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NOTE ON THE STOKES FORMULA FOR 2-DIMENSIONAL INTEGRALS IN n-SPACE

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In present note some theorems of the Stokes type concerning curvilinear and 2-dimensional integrals in n-space are established.

1. Introduction. The term path (on $\langle a, b \rangle$) is taken to mean a continuous mapping f of $\langle a, b \rangle = \{t; t \in E_1, a \leq t \leq b\}$ into E_n, the Euclidean n-space; f will be termed closed provided $f(a) = f(b)$. (For $n = 2$ we shall speak of a plane path.) The length of f on $\langle a, b \rangle$ is defined as usual; we say that f is rectifiable if its length is finite. Let f be a plane path on $\langle a, b \rangle$ and let $\Phi = [\Phi_1, \ldots, \Phi_n]$, $\Psi = [\Psi_1, \ldots, \Psi_n]$ be continuous mappings of $[f] = f(\langle a, b \rangle)$ into E_n. We put

$$\int_f \Phi \, \mathrm{d}\Psi = \sum_{i=1}^{n} \int_a^b \Phi_i(f(t)) \, \mathrm{d}\Psi_i(f(t))$$

provided the Stieltjes integrals on the right-hand side exist. If $K = \langle \alpha, \beta \rangle \times \langle \gamma, \delta \rangle$ is a rectangle then f_K will stand for the closed plane path describing simply the boundary of K in positive sense.

Let now Φ, Ψ be continuous mappings of K into E_n. General conditions are known which, imposed on Φ and Ψ, secure the existence of an integrable function γ on K with

$$\int_{f_K} \Phi \, \mathrm{d}\Psi = \iint_K \gamma \tag{1}$$

(the integral on the right-hand side is taken in the sense of Lebesgue). The aim of the present paper is, roughly speaking, to extend the validity of (1) to the case where f_K is replaced by a finite number of rectifiable closed plane paths with any number of self-intersections.

If $G \subset E_2$ is an open set and Φ, Ψ are mappings of G into E_n, then

$$\gamma = \operatorname{rot}(\Phi, \Psi) \text{ in } G$$

means that (1) holds for every rectangle $K \subset G$.

Given a closed plane path f and a point $z \in E_2 - [f]$ we shall denote by ind (z, f) the index of z with respect to f. (The reader may consult T. Radó's monograph [1], II. 4. 34 and IV. 1. 24 for a precise definition.) Our main objective is to prove the following theorem.

1,1. Theorem. *Let $f^1, \ldots, f^m$ be rectifiable closed plane paths and put $C = \bigcup_{k=1}^{m} [f^k]$,* $\omega(z) = \sum_{k=1}^{m} \operatorname{ind}(z; f^k)$ $(z \in E_2 - C)$, $G = \{z; z \in E_2 - C, \omega(z) \neq 0\}$, $G_p = \{z; z \in E_2 - C, \omega(z) = p\}$. *Let Φ, Ψ be continuous mappings of $C \cup G$ into E_n and suppose that Ψ is Lipschitzian on $C \cup G$ and $\gamma = \operatorname{rot}(\Phi, \Psi)$ in G. Then*

$$\sum_{k=1}^{m} \int_{f^k} \Phi \, \mathrm{d}\Psi = \sum_{l=1}^{\infty} \left[\sum_{p \geq l} \left(\iint_{G_p} \gamma - \iint_{G_{-p}} \gamma \right) \right] \tag{2}$$

provided the Lebesgue integrals $\iint_{G_p} \gamma$ $(p \neq 0)$ exist.

1,2. Remark. A sort of formula (2) still holds even if the Lebesgue integrals $\iint_{G_p} \gamma$ $(p \neq 0)$ fail to exist (cf. theorem 11,1 below).

The right-hand side in (2) max be replaced by the series

$$\sum_{p=1}^{\infty} p \left(\iint_{G_p} \gamma - \iint_{G_{-p}} \gamma \right) \tag{3}$$

provided (3) is convergent (possibly, non-absolutely). In [2], p. 595, an example is given showing that (3) may actually diverge even in the relatively simple case where $n = 2$ and Ψ is the identity map. If the integral

$$\iint_G \omega\gamma \tag{4}$$

happens to exist, then, in (2), we may write simply (4) instead of $\sum_{l=1}^{\infty} [\sum_{p \geq l} (\ldots)]$.

1,3. Remark. From 1,1 we obtain as a corollary a theorem of the Stokes type for 2-dimensional Lipschitzian surfaces in E_n bounded by a finite number of rectifiable curves (cf. remark 11,4 below). The reader may consult H. Whitney's monograph [3] for the rôle of analogous theorems dealing with k-dimensional integrals. An extensive bibliography concerning the Stokes formula together with corresponding comments on the subject is given in K. Krickeberg's article [4].

2. Before going into the proof of our main theorem we shall establish several auxiliary results. Let us agree to accept the following notation. H_r will stand for the r-dimensional Hausdorff measure. Given $z = [z_1, \ldots, z_{r+1}] \in E_{r+1}$ and a positive integer $i \in \langle 1, r + 1 \rangle$ we put $\hat{z}^i = [z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{r+1}]$. For $F \subset E_{r+1}$ put $\hat{F}^i = \{\hat{z}^i; z \in F\}$. For every $x \in E_r$ denote by $N_i(F, x)$ the number (possibly zero or infinite) of points in $\{z; z \in F, \hat{z}^i = x\}$.

