar{e}$, hence $a^{-1} \in H$. Since $a = aa^{-1}a$, we have $a \in aHa$.

Let L_β be a minimal left ideal of S and denote $L_\beta \cap H = L'_\beta \neq \emptyset$. Clearly L'_β is a left ideal of H.* We prove that L'_β is a minimal left ideal of H. Suppose that this were not the case. Then there exists a left ideal L'' of H such that $L'' \subset L'_\beta$, $L'' \neq L'_\beta$. Choose any element $a \in L'' \subset H$. By regularity there is an element $y \in H$ such that $a = aya$. The relation $ya = yaya$ implies that ya is an idempotent and $e'' = ya \in yL'' \subset L''$. Therefore $L'_\beta e'' \subset L'_\beta L'' \subset L''$. On the other hand every idempotent $\in L_\beta$ is a right unit of the semigroup L_β, hence $L'_\beta e'' = L'_\beta$, and finally $L'_\beta \subset L''$. This contradiction proves that L_β is a minimal left ideal of H.

Now $H = S \cap H = \bigcup_{\beta \in \Lambda_2} (L_\beta \cap H) = \bigcup_{\beta \in \Lambda_2} L'_\beta$ says that H is a union of its minimal left ideals. Since it is well known that a semigroup (without zero) containing a minimal left ideal is simple if and only if it is the sum of its minimal left ideals, we conclude that H is a simple semigroup. [Moreover it follows immediately (see [7], Lemma 1,1) that H is completely simple.]

If $\bar{a} \in \bar{G}$ and $\varphi^{-1}(\bar{a})$ contains an element $a \in S$, then $\varphi^{-1}(a)$ contains also aH, Ha, HaH; hence we have necessarily $HaH \subset \varphi^{-1}(\bar{a})$. Note also that since H contains a left unit and a right unit for every $a \in S$, we have $a \in aH \subset HaH$ and $a \in Ha \subset HaH$.

The following Lemma is of a decisive importance for all what follows:

Lemma 2. *The set $\varphi^{-1}(\bar{a})$ is exactly one class HaH (with a suitably chosen $a \in S$).*

Proof. Suppose that $\varphi^{-1}(\bar{a})$ contains at least two distinct classes Ha_1H and Ha_2H. Denote by $(\bar{a})^{-1}$ the inverse element of $\bar{a}$ in the group $\bar{G}$. Again $\varphi^{-1}[(\bar{a})^{-1}]$ contains at least one class HbH. The relation $(\bar{a})^{-1}\bar{a} = \bar{e}$ implies $\varphi^{-1}[(\bar{a})^{-1}] \cdot \varphi^{-1}(\bar{a}) \subset \varphi^{-1}(\bar{e})$, i. e.

$$\{HbH \cup \ldots\}\{Ha_1H \cup Ha_2H \cup \ldots\} \subset H.$$

Since $Hb \subset HbH$, $a_1H \subset Ha_1H$, $a_2H \subset Ha_2H$, we also have

$$\{Hb \cup \ldots\}\{a_1H \cup a_2H \cup \ldots\} \subset H,$$

and the relations $Hba_1H \subset H$, $Hba_2H \subset H$ imply $Hba_1H = Hba_2H = H$.

Without loss of generality (see above) we can suppose that b, a_1, a_2 are elements $\in G_{11}$. Denote $ba_1 = h \in H$. Denote further by a'_1 the inverse of a_1 in G_{11}. Then $ba_1a'_1 = ha'_1$, $be_{11} = ha'_1$ and $b = ha'_1$. Hence $Hb = Hha'_1 \subset Ha'_1$, which implies $Hb = Ha'_1$.

Now $(Hb)(a_2H) = H$ implies $(Ha'_1)(a_2H) = H$, hence $a'_1a_2 = h' \in H$. Further $a_1h' = a_1a'_1a_2 = e_{11}a_2 = a_2$ implies $a_2H = a_1h'H \subset a_1H$ and $Ha_2H \subset Ha_1H$, hence $Ha_2H = Ha_1H$. This proves our Lemma.

