THE INTERVAL TOPOLOGY OF AN *l*-GROUP

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Let G be an l-group, $a, c \in G$. We shall call a an archimedean element, if $a > 0$ and if for each $b \in G$ there exists a positive integer n such that $na \ngeq b$. The sets

$$I_1(c) = \{x \mid x \in G,\ x \leqq c\}, \qquad I_2(c) = \{x \mid x \in G,\ x \geqq c\}$$

are called infinite intervals (in G). The interval topology of G is defined by taking as a sub-basis for the closed sets all infinite intervals and the set G. We will consider the following condition:

(t) *G is a topological group in its interval topology.*

G. Birkhoff [1, p. 233, problem 104] has asked the question: Does any l-group satisfy the condition (t)? It is a rather trivial fact that any ordered (= linearly ordered) l-group satisfies (t). E. S. Northam [4, proposition 6] proved that the additive group A of all continuous real-valued functions defined on the closed unit interval (using the natural ordering) is an l-group which does not satisfy (t). T. H. Choe [3] has shown: If each non-empty subset $M \subset G^+$ has a minimal element and if G satisfies (t), then G is ordered. In the recent paper [2] P. Conrad studies l-groups which fufill the condition (F): Each $a \in G$, $a > 0$ is greater than or equal to at most a finite number of disjoint elements. (The elements $c, d \in G$ are called disjoint if $c \cap d = 0$.) It is proved in [2, theorem 6.3]: If G satisfies the conditions (F) and (t) then G is ordered. (Evidently this theorem includes the result of Choe but not that of Northam.) In this note we prove the following

Theorem. *If there exist disjoint archimedean elements $a, b \in G$ then G does not satisfy (t).*

Corollary. *Any archimedean l-group satisfying (t) is ordered.*

Clearly this implies the result of Northam. Since an l-group in which each non-empty subset $M \subset G^+$ has a minimal element is archimedean (this follows easily from [1, p. 236, Theorem 21]) the result of Choe is also a consequence of the corollary.

1. *Let $a, b \in G$, $a > 0$, $b > 0$, $a \cap b = 0$. Let I be the set of all integers, $A = \{x \mid x = ma,\ m \in I\}$, $B = \{y \mid y = nb,\ n \in I\}$, $C = \{z \mid z = x + y,\ x \in A,\ x \in B\}$. Then* a) *$C$ is an l-subgroup of G, and* b) *C is isomorphic with the direct product* ([1, p. 222]) *of l-groups A, B.*

Proof. Let $m, n \in I$, $m > 0$, $n > 0$, $m_i, n_i \in I$, $i = 1, 2$. From $a \cap b = 0$ follows (cf. [1, p. 219]) $ma \cap nb = 0$, $ma + nb = ma \cup nb = nb + ma$, hence $m_1a + n_1b = n_1b + m_1a$. Therefore C is a subgroup of the group G. Let $m_3 = \max(m_1, m_2)$, $n_3 = \max(n_1, n_2)$, $m_4 = \min(m_1, m_2)$, $n_4 = \min(n_1, n_2)$, $z_i = m_ia + n_ib$. If m_i, n_i $(i = 1, 2)$ are non-negative, then $z_i = m_ia \cup n_ib$, hence (because of the distributivity of G)

$$z_1 \cup z_2 = m_3a + n_3b, \qquad z_1 \cap z_2 = m_4a + n_4b. \tag{1}$$

If m_i, n_i are arbitrary, we choose m, n such that $m + m_i \geqq 0$, $n + n_i \geqq 0$, $i = 1, 2$; let $\circ \in \{\cap, \cup\}$. From

$$z_1 \circ z_2 = ((z_1 + z) \circ (z_2 + z)) - z$$

follows that in this case (1) also holds. Thus the assertion a) is proved. It is now immediate that the mapping $C \to A \times B$ defined by $ma + nb \to (ma, nb)$ is an isomorphism.

In the following C has the same meaning as above.

2. *Let a, b be archimedean elements. Let $u \in G$, $A = I_1(u) \cap C \neq \emptyset$. Then A is an infinitive interval in C.*

Proof. Let $m_0a + n_0b \in A$. Put $M = \{m | m \in I,\ ma + n_0b \leqq u\}$. Since a is archimedean, there exists the greatest element m_1 in M. Denote $N = \{n | n \in I,\ m_1a + nb \leqq u\}$; there exists the greatest element n_1 in N. If $m_1a + n_1b \leqq ma + nb \leqq u$, then $n_1 \leqq n$, $m_1a + nb \leqq u$, hence $n_1 = n$; moreover $m_1 \leqq m$, $ma + n_0b \leqq u$, thus $m = m_1$. This shows that $c_1 = m_1a + n_1b$ is the greatest element of A. Evidently each element $c \in C$, $c \leqq c_1$ belongs to A.

A similar result holds for $I_2(u) \cap C$.

3. *Let A, B be nonzero ordered groups, $D = A \times B$. Then D is not Hausdorff in its interval topology.*

This assertion is proved (though not explicitly stated) in [2, proof of the lemma 6.2].

4. Proof of the theorem. Let a, b be disjoint archimedean elements of G. Let $p, q \in C$, $p \neq q$. Suppose that G satisfies (t). Then there exist infinite intervals $I^1, \ldots, I^n$ such that $\cup I^i = G$ and no I^i contains both p q (this follows easily from the definition of the sub-basis; cf. also [2, proof of the lemma 6.5, and 6.4]). It follows from 2 that the set $I^i \cap C = J_i$ is an infinite interval in C or $J_i = \emptyset$; clearly $\cup J_i = C$ and no J_i contains both p and q. Hence C in Hausdorff in its interval topology. But from 1 and 3 we obtain that C is not Hausdorff, and we have a contradiction.

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ИНТЕРВАЛЬНАЯ ТОПОЛОГИЯ В L-ГРУППАХ

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Резюме

Пусть G — l-группа; $a, c \in G$. Элемент a называется архимедовым, если $a > 0$ и если для каждого $b \in G$ существует натуральное число n такое, что $na \nleqq b$. Множества

$$I_1(c) = \{x \,|\, x \in G,\ x \leqq c\}, \qquad I_2(c) = \{x \,|\, x \in G,\ x \geqq c\}$$

называются бесконечными интервалами в G. Интервальная топология в G определена так, что в качестве суббазы замкнутых множеств берется система, состоящая из всех бесконечных интервалов и из множества G. Мы говорим, что G обладает свойством (t), если G — топологическая группа в интервальной топологии. Доказана следующая

Теорема. Если в G существуют архимедовы элеменины $a, b, a \cap b = 0$, то G не обладает свойством (t).

Следствие. Архимедова l-группа, обладающая свойством (t), является упорядоченной.

Из этого вытекают как частные случаи теоремы Нортгама [4] и Чои [3], касающиеся интервальной топологии в l-группе.