2,1. Lemma. *Let F be an analytic set in E_{r+1}. Then $N_i(F, x)$ is Lebesgue measurable with respect to x on E_r and*

$$H_r(F) \geq \int_{E_r} N_i(F, x) \, \mathrm{d}x.$$

Proof. Write F_{nk} for the set of all $z = [z_1, \ldots, z_{r+1}] \in \hat{F}$ with $k \cdot 2^{-n} \leq z_i < (k+1)\, 2^{-n}$. Every F_{nk} is analytic and, consequently, $\hat{F}^i_{nk}$ is Lebesgue measurable. Clearly, $H_r(\hat{F}^i_{nk}) \leq H_r(F_{nk})$. Denoting by χ_{nk} the characteristic function of $\hat{F}^i_{nk}$ on E_r we obtain $\sum_{k=-\infty}^{\infty} \chi_{nk}(x) \nearrow N_i(F, x)$ $(n \to \infty)$ and

$$\int_{E_r} N_i(F, x)\, dx = \lim_{n\to\infty} \int_{E_r} \Big(\sum_{k=-\infty}^{\infty} \chi_{nk}(x) \Big)\, dx = \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} H_r(\hat{F}^i_{nk}) \leq$$

$$\leq \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} H_r(F_{nk}) = H_r(F).$$

3. Some background material. $\mathbf{D}$ will be used to denote the set of all infinitely differentiable functions with compact support on E_{r+1}. Let $\mathbf{A}_i$ be the system of all Lebesgue measurable sets $A \subset E_{r+1}$ with

$$+\infty > \| A \|_i = \sup_{\varphi} \int_A \frac{\partial \varphi(z)}{\partial z_i}\, dz, \qquad \varphi \in \mathbf{D}, \qquad \max_z |\varphi(z)| \leq 1.$$

A measurable set A belongs to $\mathbf{A}_i$ if and only if such a finite signed Borel measure $\mathbf{P}^A_i$ exists over the boundary $\dot{A}$ of A that

$$\varphi \in \mathbf{D} \Rightarrow \int_{\dot{A}} \varphi\, d\mathbf{P}^A_i = \int_A \frac{\partial \varphi(z)}{\partial z_i}\, dz.$$

$\| A \|_i$ is equal to the variation of $\mathbf{P}^A_i$ on $\dot{A}$ whenever $\dot{A} \in \mathbf{A}_i$. Further put $\mathbf{A} = \bigcap_{i=1}^{r+1} \mathbf{A}_i$. $\mathbf{A}$ is the system of all measurable $A \subset E_{r+1}$ for which the following is true: Such a vector-valued measure $\mathbf{P}^A = [\mathbf{P}^A_1, \ldots, \mathbf{P}^A_{r+1}]$ exists over $\dot{A}$ that

$$\int_{\dot{A}} v\, d\mathbf{P}^A \Big(= \sum_{i=1}^{r+1} \int_{\dot{A}} v_i\, d\mathbf{P}^A_i\Big) = \int_A \operatorname{div} v(z)\, dz$$

for every vector-valued function $v = [v_1, \ldots, v_{r+1}]$ with $v_i \in \mathbf{D}$, $1 \leq i \leq r+1$. Writing $\mathbf{V}^1$ for the set of all $v = [v_1, \ldots, v_{r+1}]$ with $v_i \in \mathbf{D}$ $(1 \leq i \leq r+1)$, $|v(z)| = \big(\sum_{i=1}^{r+1} v_i^2(z)\big)^{\frac{1}{2}} \leq 1$ on E_{r+1}, we have for a measurable set $A \subset E_{r+1}$

$$+\infty > \| A \| = \sup_v \int_A \operatorname{div} v(z)\, dz, \; v \in \mathbf{V}^1,$$

if and only if $A \in \mathbf{A}$. $\| A \|$ coincides with the total variation of the vector-valued measure $\mathbf{P}^A$ on $\dot{A}$ whenever $A \in \mathbf{A}$. $\mathbf{A}_i$ and $\mathbf{A}$ are Boolean algebrae.

$\mathbf{A}_i$ includes all measurable sets A with $\int_{E_r} N_i(\dot{A}, x)\, dx < +\infty$. In particular, every $A \subset E_{r+1}$ with $H_r(\dot{A}) < +\infty$ belongs to $\mathbf{A}$ and $\| A \| \leq H_r(\dot{A})$.

3,1. Remark. The systems $\mathbf{A}$ and $\mathbf{A}_i$ were, from different points of view, introduced by E. De Giorgi and J. Mařík. Their properties were studied by several authors. Interested reader is referred to [5] for a bibliography on the subject.

4. A set in E_{r+1} which can be represented as a union of a finite number of compact $(r+1)$-dimensional intervals which are allowed to have a void interior will be called a figure. $\bar{A}$, A°, $\dot{A}$ and diam A will stand for the closure, the interior, the boundary and the diameter of A $(\subset E_{r+1})$ respectively. $\mathbf{L}$ will denote the Lebesgue measure in E_{r+1}.