3. For convenience we introduce the following.

Definition.* A simple subsemigroup H of S is called almost normal in S if

I) H contains all idempotents $\in S$.

II) $G'_{\alpha\beta} = G_{\alpha\beta} \cap H$ is a normal subgroup of $G_{\alpha\beta}$ for at least one couple α, β.

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* For if $a \in H$, $x \in L'_\beta$, we have $ax \in aL'_\beta \subset aL_\beta \subset L_\beta$, further $ax \in H \cdot H \subset H$, hence $ax \in H \cap L_\beta = L'_\beta$.

*) For our case (the case of a completely simple semigroup S) the almost normal subsemigroups are of course the same as Dubreil's "normal unitary" subsemigroups of S, since these are just the kernels of homomorphisms of S onto a group. (See [2], p. 257.)

The restriction to one couple α, β is a formal one as the next lemma shows.

Lemma 3. *An almost normal subsemigroup intersects each group $G_{\alpha\beta}$ in a normal subgroup of $G_{\alpha\beta}$.*

Proof. Suppose that $G'_{11} = G_{11} \cap H$ is a normal subgroup of G_{11}. We prove that for any $\sigma \in \Lambda_1$, $\varrho \in \Lambda_2$ the group $G'_{\sigma\varrho} = G_{\sigma\varrho} \cap H$ is a normal subgroup of $G_{\sigma\varrho}$.

Let be $c \in G_{\sigma\varrho}$. Then $e_{11}ce_{11} \in G_{11}$ and by supposition $G'_{11}(e_{11}ce_{11}) = (e_{11}ce_{11})\, G'_{11}$, i. e. $G'_{11}ce_{11} = e_{11}cG'_{11}$. Since $c = e_{\sigma\varrho}c = ce_{\sigma\varrho}$, we have $G'_{11}(e_{\sigma\varrho}ce_{\sigma\varrho})\, e_{11} = e_{11}(e_{\sigma\varrho}ce_{\sigma\varrho})\, G'_{11}$. Multiplying both sides by $e_{\sigma\varrho}$ we get $(e_{\sigma\varrho}G'_{11}e_{\sigma\varrho})\, c(e_{\sigma\varrho}e_{11}e_{\sigma\varrho}) = (e_{\sigma\varrho}e_{11}e_{\sigma\varrho})\, c(e_{\sigma\varrho}G'_{11}e_{\sigma\varrho})$, i. e. $G'_{\sigma\varrho}cx = xcG'_{\sigma\varrho}$, where $x = e_{\sigma\varrho}e_{11}e_{\sigma\varrho} \in G'_{\sigma\varrho}$. Define x^{-1} by $x^{-1} \in G'_{\sigma\varrho}$ and $xx^{-1} = e_{\sigma\varrho}$. Then the last relation implies $(x^{-1}G'_{\sigma\varrho})\, c(xx^{-1}) = (x^{-1}x)\, c(G'_{\sigma\varrho}x^{-1})$, i. e. $G'_{\sigma\varrho}c = cG'_{\sigma\varrho}$, q. e. d.

Example. The following example enables a clearer insight into the role of the almost normal subsemigroups and the role of the double cosets. Consider the completely simple semigroup $S = \{a_1, a_2, a_3, a_4\}$ with the multiplication table:

	a_1	a_2	a_3	a_4
a_1	a_1	a_2	a_3	a_4
a_2	a_2	a_1	a_4	a_3
a_3	a_1	a_2	a_3	a_4
a_4	a_2	a_1	a_4	a_3

This semigroup admits a homomorphism φ onto a group of order two which we denote by $\bar{G} = \{\bar{e}, \bar{a}\}$. Hereby $\varphi^{-1}(\bar{e}) = \{a_1, a_3\} = H$ and $\varphi^{-1}(\bar{a}) = Ha_2H = Ha_4H = \{a_2, a_4\}$. In our notations we have $G_{11} = \{a_1, a_2\}$, $G_{12} = \{a_3, a_4\}$. The intersections $G'_{11} = H \cap G_{11} = \{a_1\}$, $G'_{12} = H \cap G_{12} = \{a_3\}$ are normal subgroups of G_{11} and G_{12} respectively. Hence H is an almost normal subsemigroup of S. The subsemigroup H is not "normal" in the sense that $Hb = bH$ since $Ha_2 = \{a_2\}$ and $a_2H = \{a_2, a_4\}$. This makes it clear that the use of double cosets is an essential one and that there is in general not possible to reduce a double coset to a unique one-sided coset.

Before proving the main theorem it is useful to prove the following.

Lemma 4. *If $a \in G_{\sigma\varrho}$, $b \in G_{\sigma_1\varrho_1}$, and H is an almost normal subsemigroup of S, then $HaH \,.\, HbH = HcH$ with $c = ae_{\sigma\varrho_1}b$.*

Proof. If $a \in G_{\sigma\varrho}$, then

$$HaH = \bigcup_{\substack{\alpha\beta \\ \gamma\delta}} G'_{\alpha\beta}aG'_{\gamma\delta} = \bigcup_{\substack{\alpha\beta \\ \gamma\delta}} (G'_{\alpha\beta}e_{\sigma\varrho}ae_{\sigma\varrho}G'_{\gamma\delta}) = \bigcup_{\alpha,\delta} G'_{\alpha\varrho}aG'_{\sigma\delta}.$$

Analogously for $b \in G_{\sigma_1\varrho_1}$ we have $HbH = \bigcup_{\alpha_1\delta_1} G'_{\alpha_1\varrho_1}bG'_{\sigma_1\delta_1}$.

Therefore

$$HaH \,.\, HbH = \bigcup_{\substack{\alpha,\delta \\ \alpha_1,\delta_1}} G'_{\alpha\varrho}aG'_{\sigma\delta}G'_{\alpha_1\varrho_1}bG'_{\sigma_1\delta_1} = \bigcup_{\alpha,\delta_1} G'_{\alpha\varrho}aG'_{\sigma\varrho_1}bG'_{\sigma_1\delta_1}.$$

Now $aG'_{\sigma\varrho_1} = (e_{\sigma\varrho}a)\, e_{\sigma\varrho_1}G'_{\sigma\varrho_1}$, and since $e_{\sigma\varrho}ae_{\sigma\varrho_1} \in G_{\sigma\varrho_1}$, we have by almost normality $aG'_{\sigma\varrho_1} = G'_{\sigma\varrho_1}(e_{\sigma\varrho}ae_{\sigma\varrho_1})$. This implies

$$HaH \,.\, HbH = \bigcup_{\alpha,\delta_1} G'_{\alpha\varrho}G'_{\sigma\varrho_1}(e_{\sigma\varrho}ae_{\sigma\varrho_1})\, bG'_{\sigma_1\delta_1} = \bigcup_{\alpha,\delta_1} G'_{\alpha\varrho_1}cG'_{\sigma_1\delta_1} = HcH$$

with $c = e_{\sigma_1\varrho_1}ae_{\sigma\varrho_1}b \in G_{\sigma_1\varrho_1}$.

4. Theorem. *If φ is a homomorphism of a completely simple semigroup S onto a group $\bar{G}$ with unit element $\bar{e}$, then $H = \varphi^{-1}(\bar{e})$ is an almost normal subsemigroup of S. For any $\bar{a} \in \bar{G}$ we have $\varphi^{-1}(\bar{a}) = HaH$ with a suitably chosen $a \in S$. The group $\bar{G}$ is isomorphic with the group of classes in the double coset decomposition*

$$S = H \cup HaH \cup HbH \cup \ldots \qquad (1)$$

Conversely: If H is an almost normal subsemigroup of S, then the classes in the decompositon (1) are congruence classes of a homomorphism of S onto a group $\bar{G}$.