4,1. Lemma.* *Let $A \subset E_{r+1}$ be a bounded set, $A^\circ \neq 0$, $H_r(\dot{A}) < +\infty$. Then there exists a sequence of figures A_k $(k = 1, 2, \ldots)$ such that $\sup_k H_r(\dot{A}_k) < +\infty$ and*

$$A_k \subset A_{k+1} \qquad (k = 1, 2, \ldots), \qquad \bigcup_k A_k = A^\circ.^{**}$$

Proof. For every positive integer p there exists a sequence $\{K_{jp}\}_{j=1}^{\infty}$ of open $(r+1)$-dimensional cubes such that $\dot{A} \subset \bigcup_{j=1}^{\infty} K_{jp}$, diam $K_{jp} < \frac{1}{p}$ $(j = 1, 2, \ldots)$ and

$$\sum_j \operatorname{diam}^r K_{jp} < 1 + cH_r(\dot{A}), \tag{5}$$

where $c > 0$ is a constant independent of p. Rearranging the sequence $\{K_{jp}\}_{j=1}^{\infty}$, if necessary, we can fix a $j(p)$ such that

$$\dot{A} \subset \bigcup_{j=1}^{j(p)} K_{jp}, \qquad A \cap K_{jp} \neq 0 \qquad \text{whenever} \qquad j \in \langle 1, j(p) \rangle.$$

Denote by p_1 the least p with $A - \bigcup_{j=1}^{j(p)} K_{jp} \neq \emptyset$ and put $A_1 = A - \bigcup_{j=1}^{j(p_1)} K_{jp_1}$. Clearly, $A_1 \subset A^\circ$ and A_1 is a figure. Suppose now that figures $A_1 \subset \ldots \subset A_k$ have already been constructed. Denote by p_{k+1} the least p for which $\bigcup_{j=1}^{j(p)} K_{jp}$ has a positive distance from A_k and put $A_{k+1} = A - \bigcup_{j=1}^{j(p_{k+1})} K_{jp_{k+1}}$. Repeating this procedure infinitely many times we arrive at a sequence of figures $A_k \nearrow A^\circ$ $(k \to \infty)$. Taking (5) into account we see that $H_r(\dot{A}_k) \leq \sum_k H_r(\dot{K}_{jp_k}) \leq 2(r+1) \sum_j \operatorname{diam}^r K_{jp_k} < 2(r+1) \times [1 + cH_r(\dot{A})]$ for every k. Thus the proof is complete.

4,2. Lemma. *Let $A \subset E_{r+1}$ be a bounded set and suppose that there exist $A_k \in \mathbf{A}$ $(k = 1, 2, \ldots)$ such that $A_k \subset A$, $\lim_{k\to\infty} \mathbf{L}(A - A_k) = 0$, $\limsup_{k\to\infty} \| A_k \| = c < +\infty$.*

* Cf. also [13], lemma 19, 26, p. 154.

** This will be expressed symbolically in the form $A_k \nearrow A^\circ$ $(k \to \infty)$.

Then $\| A \| \leq c$ *and*

$$\int_{\dot A} v \, d\mathbf{P}^A = \lim_{k\to\infty} \int_{\dot A_k} v \, d\mathbf{P}^{A_k} \tag{6}$$

for every continuous $(r+1)$*-dimensional vector-valued function* v *on* $\bar A$.*

Proof. We have for $v \in \mathbf{V}^1$

$$\int_A \operatorname{div} v(z) \, dz = \lim_{k\to\infty} \int_{A_k} \operatorname{div} v(z) \, dz = \lim_{k\to\infty} \int_{\dot A_k} v \, d\mathbf{P}^{A_k} \leq$$

$$\leq \limsup_{k\to\infty} \| A_k \| = c. \text{ Consequently, } \| A \| \leq c.$$

Noticing that

$$\Big| \int_{\dot A} v \, d\mathbf{P}^A - \int_{\dot A} \tilde v \, d\mathbf{P}^A \Big| \leq \varepsilon c, \qquad \Big| \int_{\dot A_k} v \, d\mathbf{P}^{A_k} - \int_{\dot A_k} \tilde v \, d\mathbf{P}^{A_k} \Big| \leq \varepsilon c$$

whenever $v, \tilde v$ are continuous vector-valued funtions on $\bar A$ with $\max_{z\in\bar A} | v(z) - \tilde v(z) | \leq \varepsilon$, we see at once that it is sufficient to prove (6) for $v = [v_1, \ldots, v_{r+1}]$ with $v_i \in \mathbf{D}$ $(1 \leq i \leq r+1)$ only. For such a v

$$\int_{\dot A} v \, d\mathbf{P}^A = \int_A \operatorname{div} v(z) \, dz = \lim_{k\to\infty} \int_{A_k} \operatorname{div} v(z) \, dz = \lim_{k\to\infty} \int_{\dot A_k} v \, d\mathbf{P}^{A_k}.$$

5. The scalar product of vectors $u, v \in E_n$ will be denoted by $u \circ v$. Given $M \subset E_2$ we shall denote by $\mathbf{C}_n^{(0)}(M)$ the system of all continuous mappings of M into E_n. If M happens to be open, then $\mathbf{C}_n^{(1)}(M)$ will stand for the system of all $\Phi = [\Phi_1, \ldots, \Phi_n] \in \mathbf{C}_n^{(0)}(M)$ whose components Φ_i $(1 \leq i \leq n)$ have continuous first order partial derivatives in M. We shall write simply $\mathbf{C}_n^{(1)}$ instead of $\mathbf{C}_n^{(1)}(E_2)$ and ι will beused to denote the identity map of E_2 onto itself. $V(a, b)$ is the system of all rectifiable plane paths on $\langle a, b\rangle$, $V_0(a, b)$ is the subsystem of all $f \in V(a, b)$ with $(a) = f(b)$ (i. e. of all clo sed paths in $V(a, b)$).

5,1. Lemma. *Let* $f \in V(a, b)$, $\Psi \in \mathbf{C}_n^0([f])$, $\Phi \in \mathbf{C}_n^{(1)}(O)$, *where* O *is some neighbourhood of* $[f]$ *in* E_2. *Define the mapping* $\chi = [\chi_1, \chi_2]$ *of* $[f]$ *into* E_2 *by*

$$\chi_1 = \Psi \circ \frac{\partial \Phi}{\partial x}, \qquad \chi_2 = \Psi \circ \frac{\partial \Phi}{\partial y}.^{**} \tag{7}$$

Then $\chi \in \mathbf{C}_2^{(0)}([f])$ *and*

$$\int_f \Psi \, d\Phi = \int_f \chi \, d\iota.$$

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* This assertion was communicated to us by prof. J. Mařík, compare also [13], lemma 19, 21, pp. 150—151.