Proof. a) Let $\varphi^{-1}(\bar{e}) = H$ and $H \cap G_{11} = G'_{11}$. By Lemma 2 for any $\bar{a} \in \bar{G}$ the set $\varphi^{-1}(\bar{a})$ is a double coset class of the form HaH with suitably chosen $a \in S$. Since each class HaH has a non-empty intersection with a fixed chosen group-component, say G_{11}, the homomorphism φ restricted to G_{11} is a homomorphism of the group G_{11} onto the group $\bar{G}$. Hence G'_{11} is a normal subgroup of G_{11}. Therefore H is an almost normal subsemigroup of S. The isomorphism of $\bar{G}$ with the group of cosets in (1) is an immediate consequence of the suppositions.

b) Let H be an almost normal subsemigroup of S and consider the decomposition of S into disjoint classes as given by (1).

By Lemma 4 the classes form a semigroup with H as unit element. To prove that they form a group it is sufficient to prove that to every class HaH there is a class Ha^*H such that $HaH \,.\, Ha^*H = Ha^*H \,.\, HaH = H$. Let be $a \in G_{\sigma\varrho}$. Denote by a^* the inverse element of a in $G_{\sigma\varrho}$ and consider the product $HaH \,.\, Ha^*H$. By Lemma 4 (with a^* instead of b) we have $HaH \,.\, Ha^*H = HcH$ with $c = e_{\sigma\varrho}ae_{\sigma\varrho}a^* = e_{\sigma\varrho}$, hence $= HcH = H$. Analogously $Ha^*H \,.\, HaH = H$. This proves our theorem.

5. Consider the intersection H_0 of all almost normal subsemigroups of S. The semigroup H_0 is non-empty, since it contains the subsemigroup H_{00} generated by all idempotents $\in S$. (Of course H_{00} need not be almost normal.)

We prove that *H_0 is a (uniquely determined) almost normal subsemigroup of S.* Let $\{H^{(\nu)}, \nu \in \Sigma\}$ be the set of all almost normal subsemigroups of S. Write $H^{(\nu)} = \bigcup_{\beta \in \Lambda_2} L^{(\nu)}_\beta = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G^{(\nu)}_{\alpha\beta}$. Denote $\bigcap_{\nu \in \Sigma} L^{(\nu)}_\beta = L^{(0)}_\beta$ and $\bigcap_{\nu \in \Sigma} G^{(\nu)}_{\alpha\beta} = G^{(0)}_{\alpha\beta}$. Clearly $G^{(0)}_{\alpha\beta}$

is a normal subgroup of the group $G_{\alpha\beta}$. Hence it remains only to show that $H^{(0)}$ is simple. We have*

$$H_0 = \bigcap_{\nu\in\Sigma} H^{(\nu)} = \bigcap_{\nu\in\Sigma} [\bigcup_{\beta\in\Lambda_2} L_\beta^{(\nu)}] = \bigcup_{\beta\in\Lambda_2} [\bigcap_{\nu\in\Sigma} L_\beta^{(\nu)}] = \bigcup_{\beta\in\Lambda_2} L_\beta^{(0)}.$$

The set $L_\beta^{(0)}$ is a left ideal of $H^{(0)}$. (For $H^{(0)}L_\beta^{(0)} \subset H^{(\nu)}L_\beta^{(\nu)} \subset L_\beta^{(\nu)}$ for every $\nu \in \Sigma$, hence $H^{(0)}L_\beta^{(0)} \subset \bigcap_{\nu\in\Sigma} L_\beta^{(\nu)} = L_\beta^{(0)}$.) We prove that $L_\beta^{(0)}$ is a minimal left ideal of $H^{(0)}$. Let L_β^* be a left ideal of $H^{(0)}$ such that $L_\beta^* \subset L_\beta^{(0)}$ and let be $a \in L_\beta^*$. Then a is contained in a group, say $a \in G_{\sigma\beta}^{(0)}$, $\sigma \in \Lambda_1$. A left ideal containing a contains the whole group $G_{\sigma\beta}^{(0)}$, hence $e_{\sigma\beta} \in L_\beta^*$. Now (since $e_{\sigma\beta}$ is a right unit of $L_\beta^{(0)}$) $L_\beta^{(0)} = L_\beta^{(0)} e_{\sigma\beta} \subset H^{(0)}L_\beta^* \subset L_\beta^*$, whence $L_\beta^{(0)} = L_\beta^*$. Since $H^{(0)}$ is the union of its minimal left ideals, $H^{(0)}$ is simple, which concludes the proof of our statement.

Denote by $\bar{G}$ the factor group S/H_0 (i. e. the group of classes of the decomposition $S = H_0 \cup H_0aH_0 \cup \ldots$). Denote further by φ_0 the corresponding homomorphism $S \to \bar{G}$.

Let ψ be any homomorphism of S onto a group K with unit element e^*. Then $H = \psi^{-1}(e^*)$ is an almost normal subsemigroup of S, hence $H \supset H_0$. The group K is isomorphic with the factor group $\bar{\bar{G}} = S/H$ (i. e. the group of classes of the decomposition $S = H \cup Ha'H \cup \ldots$).

Since H is itself a completely simple semigroup (and H_0 an almost normal subsemigroup of H) we have

$$H = H_0 \cup H_0a''H_0 \cup H_0b''H_0 \cup \ldots, \qquad a'', b'', \ldots \in H,$$

and each class of $\bar{\bar{G}}$ may be considered as a set theoretical union of some elements $\in \bar{G}$ (or better to say $\bar{\bar{G}}$ are classes of an equivalence relation on $\bar{G}$). Since both $\bar{G}$ and $\bar{\bar{G}}$ are groups, the class H (considered as a subset of $\bar{G}$) is a normal subgroup of $\bar{G}$. There exists therefore a homomorphism ϑ of $\bar{G}$ onto $\bar{\bar{G}}$. Now since φ_0: $S \to \bar{G}$ and ϑ: $\bar{G} \to \bar{\bar{G}}$ (and $\bar{\bar{G}} \simeq K$) we have $\psi = \varphi_0\vartheta$. This means: *Any homomorphism ψ of S onto a group K is of the form $\psi = \varphi_0\vartheta$, where ϑ is a homomorphism of $\bar{G}$ onto K. In this sense $\bar{G}$ may be considered as a maximal group homomorphic image of S.*

6. We have insisted on the use of the double cosests since they are directly the congruence classes belonging to φ. Of course the structure of the maximal group homomorphic image (as well as of other group homomorphic images) can be described in terms of coset decompositions of one group-component, say G_{11}, with respect to a certain normal subgroup.

Let H_0 be the minimal almost normal subsemigroup of S and denote $G'_{11} = G_{11} \cap H_0$. Let

$$S = H_0 \cup H_0aH_0 \cup H_0bH_0 \cup \ldots \qquad (2)$$

* Hereby we use the fact that for $\beta_1 \neq \beta_2$ we have $L_{\beta_1} \cap L_{\beta_2} = \emptyset$, and since $L_{\beta_1}^{(\nu)} \subset L_{\beta_1}$, $L_{\beta_2}^{(\mu)} \subset L_{\beta_2}$ we have $L_{\beta_1}^{(\nu)} \cap L_{\beta_2}^{(\mu)} = \emptyset$ for $\beta_1 \neq \beta_2$.

be the double coset decomposition of S modulo (H_0, H_0). Without loss of generality suppose again that $a, b, \ldots$ are elements $\in G_{11}$. The relation (2) implies

$$G_{11} = G'_{11} \cup G'_{11}aG_{11} \cup G'_{11}bG'_{11} \cup \ldots$$

With respect to almost normality of H_0 we have $G'_{11}aG'_{11} = G'_{11}(G'_{11}a) = G'_{11}a = aG'_{11}$. Consider the correspondence

$$H_0aH_0 \to G'_{11}aG'_{11}, \qquad H_0bH_0 \to G'_{11}bG'_{11}. \qquad (3)$$

Then (by Lemma 4) $H_0aH_0H_0bH_0 = H_0abH_0 \to G'_{11}abG'_{11} = (G'_{11}a)(G'_{11}b)$. Since the correspondence (3) is a one-to-one, it follows that the group of classes of G_{11} with respect to the normal subgroup G'_{11} is isomorphic with the group $\bar{G}$ of double classes as introduced above. Hence: *The maximal group homomorphic image $\bar{G}$ is isomorphic with the factor group G_{11}/G'_{11}.*

REFERENCES

[1] Clifford A. H., Preston G. B., *The algebraic theory of semigroups*, Vol. I, Math. Surveys, No. 7, Amer. Math. Society, Providence, R. I., 1961.