** We write $\dfrac{\partial\Phi}{\partial x} = \left[\dfrac{\partial\Phi_1}{\partial x}, \ldots, \dfrac{\partial\Phi_n}{\partial x}\right]$ for $\Phi(x, y) = \Phi = [\Phi_1, \ldots, \Phi_n]$; $\dfrac{\partial\Phi}{\partial y}$ has a similar meaning.

Proof. Let $f = [f_1, f_2]$ and put $t_k^m = a + k(b-a)\,m^{-1}$, $z_k^m = f(t_k^m)$ $(k = 0, \ldots, m;$ $m = 1, 2, \ldots)$. It is easily seen that $\Phi(f)$ is rectifiable on $\langle a, b\rangle$ and

$$\Phi(z_k^m) - \Phi(z_{k-1}^m) = \frac{\partial(\Phi z_{k-1}^m)}{\partial x}[f_1(t_k^m) - f_1(t_{k-1}^m)] +$$

$$+ \frac{\partial\Phi(z_{k-1}^m)}{\partial y}[f_2(t_k^m) - f_2(t_{k-1}^m)] + [f(t_k^m) - f(t_{k-1}^m)] \cdot \vec{o}_{mk},$$

where $\max_k | \vec{o}_{mk} | \to 0$ as $m \to \infty$. Hence

$$\int_f \Psi \, d\Phi = \lim_{m\to\infty} \sum_{k=1}^m \Psi(z_{k-1}^m) \circ [\Phi(z_k^m) - \Phi(z_{k-1}^m)] =$$

$$= \lim_{m\to\infty} \sum_{k=1}^m \chi(z_{k-1}^m) \circ [\iota(z_k^m) - \iota(z_{k-1}^m)] = \int_f \chi \, d\iota.$$

6. In section 3 we have recalled some basic properties of the systems $\mathbf{A}_\iota$ and $\mathbf{A}$ of subsets in E_{r+1}. Since no simplification could have been acquired by specialization to $r = 1$, we described the general situation for any $r \geq 1$. However, the special case $r = 1$ is the only one we shall deal with in the sequel. Let us agree that, from now on, the systems $\mathbf{A}_\iota$ and $\mathbf{A}$ will be considered with respect to E_2 only. (Thus every set of $\mathbf{A}_\iota$, $\mathbf{A}$ to be met below is a subset in E_2.) Further denote by $\tilde{\mathbf{A}}$ the subsystem of all $A \in \mathbf{A}$ whose boundary $\dot A$ is compact.

6,1. Definition. Let $A \in \tilde{\mathbf{A}}$, $\Phi \in \mathbf{C}_n^{(1)}$, $\Psi \in \mathbf{C}_n^{(0)}(\dot A)$. We put

$$P(A, \Phi, \Psi) = \int_{\dot A} \tilde\chi \, d\mathbf{P}^A,$$

where $\tilde\chi = [-\chi_2, \chi_1]$ and χ_1, χ_2 are defined by (7).

6,2. Lemma. *Let* $f \in V_0(a, b)$, $A \subset E_2$ *and suppose that*

$$\{z; \operatorname{ind}(z; f) = 1\} = A, \ \{z; \operatorname{ind}(z; f) = 0\} = E_2 - \bar A.$$

Then $A \in \tilde{\mathbf{A}}$ and

$$P(A, \Phi, \Psi) = \int_f \Phi \, d\Psi$$

whenever Φ, Ψ, $\dfrac{\partial\Phi}{\partial x}$, $\dfrac{\partial\Phi}{\partial y} \in \mathbf{C}_n^{(1)}$.

Proof. Since $\dot A \subset [f]$ and f is rectifiable, we have $H_1(\dot A) < +\infty$. Consequently, $A \in \tilde{\mathbf{A}}$. Using Green's formula (cf. [6]) and lemma 5,1 we obtain

$$P(A, \Phi, \Psi) = \int_{\dot A} \tilde\chi \, d\mathbf{P}^A = \iint_A \operatorname{div} \tilde\chi = -\iint_A \left(\frac{\partial\chi_2}{\partial x} - \frac{\partial\chi_1}{\partial y}\right) = -\int_f \chi \, d\iota = -\int_f \Psi \, d\Phi.$$

Finally, integration by parts for Stieltjes integrals yields $-\int_f \Psi \, d\Phi = \int_f \Phi \, d\Psi$.

6,3. Remark. In 6,2, the assumption $\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \Psi \in \mathbf{C}_n^{(1)}$ could be generalized to $\Psi \in \mathbf{C}_n^{(0)}(A)$, $\Phi \in \mathbf{C}_n^{(1)}$. As lemma 6,2 shows, $P(A, \Phi, \Psi)$ can be considered as an analogue of $\int_f \Phi \, d\Psi$. Indeed, if f is a positively oriented rectifiable simple closed curve bounding A, then these two quantities coincide with each other.