[2] Dubreil P., *Algèbre*, Gauthier-Villars, Paris, second ed., 1954.

[3] Глускин Л. М., *Вполне простые полугруппы*, Уч. зап. Харьковского пед. ин-та *18* (1956), 41—55.

[4] Ляпин Е. С., *Полугруппы*, гос. изд. физ.-мат. лит., Москва, 1960.

[5] Preston G. B., *Congruences on completely 0-simple semigroups*, Proc. London Math. Soc. (3) 11 (1961), 557—576.

[6] Stoll, R. R., *Homomorphism of a semigroup onto a group*, Amer. J. Math. 73 (1951), 475—481.

[7] Schwarz Š., *Subsemigroups of a simple semigroup*, Czechoslovak Math. J. 13 (88) (1963). To appear.

[8] Schwarz Š., *Probabilities on non-commutative semigroups*, Czechoslovak Math. J. 13 (88) (1963). To appear.

[9] Schwarz Š., *Convolution semigroup of measures on compact non-commutative semigroups*, Czechoslovak Math. J. To appear.

[10] Schwarz Š., *Probability measures on non-commutative semigroups*, Proceedings of the International symposion on topology, Prague 1961. Publ. House Czechoslovak Acad. Sci. Prague 1962.

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ГОМОМОРФИЗМЫ ВПОЛНЕ ПРОСТОЙ ПОЛУГРУППЫ НА ГРУППУ

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Резюме

Пусть S — вполне простая полугруппа без нуля. Как известно (см. [4], стр. 263), S можно написать в виде объединения $S = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G_{\alpha\beta}$, где $G_{\alpha\beta}$ — изоморфные между собой группы.

Подполугруппу H полугруппы S назовем почти нормальной, если а) H содержит все идемпотенты $\in S$; б) пересечение $G'_{\alpha\beta} = G_{\alpha\beta} \cap H$ является нормальной подгруппой группы $G_{\alpha\beta}$ хотя бы для одной пары α, β. (Оказывается, что в этом случае $G'_{\gamma\delta}$ — нормальная подгруппа $G_{\gamma\delta}$ для всякой пары $\gamma \in \Lambda_1, \delta \in \Lambda_2$).

В статье доказываются следующие утверждения.

1. Пусть φ — гомоморфизм S на некоторую группу $\bar{G}$ с единичным элементом $\bar{e}$. Тогда полный прообраз единицы $\varphi^{-1}(\bar{e}) = H$ — почти нормальная подполугруппа полугруппы S. Для всякого $\bar{a} \in \bar{G}$ имеет место $\varphi^{-1}(\bar{a}) = HaH$ с подходящим образом выбранным $a \in S$.

Группа $\bar{G}$ изоморфна группе классов в однозначно определенном разложении

$$S = H \cup HaH \cup HbH \cup \ldots ; \; a, b, \ldots \in S, \qquad (*)$$

причем произведение классов определяется естественным образом как произведение комплексов в S. (Заметим, что два класса HaH и HbH или совпадают, или не пересекаются, и произведение двух классов некоторой весь класс.)

2. Наоборот: Если H — некотороя почти нормальная подполугруппа полугруппы S, и если построим разложение (*), то существует такой гомоморфизм φ полугруппы S на некоторую группу, при котором каждый класс есть полный прообраз одного элемента группы $\varphi(S)$.

3. Если $H = H_0$ — минимальная почти нормальная подполугруппа полугруппы S, то соответствующая группа $\bar{G}$ является в естественном смысле максимальным групповым образом полугруппы S. Далее, для любой пары (α, β) $\bar{G} \cong G_{\alpha\beta}/G'_{\alpha\beta}$, где $G'_{\alpha\beta} = G_{\alpha\beta} \cap H_0$.