6,4. Lemma. *Let $A \in \tilde{\mathbf{A}}$ and let $\Phi, \Psi \in \mathbf{C}_n^{(1)}$. Then*

$$P(A, \Phi, \Psi) = -P(A, \Psi, \Phi).$$

Proof. Since $P(A, \ldots) = -P(E_2 - A, \ldots)$, we may assume that A is bounded. Let us recall that for $h \in \mathbf{C}_1^{(1)}$ and a solenoidal vector-valued function $v \in \mathbf{C}_2^{(1)}$ the formula

$$\int_{\dot{A}} hv \, d\mathbf{P}^A = \iint_A \operatorname{grad} h \circ v \tag{8}$$

is true (cf. [7], theorem 48, p. 554). Applying (8) to $h = \Psi_i$, $v = \left[-\frac{\partial \Phi_i}{\partial y}, \frac{\partial \Phi_i}{\partial x}\right]$ $(i = 1, \ldots, n)$, we obtain

$$P(A, \Phi, \Psi) = \sum_{i=1}^{n} \iint_A \left(-\frac{\partial \Psi_i}{\partial x} \cdot \frac{\partial \Phi_i}{\partial y} + \frac{\partial \Psi_i}{\partial y} \cdot \frac{\partial \Phi_i}{\partial x}\right).$$

In a similar way

$$P(A, \Psi, \Phi) = \sum_{i=1}^{n} \iint_A \left(-\frac{\partial \Phi_i}{\partial x} \cdot \frac{\partial \Psi_i}{\partial y} + \frac{\partial \Phi_i}{\partial y} \cdot \frac{\partial \Psi_i}{\partial x}\right).$$

whence our lemma follows at once.

7. Given a $M \subset E_2$ and a mapping Φ of M into E_n we put for any $N \subset M$

$$\| \Phi \|_N = \sup_{z \in N} | \Phi(z) |.$$

We say that Φ is Lipschitzian on N with constant λ provided $| \Phi(u) - \Phi(v) | \leqq \lambda | u - v |$ whenever $u, v \in N$.

7,1. Definition. Let $A \in \tilde{\mathbf{A}}$, $\Psi \in \mathbf{C}_n^{(0)}(A)$. We define

$$\alpha(A, \Psi) = \sup_{\Phi} P(A, \Phi, \Psi),$$

Φ ranging over the class of all $\Phi \in \mathbf{C}_n^{(1)}$ with $\| \Phi \|_{E_2} \leqq 1$.

7,2. Lemma. *Let $A \in \tilde{\mathbf{A}}$, $\Psi = [\Psi_1, \ldots, \Psi_n] \in \mathbf{C}_n^{(1)}$ and suppose that*

$$\left| \frac{\partial \Psi_i(z)}{\partial x} \right| \leqq \lambda, \quad \left| \frac{\partial \Psi_i(z)}{\partial y} \right| \leqq \lambda \quad (i = 1, \ldots, n)$$

whenever $z \in \dot{A}$. Then, for every $\Phi \in \mathbf{C}_n^{(1)}$,

$$| P(A, \Phi, \Psi) | \leqq \lambda \sqrt{2} \, \| \Phi \|_{\dot{A}} \cdot \| A \|. \tag{9}$$

In particular,

$$\alpha(A, \Psi) \leqq \lambda \sqrt{2} \, \| A \|. \tag{10}$$

Proof. Writing $v = \left[-\Phi \circ \frac{\partial \Psi}{\partial y}, \Phi \circ \frac{\partial \Psi}{\partial x}\right]$ we obtain by 6,4 and 6,1

$$| P(A, \Phi, \Psi) | = | P(A, \Psi, \Phi) | = \left| \int_{\dot{A}} v \, d\mathbf{P}^A \right| \leqq \| v \|_{\dot{A}} \cdot \| A \|.$$

Clearly, $\| v \|_{\dot{A}} \leqq \lambda \sqrt{2} \, \| \Phi \|_{\dot{A}}$.

7,3. Lemma. *Let h be a function which is Lipschitzian on E_2 with constant λ. Then there exists a sequence of functions $h_k \in \mathbf{C}_1^{(1)}$ $(k = 1, 2, \ldots)$ such that $h_k \to h$ uniformly on E_2 as $k \to \infty$ and*

$$\left| \frac{\partial h_k}{\partial x} \right| \leqq \lambda, \quad \left| \frac{\partial h_k}{\partial y} \right| \leqq \lambda \quad (k = 1, 2, \ldots).$$

Proof. This lemma is well known.

7,4. Proposition. *Let $A \in \tilde{\mathbf{A}}$ and let Ψ be a mapping of $\dot{A}$ into E_n, which is Lipschitzian on $\dot{A}$ with constant λ. Then* (10) *is valid.*

Proof. We may assume that $\Psi = [\Psi_1, \ldots, \Psi_n]$, where Ψ_i $(1 \leqq i \leqq n)$ are Lipschitzian on E_2 with constant λ (cf. [8], lemma 1, p. 341). According to 7,2 we have a sequence $\Psi_i^k \in \mathbf{C}_1^{(1)}$ $(k = 1, 2, \ldots)$ such that $\Psi_i^k \to \Psi_i$ $(k \to \infty)$ uniformly on E_2 and $\left| \frac{\partial \Psi_i^k}{\partial x} \right| \leqq \lambda$, $\left| \frac{\partial \Psi_i^k}{\partial y} \right| \leqq \lambda$. Put $\Psi^k = [\Psi_1^k, \ldots, \Psi_n^k]$. Clearly, $\Psi^k \in \mathbf{C}_n^{(1)}$ and, in view of 7,2, $| P(A, \Phi, \Psi^k) | \leqq \lambda \sqrt{2} \, \| \Phi \|_{\dot{A}} \cdot \| A \|$ for an arbitrary $\Phi \in \mathbf{C}_n^{(1)}$. Making $k \to \infty$ we obtain (9) (cf. the definition 6,1). Hence (10) easily follows.

8,1. Lemma. *Let $A \in \tilde{\mathbf{A}}$ and suppose that $\Psi \in \mathbf{C}_n^{(0)}(\dot{A})$, $\alpha(A, \Psi) < \infty$. Then, for every $\Phi \in \mathbf{C}_n^{(1)}$,*

$$| P(A, \Phi, \Psi) | \leqq \| \Phi \|_{\dot{A}} \cdot \alpha(A, \Psi).$$

Proof. Given $\varepsilon > 0$ and $\Phi \in C_n^{(1)}$ we can fix a $\tilde{\Phi} \in C_n^{(1)}$ such that $\| \tilde{\Phi} \|_{E_2} \leqq \varepsilon + \| \Phi \|_{\dot{A}}$ and $\tilde{\Phi} = \Phi$ in some neighbourhood of $\dot{A}$ (cf. lemma 5 in [7]). According to the definition 7,1 we have $| P(A, \Phi, \Psi) | = | P(A, \tilde{\Phi}, \Psi) | \leqq \| \tilde{\Phi} \|_{E_2} \cdot \alpha(A, \Psi) \leqq (\varepsilon + \| \Phi \|_{\dot{A}}) \cdot \alpha(A, \Psi)$. Since ε was an arbitrary positive number, the proof is complete.

8,2. Remark. Let $A \in \tilde{\mathbf{A}}$, $\Psi \in \mathbf{C}_n^{(0)}(\dot{A})$, $\alpha(A, \Psi) < +\infty$. Fix $\Phi \in \mathbf{C}_n^{(0)}(A)$ and suppose that $\Phi^k \in \mathbf{C}_n^{(1)}$ $(k = 1, 2, \ldots)$,

$$\lim_{k \to \infty} \| \Phi - \Phi^k \|_{\dot{A}} = 0. \tag{11}$$

It follows easily from 8,1 that the limit $\lim_{k\to\infty} P(A, \Phi^k, \Psi)$ exists and is independent of the choice of the sequence $\{\Phi^k\}_{k=1}^{\infty}$ fulfilling (11). We are thus justified to introduce the following definition:

8,3. Definition. Let $A \in \tilde{\mathbf{A}}$, $\Psi \in \mathbf{C}_n^{(0)}(\dot{A})$, $\alpha(A, \Psi) < +\infty$. For any $\Phi \in \mathbf{C}_n^{(0)}(\dot{A})$ put

$$P(A, \Phi, \Psi) = \lim_{k\to\infty} P(A, \Phi^k, \Psi),$$

where $\{\Phi^k\}_{k=1}^{\infty}$ is a sequence of mappings in $\mathbf{C}_n^{(1)}$ fulfilling (11).

9. The symbols $f^1, \ldots, f^m, C, \omega, G_p, G$ will have the same meaning as in the theorem 1,1. Further put $U_l = \{z;\ z \in E_2 - C,\ \omega(z) \geqq l\}$.

9,1. Lemma.

$$\sum_{l=-\infty}^{\infty} \| U_l \| < +\infty, \qquad \sum_{p=-\infty}^{\infty} \| G_p \| < +\infty. \tag{12}$$

In particular, $U_p, G_p \in A$ for every integer p and, by proposition 7,4,

$$\sum_{l=-\infty}^{\infty} \alpha(U_l, \Psi) < +\infty, \qquad \sum_{p=-\infty}^{\infty} \alpha(G_p, \Psi) < +\infty \tag{13}$$

for every Lipschitzian mapping Ψ of C into E_n.

Proof. Since $\sum_l \| U_l \|_i \leqq \sum_l \int_{E_1} N_i(\dot{U}_l, y)\,dy$, $\sum_p \| G_p \|_i \leqq \sum_p \int_{E_1} N_i(\dot{G}_p, y)\,dy$, it is sufficient to prove that the functions

$$\sum_l N_i(\dot{U}_l, y), \qquad \sum_p N_i(\dot{G}_p, y) \qquad (i = 1, 2)$$

are integrable (with respect to the variable y) on E_1. Clearly, we may consider the case $i = 2$ only. Let us keep the notation introduced in [2], section 15, pp. 589–591. From investigations described there we obtain for every $y \in E_1 - M$

$$N_2(\dot{U}_l, y) \leqq \sum_{j=1}^{n} |s_y(U_l, u_j)|, \qquad \sum_l \sum_{j=1}^{n} |s_y(U_l, u_j)| \leqq \psi(y).$$

Noticing that M has measure zero and ψ is integrable on E_1 we see that integrability of $\sum_l N_2(\dot{U}_l, y)$ is checked. Similarly, investigations described in [2], p. 592, imply the inequality

$$\sum_p N_2(\dot{G}_p, y) \leqq 2\psi(y) \qquad (y \in E_1 - M)$$

showing that $\sum_p N_2(\dot{G}_p, y)$ is integrable on E_1.

9,2. Theorem. *Let $\Phi \in \mathbf{C}_n^{(0)}(C)$ and let Ψ be a Lipschitzian mapping of C into E_n. Then*

$$\sum_{k=1}^{m} \int_{f^k} \Phi \, d\Psi = \sum_{l=1}^{\infty} \{ \sum_{p \geqq l} [P(G_p, \Phi, \Psi) - P(G_{-p}, \Phi, \Psi)]\}. \tag{14}$$

Proof. We shall first prove

$$\sum_{k=1}^{m} \int_{f^k} \Phi \, d\Psi = \sum_{l=1}^{\infty} [P(U_l, \Phi, \Psi) + P(U_{1-l}, \Phi, \Psi)], \tag{15}$$

$$P(U_l, \Phi, \Psi) + P(U_{1-l}, \Phi, \Psi) = \sum_{p \geqq l} [P(G_p, \Phi, \Psi) - P(G_{-p}, \Phi, \Psi)], \tag{16}$$

whence (14) follows at once. In view of (13) we may assume that $\Phi \in \mathbf{C}_n^{(1)}$ (cf. also 8,2). Define $\chi = [\chi_1, \chi_2]$ by (7). We obtain from 5,1

$$\sum_{k=1}^{m} \int_{f^k} \Phi \, d\Psi = -\sum_{k=1}^{m} \int_{f^k} \chi \, d\iota. \tag{18}$$

Keeping the notation introduced in [2], section 17, we derive from theorem 15 and remark 17 in [2]

$$\sum_{k=1}^{m} \int_{f^k} \chi \, di = \sum_{l=1}^{\infty} [P_0(U_l, \chi) + P_0(U_{1-l}, \chi)]. \tag{15bis}$$

In a similar way we obtain from investigations on p. 592 in [2]

$$P_0(U_l, \chi) + P_0(U_{1-l}, \chi) = \sum_{p \geqq l} [P_0(G_p, \chi) - P_0(G_{-p}, \chi)]. \tag{16bis}$$

Comparing the definition 6,1 of the present note with the remark 17 in [2] we see that

$$P_0(U_l, \chi) = -\int_{\dot{U}_l} \tilde{\chi} \, d\mathbf{P}^{U_l} = -P(U_l, \Phi, \Psi),$$

$$P_0(G_p, \chi) = -P(G_p, \Phi, \Psi).$$

Thus (15^{bis}), (16^{bis}) and (18) imply (15), (16).

10,1. Lemma. *Let $A \subset E_2$ be a bounded set, $A_k \subset A$ $(k = 1, 2, \ldots)$ and suppose that*

$$\lim_{k\to\infty} \mathbf{L}(A - A_k) = 0, \quad \limsup_{k\to\infty} \| A_k \| < +\infty.$$

Let Ψ be a Lipschitzian mapping of $\bar{A}$ into E_n. Then

$$\alpha(A, \Psi) < +\infty, \quad \limsup_{k\to\infty} \alpha(A_k, \Psi) < +\infty \tag{19}$$

and, for every $\Phi \in \mathbf{C}_n^{(0)}(\dot{A})$,

$$\lim_{k\to\infty} P(A_k, \Phi, \Psi) = P(A, \Phi, \Psi). \tag{20}$$

Proof. By lemma 4,2, (20) is true for any $\Phi \in \mathbf{C}_n^{(1)}$ (cf. 6,1). In view of 7,4 we obtain (19). Hence it folows easily that (20) can be extended, by continuity, to any $\Phi \in \mathbf{C}_n^{(0)}(\bar{A})$.

10,2. Definition. Let $A \subset E_2$ be a bounded set, $\mathbf{L}(\dot{A}) = 0$. Let γ be a function defined almost everywhere on A such that the Lebesgue integral $\iint_K \gamma$ is available for every rectangle $K \subset A^\circ$. (Consequently, $\iint_B \gamma$ exists for every two-dimensional figure $B \subset A^\circ$ as well.) If

$$\lim_{k\to\infty} \iint_{A_k} \gamma \tag{21}$$

exists for every sequence of figures $A_k \nearrow A^\circ$ $(k \to \infty)$ with

$$\sup_k H_1(\dot{A}_k) < +\infty, \tag{22}$$

then the limit (21) is independent of the choice of figures $A_k \to A^\circ$ fulfilling (22) and its value will be denoted by $\mathbf{L}(A, \gamma)$.

10,3. Remark. Of course, $\mathbf{L}(A, \gamma) = \iint_A \gamma$ whenever γ happens to be Lebesgue integrable on A, so that $\mathbf{L}(A, \gamma)$ may be considered as an extension of the Lebesgue integral. For more general study of analogous extensions the reader may consult [9].

The articles [10], [11] reviewed in Ref. jour. 1959 which seem to deal with similar problems were not available to us.

10,4. Proposition. *Let $A \subset E_2$ be a bounded set, $H_1(\dot{A}) < +\infty$. Let $\Phi \in \mathbf{C}_n^{(0)}(\bar{A})$ and suppose that Ψ is a Lipschitzian mapping of $\bar{A}$ into E_n.*

If $\gamma = \text{rot}(\Phi, \Psi)$ in A°, then

$$P(A, \Phi, \Psi) = \mathbf{L}(A, \gamma).$$

This proposition follows easily from 10,1 and 10,2.

11. As an easy consequence of 9,2 and 10,4 we obtain the following theorem.

11,1. Theorem. *Let us keep all the assumptions and notation of the theorem* 1,1. *Then*

$$\sum_{k=1}^{m} \int_{f^k} \Phi \, d\Psi = \sum_{l=1}^{m} \{\sum_{p \geqq l} [\mathbf{L}(G_p, \gamma) - \mathbf{L}(G_{-p}, \gamma)]\}. \tag{23}$$

11,2. Remark. Theorem 1,1 is merely a corollary of 11,1.

11,3. Remark. The right hand side in (23) may be replaced by $\sum_{p=1}^{n} p[\mathbf{L}(G_p, \gamma) - \mathbf{L}(G_{-p}, \gamma)]$ if this series happens to converge.

11,4. Remark. Let $\Psi = [\Psi_1, \ldots, \Psi_n]$ be a Lipschitzian mapping of G into E_n and let $\Gamma = [\Gamma_1, \ldots, \Gamma_n]$ be a Lipschitzian mapping of $V = \Psi(G)$ into E_n. Suppose that $M \subset V$, $\mathbf{L}\Psi^{-1}(M) = 0$ and that $\sum_{k=1}^{n} \tau_k^i(u^\circ)(u_k - u_k^0)$ is the differential of Γ^i with respect to V at any $u^0 = [u_1^0, \ldots, u_n^0] \in V - M$. Put $\Phi(z) = \Gamma(\Psi(z))$,

$$\gamma(z) = \sum_{\substack{i,k=1 \\ i<k}}^{n} [\tau_i^k(\Psi(z)) - \tau_k^i(\Psi(z))] \cdot \begin{vmatrix} \frac{\partial \Psi_i(z)}{\partial x}, & \frac{\partial \Psi_i(z)}{\partial y} \\ \frac{\partial \Psi_k(z)}{\partial x}, & \frac{\partial \Psi_k(z)}{\partial y} \end{vmatrix}$$

as far as the symbols involved are meaningful. Then $\gamma = \text{rot}(\Phi, \Psi)$ in G.

This follows at once from theorem 12 in [12].

This assertion can be combined with 11,1.

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ЗАМЕТКА К ТЕОРЕМЕ СТОКЕСА ДЛЯ ДВУМЕРНЫХ ИНТЕГРАЛОВ В n-МЕРНОМ ПРОСТРАНСТВЕ

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Выводы

Если f — непрерывное отображение отрезка $\langle a, b\rangle$ в плоскость E_2, то символом $v(a, b, f)$ обозначим длину пути f, определенную обыкновенным образом. Если $v(a, b, f) < +\infty$, то мы будем говорить, что $f \in V(a, b)$. Подсистему всех $f \in V(a, b)$ удовлетворяющих условию $f(a) = f(b)$, обозначим символом $V_0(a, b)$. Если $f \in V(a, b)$ и $\Phi = [\Phi_1, \ldots, \Phi_n]$, $\Psi = [\Psi_1, \ldots, \Psi_n]$ — непрерывные отображения множества $f(\langle a, b\rangle)$ в пространство E_n, то полагаем по определению $\int_f \Phi \, d\Psi = \sum_{i=1}^{n} \int_a^b \Phi_i(f(t)) \, d\Psi_i(f(t))$ в предложении, что существуют соответствующие интегралы Стиль тьеса. Словом „интервал" мы будем подразумевать двумерный компактный интервал. Если K — интервал, то обозначим f_K отображение из $V_0(a, b)$, которое даст параметрическое представление контура K (описываемого в положительном направлении при изменении параметра от a до b). Пусть теперь G — открытое множество в E_2 и пусть Φ, Ψ — непрерывные отображения множества G в пространство E_n. Пусть, далее, γ — функция определенная почти всюду на G и интегрируемая по Лебегу на каждом интервале $K \subset G$. Будем говорить, что $\gamma = \mathrm{rot}\,(\Phi, \Psi)$ на G, если для каждого интервала $K \subset G$ справедливо равенство $\int_{f_K} \Phi \, d\Psi = \iint_K \gamma$. Множество, которое является соединением конечного числа интервалов будем называть фигурой. Символом H_1, L обозначим линейную меру Хаусдорфа и двумерную меру Лебега соответственно. Пусть A — ограниченное множество в E_2, $\dot{A}$ — его граница, $A^\circ = A - \dot{A}$. Пусть, далее, $L(\dot{A}) = 0$ и пусть γ — функция, определенная почти всюду на A и интегрируемая на каждом интервале $K \subset A^\circ$. Если для каждой последовательности фигур $F_k \subset A^\circ$, удовлетворяющей требованию $\sup H_1(\dot{F}_k) < \infty$ существует предел $\lim_{k \to \infty} \iint_{F_k} \gamma$, то этот предел не зависит от последовательности $\{F_k\}_{k=1}^{\infty}$ и мы его обозначим через $\mathbf{L}(A, \gamma)$; разумеется $\mathbf{L}(A, \gamma) = \iint_A \gamma$ если γ интегрируемая на A.

Теорема. Пусть $f^j \in V_0(a_j, b_j)$ $(1 \leqq f \leqq m)$, $C = \bigcup_{j=1}^{m} f^j(\langle a_j, b_j\rangle)$. Для $z \in E_2 - C$ положим $\omega(z) = \sum_{j=1}^{m} \mathrm{ind}(z, f^j)$, где $\mathrm{ind}(z, f^j)$ обозначает порядок точки z относительно пути f^j. Пусть $G_p = \{z;\, z \in E_2 - C,\, \omega(z) = p\}$, $G = \bigcup_{p \neq 0} G_p$ и пусть на $C \cup G$ определены непрерывные отображения Φ, Ψ в пространство E_n, причем Ψ удовлетворяет условию Липшица. Если $\gamma = \mathrm{rot}\,(\Phi, \Psi)$ на G, тогда существуют несобственные интегралы $\mathbf{L}(G_p, \gamma)$ $(p \neq 0)$ и имеет место формула

$$\sum_{j=1}^{m} \int_{f^j} \Phi \, d\Psi = \sum_{l=1}^{\infty} \Big\{ \sum_{p \geqq l} [\mathbf{L}(G_p, \gamma) - \mathbf{L}(G_{-p}, \gamma)] \Big\}. \qquad (*)$$

Правую часть равенства (*) можно заменить на $\sum_{p=1}^{\infty} p[\mathbf{L}(G_p, \gamma) - \mathbf{L}(G_{-p}, \gamma)]$ соотв. на $\iint_G \omega\gamma$, если последние символы имеют смысл.